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On totally umbilical submanifolds of Finsler spaces

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Abstract. The notion of a totally umbilical submanifold of a Finsler manifold is introduced. Some Gauss equations are given and some results on totally umbilical submanifolds of Riemannian manifolds are generalized. Totally umbilical submanifolds of Randers spaces are studied; a rigidity theorem and an example are given.

1. Introduction. In recent decades, Finsler geometry has been rapidly developed. The study of the geometry of submanifolds has also made some progress ([HS1], [HS2], [S], [SST], [ST]). By using the Busemann–Hausdorff volume form, Z. Shen ([S]) investigated the geometry of Finsler submanifolds. Avoiding any connections in Finsler geometry, he introduced the notions of mean curvature and normal curvature for Finsler submanifolds. By using the Holmes–Thompson volume form, i.e., the volume form induced from the projective sphere bundle of the Finsler manifold, Q. He and Y. B. Shen ([HS1]) introduced the notions of another mean curvature and the second fundamental form, which coincide with the usual notions in the Riemannian case.

The usual approach in the geometry of submanifolds is to consider the induced (resp. intrinsic) connections and to establish some equations related to the curvatures of submanifolds and the curvatures of the ambient space. These equations are usually too complicated to use. In this paper, first of all, we shall establish some straightforward equations and use them to study totally umbilical submanifolds of Finsler manifolds, which are defined by using the second fundamental form introduced in [HS1]. Secondly, we shall study submanifolds of Randers spaces, give relations between totally umbilical submanifolds of the Randers space $(\widetilde{M}, \tilde{\alpha} + \tilde{\beta})$ and of the Riemannian manifold $(\widetilde{M}, \tilde{\alpha})$, and obtain a rigidity theorem for complete and connected totally umbilical submanifolds of a special Randers space.

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Lastly, an example of a totally umbilical submanifold of a Randers spaces is given.

2. Finsler volume forms and minimal immersions. Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection. A *Finsler metric* on M is a function $F : TM \to [0,\infty)$ with the following properties: (i) F is smooth on $TM \setminus \{0\}$; (ii) $F(x,\lambda y) = \lambda F(x,y)$ for all $\lambda > 0$; (iii) the induced quadratic form g is positive definite, where

$$g := g_{ij}(x,y)dx^i \otimes dx^j, \quad g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}. \tag{2.1}$$

Here and from now on, $[F]_{y^i}$, $[F]_{y^i y^j}$ mean $\frac{\partial F}{\partial y^i}$, $\frac{\partial^2 F}{\partial y^i \partial y^j}$, etc., and we shall use the following convention of index ranges unless otherwise stated:

$$1 \le i,j,\ldots \le n; \quad n+1 \le a,b,\ldots \le m; \quad 1 \le \alpha,\beta,\ldots \le m\ (> n).$$

The projection $\pi : TM \to M$ gives rise to the pull-back bundle π^*TM and its dual π^*T^*M. We shall work on $TM\setminus\{0\}$ and rigidly use only objects that are invariant under rescaling $y \mapsto \lambda y$ ($\lambda > 0$), so that one may view them as objects on the projective sphere bundle SM using homogeneous coordinates (see also [BCS, p. 29, lines 31–39]).

In π^*T^*M there is a global section $\omega = [F]_{y^i}dx^i$, called the *Hilbert form*, whose dual is $l = l^i \frac{\partial}{\partial x^i}$, $l^i = y^i/F$, called the *distinguished field*. Set

$$\delta y^i = \frac{1}{F}(dy^i + N^i_j dx^j), \quad \frac{\delta}{\delta x^j} = \frac{\partial}{\partial x^j} - N^k_j \frac{\partial}{\partial y^k}. \tag{2.2}$$

The volume element dV_{SM} of SM with the Riemannian metric $\hat{g}$ is

$$dV_{SM} = \Omega d\tau \wedge dx, \tag{2.3}$$

where

$$\Omega := \det\left(\frac{g_{ij}}{F}\right), \quad dx = dx^1 \wedge \cdots \wedge dx^n, \tag{2.4}$$

$$d\tau := \sum_{i=1}^{n} (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n. \tag{2.5}$$

