ANNALES
POLONICI MATHEMATICI
100.2 (2011)

On zeros of differences of meromorphic functions

by Yong Liu and HongXun Yi (Jinan)

Abstract. Let f be a transcendental meromorphic function and $g(z) = f(z + c_1) + \cdots + f(z+c_k) - kf(z)$ and $g_k(z) = f(z+c_1)\cdots f(z+c_k) - f^k(z)$. A number of results are obtained concerning the exponents of convergence of the zeros of $g(z)$, $g_k(z)$, $g(z)/f(z)$, and $g_k(z)/f^k(z)$.

1. Introduction and main results. In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notation of Nevanlinna's value distribution theory of meromorphic functions (see, e.g., [11, 16, 18]). In addition, we will use $\sigma(f)$, $\mu(f)$, $\lambda(f)$, $\overline{\lambda}(f)$ to denote the order, the lower order, the exponent of convergence of the zero-sequence and the exponent of convergence of the distinct zeros of a meromorphic function $f(z)$ respectively.

Recently, a number of papers (including [1, 3, 5, 8, 9, 13, 15, 17]) have focused on complex difference equations and difference analogues of Nevanlinna's theory. Bergweiler and Langley [3] were the first to investigate the existence of zeros of $\triangle f$ and $\triangle f(z)/f(z)$, and obtained many profound and significant results. Those results may be viewed as discrete analogues of the following theorem on the zeros of f'.

Theorem A ([2, 7, 14]). *Let f be transcendental and meromorphic in the plane with*
$$\liminf_{r\to\infty} \frac{T(r,f)}{r} = 0.$$
Then f' has infinitely many zeros.

Theorem A is sharp. If f satisfies the hypotheses of Theorem A, it follows from Hurwitz's theorem that if z_0 is a zero of f' then $f(z+c) - f(z)$ has a zero near z_0 for all sufficiently small $c \in \mathbb{C} \setminus \{0\}$. This makes it natural to ask whether $f(z+c) - f(z)$ must have infinitely many zeros. Bergweiler and Langley [3] answered this problem, and obtained the following theorems.

2010 *Mathematics Subject Classification*: Primary 30D35; Secondary 39B12.
Key words and phrases: complex difference, zero, exponents of convergence.

DOI: 10.4064/ap100-2-6

Theorem B ([3]). *There exists $\delta_0 \in (0, 1/2)$ with the following property. Let f be a transcendental entire function with order*

$$\sigma(f) \le \sigma < 1/2 + \delta_0 < 1.$$

Then

$$G(z) = \frac{f(z+1) - f(z)}{f(z)}$$

has infinitely many zeros.

Theorem C ([3]). *Let f be a function transcendental and meromorphic of lower order $\mu(f) < 1$ in the plane. Let $c \in \mathbb{C} \setminus \{0\}$ be such that at most finitely many poles z_j, z_k of f satisfy $z_j - z_k = c$. Then $g(z) = f(z+c) - f(z)$ has infinitely many zeros.*

The results above show that there are a large number of zeros of differences and divided differences in the complex plane.

Recently, differences of the forms $f(z_j+c_1)+f(z_j+c_2)$, $f(z_j+c_1)f(z_j+c_2)$ appear in a number of papers (see [1, 5, 13, 17]).

Thus, it is natural to ask the following questions.

Problem 1.1. *What are the exponents of convergence of zeros of differences and divided differences?*

Problem 1.2. *What can be said about the zeros of the differences $g(z) = f(z+c_1)+\cdots+f(z+c_k)-kf(z)$ and $g_k(z) = f(z+c_1)\cdots f(z+c_k) - f^k(z)$?*

For $k = 2$, Chen and Shon [4, Theorems 1–6] get some estimates for the zeros of $g(z) = f(z+c_1)+f(z+c_2)-2f(z)$, $g_2(z) = f(z+c_1)f(z+c_2)-f^2(z)$.

For the general case, we obtain the following results.

Theorem 1.1. *Let $f(z)$ be a transcendental entire function of order of growth $\sigma(f) = \sigma < 1$. Let $c_1, \ldots, c_k \in \mathbb{C} \setminus \{0\}$ be such that $c_1 + \cdots + c_k \ne 0$. Then $g(z)$ has infinitely many zeros and satisfies $\lambda(g) = \sigma(g) = \sigma$.*

In particular, if f has at most finitely many zeros z_j satisfying $f(z_j + c_1) + \cdots + f(z_j + c_k) = 0$, then $G(z) = g(z)/f(z)$ satisfies $\lambda(G) = \sigma(G) = \sigma$.