The *volume form* of a Finsler n-manifold (M,F) is defined by ([HS1])

$$dV_M := \sigma(x)dx, \quad \sigma(x) := \frac{1}{c_{n-1}} \int_{S_xM} \Omega\, d\tau, \tag{2.6}$$

where c_{n-1} denotes the volume of the unit Euclidean $(n-1)$-sphere S^{n-1}, $S_xM = \{[y] \mid y \in T_xM\}$. It is well known that there exists a unique Chern connection ∇ on π^*TM with $\nabla \frac{\partial}{\partial x^j} = \omega^i_j \frac{\partial}{\partial x^i}$ and $\omega^i_j = \Gamma^i_{jk} dx^k$, satisfying

$$\begin{aligned} &d(dx^i) - dx^j \wedge \omega^i_j = 0, \\ &dg_{ij} - g_{ik}\omega^k_j - g_{jk}\omega^k_i = 2A_{ijk}\delta y^k, \end{aligned} \tag{2.7}$$

where $A_{ijk} = FC_{ijk}$ and $C_{ijk} = \frac{1}{4}[F^2]_{y^iy^jy^k}$ are called the Cartan tensors. The curvature 2-forms of the Chern connection ∇ are

$$d\omega_j^i - \omega_j^k \wedge \omega_k^i = \Omega_j^i := \frac{1}{2} R_j{}^i{}_{kl} dx^k \wedge dx^l + P_j{}^i{}_{kl} dx^k \wedge \delta y^l, \tag{2.8}$$

where $R_j{}^i{}_{kl} = -R_j{}^i{}_{lk}$ and $P_j{}^i{}_{kl} = P_k{}^i{}_{jl}$ are the hh-curvature and the hv-curvature respectively. The Riemannian curvature tensor and the Landsberg curvature tensor are defined by

$$R_j^{\ i} := R_s{}^i{}_{jk} l^s l^k, \qquad L_{jk}^i := -l^s P_s{}^i{}_{jk}, \tag{2.9}$$

respectively, and we have

$$L_{ijk} = g_{il} L_{jk}^l = \dot{A}_{ijk}, \tag{2.10}$$

where "$\cdot$" denotes the covariant derivative along the Hilbert form. There is another torsion-free Berwald connection ${}^b\nabla$ defined by

$${}^b\nabla = \nabla + \dot{A}, \qquad {}^b\Gamma_{jk}^i = \Gamma_{jk}^i + \dot{A}_{jk}^i. \tag{2.11}$$

It is obvious that $\nabla_l = {}^b\nabla_l$.

Let (M, F) and $(\widetilde{M}, \widetilde{F})$ be Finsler manifolds, and $f : M \to \widetilde{M}$ be an immersion. If $F(x, y) = \widetilde{F}(f(x), df(y))$ for all $(x, y) \in TM \setminus \{0\}$, then f is called an *isometric immersion*. It is clear that

$$g_{ij}(x, y) = \tilde{g}_{\alpha\beta}(\tilde{x}, \tilde{y}) f_i^\alpha f_j^\beta \tag{2.12}$$

for the isometric immersion $f : (M, F) \to (\widetilde{M}, \widetilde{F})$, where

$$\tilde{x}^\alpha = f^\alpha(x), \qquad \tilde{y}^\alpha = f_i^\alpha y^i, \qquad f_i^\alpha = \frac{\partial f^\alpha}{\partial x^i}. \tag{2.13}$$

From (2.7)–(2.11), we have

Lemma 2.1 (see also [HS1]). *Let ${}^b\tilde{\nabla}$ be the pullback Berwald connection on $\pi^*(f^{-1}T\widetilde{M})$ and $\tilde{h} = {}^b\tilde{\nabla} df$ be the second fundamental form with respect to the Berwald connection. For any $X \in \mathcal{C}(\pi^*TM)$, $X^H = X^i \frac{\delta}{\delta x^i}$ denotes the horizontal part of X. Then*

$$\begin{aligned} &\tilde{h}(X, Y) = {}^b\tilde{\nabla}_X(dfY) - df({}^b\nabla_X Y) = {}^b\tilde{\nabla}_{X^H}(dfY) - df({}^b\nabla_{X^H} Y), \\ &({}^b\tilde{\nabla}_{X^H}\tilde{g})(U, W) = 2\tilde{A}(U, W, \tilde{h}(l, X)) - 2\tilde{L}(U, W, dfX), \end{aligned} \tag{2.14}$$

for any $U, W \in \mathcal{C}(\pi^* \circ f^{-1}(T\tilde{M}))$ and $X, Y \in \mathcal{C}(\pi^*TM)$.

$(2.14)_1$ can be rewritten as

$$\tilde{h}_{ij}^\alpha = f_{ij}^\alpha - {}^b\Gamma_{ij}^k f_k^\alpha + {}^b\tilde{\Gamma}_{\beta\gamma}^\alpha f_i^\beta f_j^\gamma, \tag{2.15}$$

where $f_{ij}^\alpha = \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j}$. Set

$$\begin{aligned} &h^\alpha = \tilde{h}_{ij}^\alpha y^i y^j = f_{ij}^\alpha y^i y^j - f_k^\alpha G^k + \widetilde{G}^\alpha, \qquad h_\alpha = \tilde{g}_{\alpha\beta} h^\beta, \\ &h := \frac{h^\alpha}{F^2} \frac{\partial}{\partial \tilde{x}^\alpha}, \qquad h^* := \frac{1}{F^2} h_\alpha d\tilde{x}^\alpha, \end{aligned} \tag{2.16}$$

where G^k and $\widetilde{G}^\alpha$ are the geodesic coefficients for (M,F) and $(\widetilde{M},\widetilde{F})$, respectively, and h is the *normal curvature*. Then we see easily that

$$\tilde{h}^\alpha_{ij} = \frac{1}{2}[h^\alpha]_{y^iy^j}, \qquad \tilde{h}^\alpha_{ij}y^j = \frac{1}{2}[h^\alpha]_{y^i}. \tag{2.17}$$