Theorem 1.2. *Let f, c_j $(j = 1, \ldots, k)$ satisfy the conditions of Theorem 1.1. Then $g_k(z)$ has infinitely many zeros and satisfies $\lambda(g_k) = \sigma(g_k) = \sigma$.*

In particular, suppose that the set $H = \{z_j\}$ of all distinct zeros of $f(z)$ satisfies one of the following two conditions:

(i) *at most finitely many zeros z_i, z_l satisfy $z_i - z_l = c_j$ $(j = 1, \ldots, k)$;*
(ii) $\liminf_{j\to\infty} |z_{j+1}/z_j| = l > 1$.

Then $G_k(z) = g_k(z)/f^k(z)$ has infinitely many zeros and satisfies $\lambda(G_k) = \sigma(G_k) = \sigma$.

THEOREM 1.3. *Let $f(z)$ be a transcendental entire function of order of growth $\sigma(f) = \sigma < 1$. Let $c_1, \ldots, c_k \in \mathbb{C} \setminus \{0\}$ be such that $c_1 + \cdots + c_k \neq 0$. If f has at most finitely many poles b_j, b_s satisfying*

$$b_j - b_s = k_1 c_{l_1} + k_2 c_{l_2} \quad (k_d = 0, \pm 1,\ d = 1, 2;\ l_1, l_2 \in \{1, \ldots, k\}),$$

then $g(z)$ has infinitely many zeros and satisfies $\lambda(g) = \sigma(g) = \sigma$.

In particular, if f has at most finitely many zeros z_j satisfying $f(z_j + c_1) + \cdots + f(z_j + c_k) = 0$, then $G(z) = g(z)/f(z)$ has infinitely many zeros and satisfies $\lambda(G) = \sigma(G) = \sigma$.

THEOREM 1.4. *Let f, c_j $(j = 1, \ldots, k)$ satisfy the conditions of Theorem 1.3. If f has at most finitely many poles b_j satisfying*

$$f(b_j + k_1 c_{l_1} + k_2 c_{l_2}) = 0, \infty \quad (k_d = 0, \pm 1,\ d = 1, 2;\ l_1, l_2 \in \{1, \ldots, k\}),$$

then $g_k(z)$ has infinitely many zeros and satisfies $\lambda(g_k) = \sigma(g_k) = \sigma$.

In particular, suppose that the set $H = \{z_j\}$ of all distinct zeros of $f(z)$ satisfies one of the following two conditions:

(i) *at most finitely many zeros z_i, z_l satisfy $z_i - z_l = c_j$ $(j = 1, \ldots, k)$;*
(ii) $\liminf_{j \to \infty} |z_{j+1}/z_j| = l > 1$.

Then $G_k(z) = g_k(z)/f^k(z)$ has infinitely many zeros and satisfies $\lambda(G_k) = \sigma(G_k) = \sigma$.

2. Some lemmas. In order to prove our theorems, we need the following lemmas.

LEMMA 2.1 ([3]). *Let f be transcendental and meromorphic of order less than 1 in the plane. Let $h > 0$. Then there exists an ε-set E_n such that*

$$f(z + c) - f(z) = cf'(z)(1 + o(1)) \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E_n,$$

uniformly in c for $|c| \leq h$.

REMARK 2.1. Following Hayman [12, pp. 75–76], we define an *ε-set* to be a countable union of open discs not containing the origin and subtending angles at the origin, whose sum is finite. If E is an ε-set then the set of $r \geq 1$ for which the circle $S(0, r)$ meets E has finite logarithmic measure, and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

LEMMA 2.2. *Let f be a transcendental and meromorphic function of order less than 1. Let $c_1, \ldots, c_k \in \mathbb{C} \setminus \{0\}$ be such that $c_1 + \cdots + c_k \neq 0$. Then $g(z)$ and $G(z) = g(z)/f(z)$ are both transcendental.*

Proof. Assume that $g(z)$ is a rational function. Then

$$f(z + c_1) + \cdots + f(z + c_k) = R(z) + kf(z), \tag{2.1}$$

where $R(z)$ is a rational function. Now we prove that $f(z)$ has at most finitely many poles. Suppose the contrary. Choose a pole z_0 of $f(z)$ of multiplicity

$m \geq 1$ such that z_0 is not a pole of $R(z)$. Then the right-hand side of (2.1) has a pole of multiplicity m at z_0. Hence, there exists at least one index $l_1 \in \{1, \ldots, k\}$ such that $z_0 + c_{l_1}$ is a pole of $f(z)$ of multiplicity $m_1 \geq m$. Substituting $z_0 + c_{l_1}$ for z into (2.1), we obtain

$$f(z_0 + c_1 + c_{l_1}) + \cdots + f(z_0 + c_k + c_{l_1}) = R(z_0 + c_{l_1}) + kf(z_0 + c_{l_1}).$$

Then there are the following two possibilities:

If $z_0 + c_{l_1}$ is a pole of $R(z)$, we terminate this process and choose another pole z_0 of $f(z)$ in the way we did above.