LEMMA 2.2 (see also [HS3, Proposition 3.1]). *Let $f:(M,F)\to(\widetilde{M},\widetilde{F})$ be an immersion. Then*

$$(\tilde{\nabla}_{l^H}\tilde{\nabla}_l df)(X) = df\mathfrak{R}(X) - \frac{\tilde{F}^2}{F^2}\tilde{\mathfrak{R}}(dfX) + {}^b\tilde{\nabla}_{X^H}h, \tag{2.18}$$

for any $X\in\mathcal{C}(\pi^*TM)$, where $\mathfrak{R} = R^i_j dx^j\otimes\frac{\partial}{\partial x^i}$.

Let $B^\alpha_{ij} := \frac{1}{2}\tilde{g}^{\alpha\beta}[h_\beta]_{y^iy^j}$ be the second fundamental form as defined in [HS1]. Then

$$\begin{aligned} B(X,Y) &= \tilde{h}(X,Y) + 2\tilde{A}(\tilde{h}(l,X), dfY) \\ &\qquad + 2\tilde{A}(\tilde{h}(l,Y), dfX) + \tilde{C}^\#(dfX, dfY, h), \\ B(X,l) &= \tilde{h}(X,l) + \tilde{A}(h, dfX), \end{aligned} \tag{2.19}$$

where $\tilde{C}^\# = F^2\tilde{C}_{\lambda\beta\gamma\delta}\tilde{g}^{\lambda\alpha}\frac{\partial}{\partial\tilde{x}^\alpha}\otimes d\tilde{x}^\beta\otimes d\tilde{x}^\gamma\otimes d\tilde{x}^\delta$, $\tilde{C}_{\lambda\beta\gamma\delta} = \frac{\partial^2\tilde{g}_{\lambda\beta}}{\partial\tilde{y}^\gamma\partial\tilde{y}^\delta}$. The trace of B is $H = (1/n)\operatorname{tr}_{\tilde{g}} B$, which is called the *mean curvature vector field* in [HS1]. From $\tilde{A} = \tilde{A}^\alpha_{\beta\gamma}\frac{\partial}{\partial\tilde{x}^\alpha}\otimes d\tilde{x}^\beta\otimes d\tilde{x}^\gamma$, *the Cartan normal curvature operator* $A_h:\mathcal{C}(\pi^*TM)\to\mathcal{C}(\pi^*\circ f^{-1}(T\tilde{M}))$ is defined by

$$A_h(X) = \tilde{A}(h, dfX) \quad \text{for any } X\in\mathcal{C}(\pi^*TM). \tag{2.20}$$

Let $(\pi^*TM)^\perp$ be the orthogonal complement of π^*TM in $\pi^*\circ f^{-1}(T\tilde{M})$ with respect to $\tilde{g}$, and let

$$\mathcal{V}^* = \{\xi\in\mathcal{C}(f^*T^*\widetilde{M}) \mid \xi(df(X)) = 0,\ \forall X\in\mathcal{C}(TM)\},$$

which are both called the *normal bundle* of f in [S].

We know that $h, H, B(X,Y)\in\mathcal{C}(\pi^*TM)^\perp$ in [HS1]. Then from (2.16) we have

$$G^k = \phi^k_\beta(f^\beta_{ij}y^iy^j + \widetilde{G}^\beta), \tag{2.21}$$

where $\phi^k_\beta = f^\alpha_l g^{lk}\tilde{g}_{\alpha\beta}$. Let $p^\perp:\pi^*\circ f^{-1}(T\tilde{M})\to(\pi^*TM)^\perp$ be the orthogonal projection with respect to $\tilde{g}$. Then (2.21) and (2.16) show that

$$h^\beta = p^{\perp\beta}_\alpha(f^\alpha_{ij}y^iy^j + \widetilde{G}^\alpha), \tag{2.22}$$

where $p^{\perp\beta}_\alpha := \delta^\beta_\alpha - f^\beta_i\phi^i_\alpha$. Set

$$\mu = \frac{1}{c_{n-1}\sigma}\bigg(\int_{S_xM}\frac{h_\alpha}{F^2}\Omega\, d\tau\bigg)d\tilde{x}^\alpha. \tag{2.23}$$

Then $\mu\in\mathcal{V}^*$, and it is called the *mean curvature form* of f. An isometric immersion $f:(M,F)\to(\widetilde{M},\widetilde{F})$ is called a *minimal immersion* if any compact domain of M is a critical point of its volume functional with respect to any variation vector field.