If $z_0 + c_{l_1}$ is not a pole of $R(z)$, then the right-hand side of (2.1) has a pole of multiplicity m_1 at $z_0 + c_{l_1}$. Hence, there exists at least one index $l_2 \in \{1, \ldots, k\}$ such that $z_0 + c_{l_1} + c_{l_2}$ is a pole of multiplicity $m_2 \geq m_1 \geq m$. We know that $R(z)$ has only finitely many poles (so the process above terminates), all of which are in a finite disc $|z| < R$.

Since $f(z)$ has infinitely many poles, we will find a pole z_0 of $f(z)$ such that

$$z_0 + c_{l_1} + \cdots + c_{l_n} = \omega_n \quad (n \in \mathbb{N})$$

is a pole of $f(z)$ of multiplicity m_n for all $n \in \mathbb{N}$. Hence, $f(z)$ has a sequence of poles

$$\{\omega_n = z_0 + c_{l_1} + \cdots + c_{l_n} : n = 1, 2, \ldots\},$$

so that $\lambda(1/f) = 1$. This is a contradiction. Hence f has at most finitely many poles.

Thus, there exists a rational function R_1 such that $h(z) = f(z) - R_1(z)$ is a transcendental entire function. By (2.1), we have

$$h(z + c_1) + \cdots + h(z + c_k) = kh(z) + P(z), \tag{2.2}$$

where $P(z) = R(z) + kR_1(z) - R_1(z + c_1) - \cdots - R_k(z + c_k)$. As $h(z + c_j)$ $(j = 1, \ldots, k)$ and $h(z)$ are entire functions, we see that $P(z)$ is a polynomial. By Lemma 2.1, there exists an ε-set E such that

$$h(z + c_j) - h(z) = c_j h'(z)(1 + o(1)) \quad (j = 1, \ldots, k) \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E. \tag{2.3}$$

If $P(z) \equiv 0$, by (2.2) and (2.3), as $z \to \infty$ in $\mathbb{C} \setminus E$, we have

$$(c_1 + \cdots + c_k)h'(z)(1 + o(1)) = 0;$$

since $c_1 + \cdots + c_k \neq 0$, this yields $h'(z) = 0$ (as $z \notin E$), which is impossible. Hence $P(z) \not\equiv 0$. Set $\deg P = l \geq 0$. Then $P(z) = cz^m(1 + o(1))$, where c $(\neq 0)$ is a constant. By (2.2) and (2.3), as $z \to \infty$ in $\mathbb{C} \setminus E$, we get

$$(c_1 + \cdots + c_k)h'(z)(1 + o(1)) = cz^l(1 + o(1)),$$

which contradicts the fact that $h'(z)$ is transcendental.

Next, we assume that $G(z)$ is a rational function. Then

$$\frac{f(z+c_1)+\cdots+f(z+c_k)-kf(z)}{f(z)}=\theta(z),$$

where $\theta(z)$ is a rational function. By Lemma 2.1, there exists an ε-set E such that

$$(2.4)\qquad \frac{(c_1+\cdots+c_k)f'(z)(1+o(1))}{f(z)}=\theta(z)\quad \text{as } z\to\infty \text{ in } \mathbb{C}\setminus E.$$

But since $f(z)$ is transcendental and has either infinitely many poles or infinitely many zeros, $f'(z)/f(z)$ must be transcendental. Thus (2.4) is impossible.

Lemma 2.3. *Let f, c_j $(j=1,\dots,k)$ satisfy the conditions of Lemma 2.2. Then $g_k(z)$ is transcendental.*

Proof. Assume

$$(2.5)\qquad f(z+c_1)\cdots f(z+c_k)=\theta(z)+f^k(z),$$

where $\theta(z)$ is a rational function. Using a similar method to the proof of Lemma 2.2, we find that f has at most finitely many poles. By Lemma 2.1, there exists an ε-set E such that

$$(2.6)\qquad f(z+c_j)-f(z)=c_jf'(z)(1+o(1))\quad (j=1,\dots,k)\quad \text{as } z\to\infty \text{ in } \mathbb{C}\setminus E.$$