LEMMA 2.3 (see also [HS1, Theorem 2.2]). *Let $f:(M,F)\to(\widetilde{M},\widetilde{F})$ be an isometric immersion. Then f is minimal if and only if $\mu=0$.*

3. Gauss equations and totally umbilical submanifolds. First, from Lemma 2.1, Lemma 2.2 and (3.8.3)–(3.8.5) in [BCS], we have

$$
\begin{aligned}
(\tilde{\nabla}_{l^H}\tilde{\nabla}_l df)(X) &= \tilde{\nabla}_{l^H}[(\tilde{\nabla}_l df)(X)] - (\tilde{\nabla}_l df)(\tilde{\nabla}_{l^H}X)\\
&= \tilde{\nabla}_{l^H}[\tilde{h}(X,l)] - (\tilde{\nabla}_l df)(\tilde{\nabla}_{l^H}X)\\
&= (\tilde{\nabla}_{l^H}\tilde{h})(X,l) = (\tilde{\nabla}_{l^H}B)(X,l) - (\tilde{\nabla}_{l^H}A_h)(X),\\
\tilde{F}_{y^\alpha}(h^\alpha_{ij})_{y^k} &= -\frac{\tilde{F}_{y^\alpha}}{F}({}^b\tilde{P}_{\beta\ \gamma\delta}^{\ \alpha} f^\delta_k f^\beta_i f^\gamma_j - {}^bP_{i\ jk}^{\ l} f^\alpha_l)\\
&= \frac{1}{F}(-2\dot{\tilde{A}}_{\beta\gamma\delta} f^\delta_k f^\beta_i f^\gamma_j + g_{ls} l^{s\,b}P_{i\ jk}^{\ l})\\
&= \frac{1}{F}(-2\dot{\tilde{A}}_{\beta\gamma\delta} f^\delta_k f^\beta_i f^\gamma_j + 2\dot{A}_{ijk}).
\end{aligned}
$$

From the formulas above, we have

THEOREM 3.1 (Gauss equations). *Let $f:(M,F)\to(\widetilde{M},\widetilde{F})$ be an isometric immersion. Then*

$$
\begin{aligned}
(3.1)\quad K(X) &= \tilde{K}(df X) + \tilde{g}((\tilde{\nabla}_{l^H}\tilde{h})(X,l) - {}^b\tilde{\nabla}_{X^H}h, df X)\\
&= \tilde{K}(df X) + \tilde{g}((\tilde{\nabla}_{l^H}B)(X,l) - (\tilde{\nabla}_{l^H}A_h)(X) - {}^b\tilde{\nabla}_{X^H}h, df X)
\end{aligned}
$$

for any $X\in\mathcal{C}(\pi^*TM)$ satisfying $X\perp l$ and $\|X\|=1$, where $\tilde{h}$ and h are defined in (2.16) *and* (2.18) *respectively, and*

$$
(3.2)\qquad L(X,Y,Z) = \tilde{L}(df X, df Y, df Z) + \frac{1}{2}\tilde{g}((\nabla_{Y^v}\tilde{h})(X,Z), df l)
$$

for any $X,Y,Z\in\mathcal{C}(\pi^*TM)$, where $Y^v = F dx^i(Y)\frac{\partial}{\partial y^i}$.

From Theorem 3.1 and the fact that ${}^b\tilde{\nabla}_{X^H}h = ({}^b\tilde{\nabla}_{X^H}\tilde{h})(l,l)$, we deduce easily

PROPOSITION 3.2. *Let $f:(M,F)\to(\widetilde{M},\widetilde{F})$ be an isometric immersion.*

1. *If $(\widetilde{M},\widetilde{F})$ has constant flag curvature c, and the second fundamental form $\tilde{h}$ with respect to the Berwald connection is parallel along the horizontal directions, then (M,F) also has the constant flag curvature c.*
2. *If $(\widetilde{M},\widetilde{F})$ is a Landsberg manifold, and the second fundamental form $\tilde{h}$ with respect to the Berwald connection is parallel along the vertical directions, then (M,F) is also a Landsberg manifold.*

An isometric immersion $f:(M,F)\to(\widetilde{M},\widetilde{F})$ is called *totally umbilical* if there exists a vector field $v\in\mathcal{C}((\pi\circ f)^*T\widetilde{M})$ such that

$$B(X,Y)=g(X,Y)v \tag{3.3}$$

for any $X,Y\in\mathcal{C}(\pi^*TM)$.

LEMMA 3.3. *Let $f:(M,F)\to(\widetilde{M},\widetilde{F})$ be an isometric immersion. Then M is totally umbilical if and only if $v=h=H$ and h^* is independent of y.*

Proof. Necessity: It is obvious that

$$H=\frac{1}{n}\operatorname{tr}_{\tilde{g}}B=v,\qquad h=B(l,l)=v.$$

So, (3.3) means $B^{\alpha}_{ij}=(1/F^2)h^{\alpha}g_{ij}$. On the other hand, from (2.16), (2.17) and (2.19), we see that

$$\begin{aligned}B^{\alpha}_{ij}y^j&=(\tilde{h}^{\alpha}_{ij}+2\tilde{A}^{\alpha}_{\beta\gamma}\tilde{h}^{\beta}_{ik}l^kf^{\gamma}_j+2\tilde{A}^{\alpha}_{\beta\gamma}\tilde{h}^{\beta}_{kj}l^kf^{\gamma}_i+\tilde{C}^{\alpha}_{\beta\ \gamma\sigma}h^{\beta}f^{\gamma}_if^{\sigma}_j)y^j\\&=\tilde{h}^{\alpha}_{ij}y^j+\tilde{C}^{\alpha}_{\beta\gamma}h^{\beta}f^{\gamma}_i,\end{aligned}$$

from which we have

$$[h_\alpha]_{y^i}=2\tilde{g}_{\alpha\beta}B^{\beta}_{ij}y^j=\frac{2}{F^2}\tilde{g}_{\alpha\beta}h^{\beta}g_{ij}y^j=2\frac{1}{F}h_\alpha F_{y^i}.$$

Hence,

$$\left[\frac{h_\alpha}{F^2}\right]_{y^i}=F^{-3}(F[h_\alpha]_{y^i}-2F_{y^i}h_\alpha)=0.$$

Conversely, if $\left[\frac{h_\alpha}{F^2}\right]_{y^i}=0$, then

$$\begin{aligned}(h_\alpha)_{y^i}&=2\frac{1}{F}h_\alpha F_{y^i},\\(h_\alpha)_{y^iy^j}&=\frac{2}{F}(h_\alpha F_{y^iy^j}+F_{y^i}(h_\alpha)_{y^j})-\frac{2}{F^2}h_\alpha F_{y^i}F_{y^j}\\&=\frac{2}{F^2}(FF_{y^iy^j}+F_{y^i}F_{y^j})h_\alpha=\frac{2}{F^2}g_{ij}h_\alpha.\end{aligned}$$

So $B^{\alpha}_{ij}=(1/F^2)h^{\alpha}g_{ij}$. ∎

Similar to the Riemannian case, we have

PROPOSITION 3.4. *An isometric immersion $f:(M,F)\to(\widetilde{M},\widetilde{F})$ is totally umbilical minimal if and only if f is totally geodesic.*

Proof. If f is a totally umbilical minimal immersion, then h^* is independent of y and

$$0=\mu_\alpha=\frac{1}{c_{n-1}\sigma}\bigg(\int\limits_{S_xM}\frac{1}{F^2}h_\alpha\Omega\,d\tau\bigg)=\frac{h_\alpha}{F^2},$$

from which we see that $h_\alpha=0$, i.e., $h=0$ and f is totally geodesic. ∎

Theorem 3.5. *Any totally umbilical submanifold of a Landsberg manifold with $A_h = 0$ is also a Landsberg manifold.*

Proof. From (3.3), (2.20) and Lemma 3.3, we know that $B(X,Y) = g(X,Y)h$, and

$$\tilde{C}^{\#}(dfX, dfY, h) = 2\tilde{A}(\tilde{h}(l,Y), dfX)$$

for any $X, Y \in \mathcal{C}(\pi^*TM)$. We infer from (2.19) that

$$\begin{aligned}\tilde{h}(X,l) &= B(l,X) - A_h(X) = g(X,l)h,\\ \tilde{h}(X,Y) &= B(X,Y) - 2\tilde{A}(\tilde{h}(l,X), dfY)\\ &\quad - 2\tilde{A}(\tilde{h}(l,Y), dfX) - \tilde{C}^{\#}(dfX, dfY, h)\\ &= g(X,Y)h,\end{aligned}$$

from which and (3.2) we have

$$L(X,Y,Z) = \tilde{L}(dfX, dfY, dfZ) + \tilde{g}\left(\frac{1}{2}A(X,Y,Z)h + g(X,Z)\tilde{h}(Y,l), dfl\right) = 0$$

for any $X, Y, Z \in \mathcal{C}(\pi^*TM)$. ∎

Theorem 3.6. *Let (M,F) be an n-dimensional totally umbilical submanifold of a locally Minkowski space $(\tilde{M}, \tilde{F})$ $(n \geq 3)$. If there exists a function λ such that $A_h(X) = \lambda(dfX - \omega(X)\tilde{l})$ for any $X \in \mathcal{C}(\pi^*TM)$, then M has scalar flag curvature $\|h\|^2 - l^H(\lambda) - \lambda^2$.*

Proof. From (2.19), for any $X \in \mathcal{C}(\pi^*TM)$ satisfying $X \perp l$ and $\|X\| = 1$, we have

$$\begin{aligned}\tilde{h}(X,l) &= g(X,l)h - \lambda(dfX - \omega(X)\tilde{l}) = -\lambda dfX,\\ (\tilde{\nabla}_{l^H}\tilde{h})(X,l) &= -l^H(\lambda)dfX + \lambda^2 dfX.\end{aligned}$$

From (3.1), (3.2), (2.16) and (2.19), we obtain

$$\begin{aligned}K(X) &= \lambda^2 - l^H(\lambda) + 2\tilde{A}(h, dfX, \tilde{h}(l,X)) + \tilde{g}(h, \tilde{h}(X,X))\\ &= -\lambda^2 - l^H(\lambda) + \tilde{g}(h,h) + 4\lambda\tilde{A}(dfX, dfX, h) - \tilde{C}^{\#}(dfX, dfX, h, h).\end{aligned}$$

From $A_h(X) = \lambda(dfX - \omega(X)\tilde{l})$, we obtain

$$\tilde{C}^{\#}(dfX, dfX, h, h) = 4\lambda\tilde{A}(dfX, dfX, h) = 4\lambda^2.$$

Hence,

$$K(X) = \|h\|^2 - l^H(\lambda) - \lambda^2. \ \blacksquare$$

4. Totally umbilical submanifolds in Randers spaces. Let $f : (M,F) \to (\widetilde{M}, \widetilde{F})$ be an isometric immersion into a Randers $(n+p)$-space

$(\widetilde{M}, \widetilde{F})$ with

$$\widetilde{F} = \tilde{\alpha} + \tilde{\beta} = \sqrt{\tilde{a}_{\alpha\beta}(\tilde{x})\tilde{y}^\alpha \tilde{y}^\beta} + \tilde{b}_\alpha(\tilde{x})\tilde{y}^\alpha,$$
$$\|\tilde{\beta}\| = \sqrt{\tilde{a}^{\alpha\beta}\tilde{b}_\alpha \tilde{b}_\beta} = \tilde{b} \quad (0 \le \tilde{b} < 1).$$

Clearly, we have

$$F = f^*\widetilde{F} = \alpha + \beta = \sqrt{a_{ij}y^i y^j} + b_i y^i, \tag{4.1}$$

where

$$a_{ij} = \tilde{a}_{\alpha\beta} f_i^\alpha f_j^\beta, \qquad b_i = \tilde{b}_\alpha f_i^\alpha, \tag{4.2}$$

which means (M, F) is also a Randers n-space.

LEMMA 4.1. *Let $f : (M, F) \to (\widetilde{M}, \tilde{\alpha} + \tilde{\beta})$ be an isometric immersion into a Randers $(n + p)$-space, and let $\{\mathbf{n}_a\}$ be a local orthonormal frame of the normal bundle $TM^\perp$ of f with respect to the Riemannian metric $\tilde{\alpha}$. Denote*

$$\tilde{\mathbf{n}}_a = \sqrt{\frac{\alpha}{F}}\,[\mathbf{n}_a - \tilde{\beta}(\mathbf{n}_a)\tilde{l}]. \tag{4.3}$$

Then $\{\tilde{\mathbf{n}}_a\}$ is a local orthonormal frame of the normal bundle $[\pi^*TM]^\perp$ of f with respect to $\tilde{g}_{\tilde{y}}$ for $\tilde{y} = df(y)$ in $(\widetilde{M}, \widetilde{F})$.

Proof. Let $\mathbf{n}_a = n_a^\sigma \frac{\partial}{\partial \tilde{x}^\sigma}$ and $\tilde{\mathbf{n}}_a = \tilde{n}_a^\sigma \frac{\partial}{\partial \tilde{x}^\sigma}$. Then from [BCS], we have

$$\tilde{n}_a^\alpha = \sqrt{\frac{\alpha}{F}}\,[n_a^\alpha - \tilde{\beta}(\mathbf{n}_a)\tilde{l}^\alpha] = \sqrt{\frac{F}{\alpha}}\,\tilde{g}^{\alpha\beta}\tilde{a}_{\sigma\beta}n_a^\sigma,$$

$$\begin{aligned} \tilde{g}_{\alpha\beta}\tilde{n}_a^\alpha f_i^\beta &= \sqrt{\frac{F}{\alpha}}\,\tilde{a}_{\alpha\beta}n_a^\alpha f_i^\beta = 0, \\ \tilde{g}(\tilde{\mathbf{n}}_a, \tilde{\mathbf{n}}_b) = \tilde{g}_{\alpha\beta}\tilde{n}_a^\alpha \tilde{n}_b^\beta &= \frac{F}{\alpha}\tilde{g}^{\alpha\tau}\tilde{a}_{\tau\sigma}n_a^\sigma \tilde{a}_{\alpha\mu}n_b^\mu \\ &= [n_a^\alpha - \tilde{\beta}(\mathbf{n}_a)\tilde{l}^\alpha]\tilde{a}_{\alpha\beta}n_b^\beta = \tilde{a}_{\alpha\beta}n_a^\alpha n_b^\beta \\ &= \langle \mathbf{n}, \mathbf{n}\rangle_{\tilde{\alpha}} = \delta_{ab}. \ \blacksquare \end{aligned} \tag{4.4}$$

Let $\overline{G}^i$ and $\overline{\widetilde{G}}^\alpha$ be the geodesic coefficients for (M, F) and $(\widetilde{M}, \widetilde{F})$ with respect to the Riemannian metrics α and $\tilde{\alpha}$, respectively, and $\bar{h}$ be the normal curvature of f with respect to $\tilde{\alpha}$, that is,

$$\bar{h}^\alpha = f_{ij}^\alpha y^i y^j - f_k^\alpha \overline{G}^k + \overline{\widetilde{G}}^\alpha.$$

From (2.21), (2.22) and [BCS], we have

$$\begin{aligned}\widetilde{G}^\alpha &= \overline{\widetilde{G}}^\alpha + \tilde{b}_{\beta|\gamma}\tilde{y}^\beta\tilde{y}^\gamma\tilde{l}^\alpha + (\tilde{a}^{\alpha\beta} - \tilde{l}^\alpha\tilde{b}^\beta)(\tilde{b}_{\beta|\gamma} - \tilde{b}_{\gamma|\beta})\tilde{\alpha}\tilde{y}^\gamma,\\ \overline{G}^k &= f_l^\alpha a^{lk}\tilde{a}_{\alpha\beta}(f_{ij}^\beta y^i y^j + \overline{\widetilde{G}}^\beta),\\ \bar{h}^\alpha &= \sum_a (f_{ij}^\beta y^i y^j + \overline{\widetilde{G}}^\beta)\tilde{a}_{\beta\sigma}n_a^\sigma n_a^\alpha.\end{aligned}$$

Then, from (2.22) and (4.4), we have

$$\begin{aligned}h^\alpha &= \sum_a (f_{ij}^\beta y^i y^j + \widetilde{G}^\beta)\tilde{g}_{\beta\sigma}\tilde{n}_a^\sigma\tilde{n}_a^\alpha\\ &= \sum_a (f_{ij}^\beta y^i y^j + \widetilde{G}^\beta)\tilde{a}_{\beta\sigma}n_a^\sigma[n_a^\alpha - \tilde{\beta}(\mathbf{n}_a)\tilde{l}^\alpha]\\ &= \bar{h}^\alpha - \alpha^2\tilde{\beta}(\bar{h})\tilde{l}^\alpha - \sum_a \alpha(\tilde{b}_{\beta|\gamma} - \tilde{b}_{\gamma|\beta})\tilde{y}^\beta n_a^\gamma[n_a^\alpha - \tilde{\beta}(\mathbf{n}_a)\tilde{l}^\alpha],\\ h_\alpha &= \frac{F}{\alpha}\Big[\bar{h}_\alpha + \sum_a \alpha n_a^\delta(\tilde{b}_{\delta|\tau} - \tilde{b}_{\tau|\delta})\tilde{y}^\tau n_a^\beta a_{\alpha\beta}\Big].\end{aligned} \tag{4.5}$$

From the formulas above, we see that

Proposition 4.2. *Let $f : (M, \alpha + \beta) \to (\widetilde{M}, \tilde{\alpha} + \tilde{\beta})$ be an isometric immersion into a Randers $(n+p)$-space. If $\tilde{\beta}$ is a closed 1-form, then*

$$h = \frac{\alpha^2}{F^2}[\bar{h} - \tilde{\beta}(\bar{h})\tilde{l}], \qquad h^* = \frac{\alpha}{F}\bar{h}^*, \tag{4.6}$$

where $\bar{h}^* = (\bar{h}_\alpha/\alpha^2)d\tilde{x}^\alpha$. Hence $(M, \alpha + \beta)$ is a totally geodesic submanifold of $(\widetilde{M}, \tilde{\alpha} + \tilde{\beta})$ iff (M, α) is a totally geodesic submanifold of $(\widetilde{M}, \tilde{\alpha})$.

Theorem 4.3. *Let $f : (M, \alpha + \beta) \to (\widetilde{M}, \tilde{\alpha} + \tilde{\beta})$ be an isometric immersion into a Randers $(n+p)$-space with closed 1-form $\tilde{\beta}$. Then $(M, \alpha + \beta)$ is totally umbilical if and only if either $(M, \alpha + \beta)$ is a totally geodesic submanifold of $(\widetilde{M}, \tilde{\alpha} + \tilde{\beta})$, or $\beta = 0$ and (M, α) is a totally umbilical submanifold of $(\widetilde{M}, \tilde{\alpha})$.*

Proof. From Lemma 3.3 and (4.6), we know that $H_\alpha = h_\alpha/F^2$ is independent of y, and

$$\bar{h}_\alpha = \alpha F H_\alpha = H_\alpha\left(\frac{1}{2}\alpha^2 + \frac{1}{2}F^2 - \frac{1}{2}\beta^2\right), \qquad (\bar{h}_\alpha)_{y^iy^j} = H_\alpha(a_{ij} + g_{ij} - b_ib_j).$$

Since $\bar{h}_{ij}^\alpha = a^{\alpha\beta}\big(\frac{1}{2}\bar{h}_\beta\big)_{y^iy^j}$ is the second fundamental form with respect to Riemannian connections, it is independent of y, which implies that

$$0 = [\bar{h}_{ij}^\alpha]_{y^k} = \frac{1}{2}a^{\alpha\beta}H_\beta(2C_{ijk}) = a^{\alpha\beta}H_\beta C_{ijk}.$$

Thus, either $H = 0$ or $C = 0$.

If $H = h = 0$, then $(M, \alpha + \beta)$ is a totally geodesic submanifold of $(\widetilde{M}, \tilde{\alpha} + \tilde{\beta})$.

If $C = 0$, then M is a Riemannian manifold with $F = \alpha$, so $\bar{h}^*$ is independent of y iff h^* is independent of y. From Lemma 4.1, we know that (M, α) is a totally umbilical submanifold of $(\widetilde{M}, \tilde{\alpha})$ iff (M, F) is a totally umbilical submanifold of $(\widetilde{M}, \tilde{F})$. ∎

From Theorem 4.3 and Proposition 4.2, we immediately deduce

THEOREM 4.4. *Let $(V^{n+1}, \tilde{\alpha} + \tilde{\beta})$ be a Randers space, where $\tilde{\alpha}$ is a Euclidean metric and $\tilde{\beta}$ is a closed 1-form. Then any complete and connected n-dimensional totally umbilical submanifold of $(V^{n+1}, \tilde{\alpha} + \tilde{\beta})$ must be either a plane or a Euclidean sphere. The latter case happens only when there exist a point $\tilde{x}_0$ and a function $\lambda(\tilde{x})$ on V^{n+1} such that $\tilde{\beta} = \lambda(\tilde{x})d(\|\tilde{x} - \tilde{x}_0\|^2_{\tilde{\alpha}})$ and the sphere is centered at $\tilde{x}_0$.*

EXAMPLE 4.5. Let $(V^{n+1}, \tilde{F})$ be a Randers manifold with $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$, where

$$\tilde{\alpha} = \sqrt{\sum_\alpha (\tilde{y}^\alpha)^2}, \qquad \tilde{\beta} = \sum_\alpha \frac{b\tilde{x}^\alpha d\tilde{x}^\alpha}{\sqrt{\sum_\alpha (\tilde{x}^\alpha)^2}}$$

where b is a constant and $0 < |b| < 1$. Then $d\tilde{\beta} = 0$.

Let

$$M = \Big\{ \tilde{x} \in V^{n+1} \;\Big|\; \sum_\alpha (\tilde{x}^\alpha - \tilde{x}_0^\alpha)^2 = r^2 \Big\},$$

and $f : (M, F) \hookrightarrow (V^{n+1}, \tilde{F})$ be an isometric immersion. It is obvious that

$$\sum_\alpha (f^\alpha - \tilde{x}_0^\alpha) f_i^\alpha = 0.$$

Then the unit normal vector with respect to $\tilde{\alpha}$ is $\mathbf{n} = (1/r)(f - \tilde{x}_0)$, and the unit normal vector with respect to $\tilde{F}$ is $\tilde{\mathbf{n}} = \sqrt{\alpha/F}[\mathbf{n} - \tilde{\beta}(\mathbf{n})\tilde{l}]$.

Let $F = \alpha + \beta$, where

$$\alpha = \sqrt{\sum_\alpha f_i^\alpha f_j^\alpha y^i y^j}, \qquad \beta = \sum_\alpha f_i^\alpha \frac{b\tilde{x}^\alpha y^i}{\sqrt{\sum_\alpha (\tilde{x}^\alpha)^2}}.$$

It is obvious that $\beta = 0$ for any $x \in M$ if and only if $\tilde{x}_0 = 0$. It is well known that (M, α) is a totally umbilical submanifold of $(V^{n+1}, \tilde{\alpha})$ in any case. But from Theorem 4.3, we see that (M, F) is a totally umbilical submanifold of $(V^{n+1}, \tilde{\alpha} + \tilde{\beta})$ if and only if $\tilde{x}_0 = 0$.

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