By (2.5) and (2.6), we have

$$(2.7)\qquad c_1\cdots c_k(f')^k(1+o(1))+A_{k-1}(f')^{k-1}f(z)(1+o(1))+\cdots + A_2f''f^{k-2}(z)(1+o(1))+(c_1+\cdots+c_k)f'f^{k-1}(z)(1+o(1))=\theta(z),$$

where A_j $(j=2,\dots,k-1)$ are constant in $c_1,\dots,c_k$. Set $f_1(z)=f(z)l(z)$, where $l(z)$ is a polynomial whose zeros are all poles of $f(z)$. Obviously, $f_1(z)$ is a transcendental entire function with $\sigma(f_1)=\sigma(f)=\sigma<1$. From the Wiman–Valiron theory, there exists a subset $E_1\subset(1,\infty)$ of finite logarithmic measure such that for large $r\notin E_1$, for all z satisfying $|z|=r$ and $|f_1(z)|=M(r,f_1)$, we get

$$(2.8)\qquad \frac{f_1'(z)}{f_1(z)}=\frac{\upsilon(r)}{z}(1+o(1)),$$

where $\upsilon(r)$ is the central index of $f_1(z)$. From (2.8) and $f_1(z)=f(z)l(z)$ for all z satisfying $|z|=r$ and $|f_1(z)|=M(r,f_1)$, we have

$$(2.9)\qquad \frac{f'(z)}{f(z)}=\frac{f_1'(z)}{f_1(z)}-\frac{l'(z)}{l(z)}=\frac{\upsilon(r)}{z}(1+o(1)).$$

Set $E_2=\{|z|: z\in E\}$. Since E is an ε-set, E_2 has finite logarithmic measure. By (2.7) and (2.9), for all z satisfying $|z|=r\notin[0,1]\cup E_1\cup E_2$ and

$|f_1(z)| = M(r, f_1)$, we get

$$(2.10)\qquad c_1 \cdots c_k \left(\frac{v(r)}{z}\right)^{k-1}(1+o(1)) + A_{k-1}\left(\frac{v(r)}{z}\right)^{k-2}(1+o(1)) + \cdots + (c_1 + \cdots + c_k)(1+o(1)) = \frac{\theta(z)l^k(z)z}{[M(r,f_1)]^k}\,\frac{1}{v(r)}(1+o(1)).$$

Since $f_1(z)$ is transcendental and $\sigma(f_1) < 1$, we obtain

$$(2.11)\qquad v(r)/z \to 0, \quad v(r) \to \infty \quad (z \to \infty),$$

and

$$(2.12)\qquad \frac{\theta(z)l^k(z)z}{[M(r,f_1)]^k}\,\frac{1}{v(r)}((1+o(1)) \to 0 \quad (z \to \infty).$$

From (2.11), (2.12) and $c_1+\cdots+c_k \neq 0$, we deduce that (2.10) is impossible. Hence $g(z)$ is transcendental.

Lemma 2.4 ([3]). *Let f be a function transcendental and meromorphic in the plane of lower order $\mu(f) < \mu < 1$. Then there exists an arbitrarily large R with the following properties. First,*

$$T(32R, f') < R^{\mu}.$$

Second, there exists a set $J_R \subseteq [R/2, R]$ of linear measure $(1 - o(1))R/2$ such that, for $r \in J_R$,

$$f(z+c) - f(z) \sim cf'(z) \quad \text{on } |z| = r.$$

Lemma 2.5. *Let f be a transcendental and meromorphic function of order of growth $\sigma(f) = \sigma < 1$. Let a_j $(j = 0, 1, \ldots, k) \in \mathbb{C}$ and $a_i \neq 0$ $(i = 0, k)$. If $\overline{\lambda}(1/f) = \lambda(1/f)$, then*

$$\max\{\lambda(f'), \lambda(a_k(f')^k + a_{k-1}(f')^{k-1}f + \cdots + a_0 f^k)\} = \sigma(f) = \sigma.$$

Proof. If $\lambda(f') < \sigma$, then $\lambda(1/f') = \sigma$. By hypothesis,

$$(2.13)\qquad \lambda(1/f') = \lambda(1/f) = \overline{\lambda}(1/f) = \sigma(f) = \sigma.$$

Set

$$f(z) = q(z)/p(z), \qquad f'(z) = q_1(z)/p_1(z),$$

where $q(z)$ $[q_1(z)]$ and $p(z)$ $[p_1(z)]$ are canonical products (or polynomials) formed by the zeros and poles of $f(z)$ $[f'(z)]$ respectively, such that $q(z)$ and $p(z)$ $[q_1(z)$ and $p_1(z)]$ are irreducible. From $\lambda(f') < \sigma$ and (2.13), we get

$$\sigma(p) = \sigma(p_1) = \sigma(f), \qquad \lambda(q_1) = \sigma(q_1) < \sigma(f).$$

If $f(z)$ has a pole of multiplicity m at z_0, then f' has a pole of multiplicity $m+1$ at z_0, so we have

$$p_1(z) = p(z)d(z),$$

where $d(z)$ is the canonical product formed by different poles of $f(z)$. By (2.13), we have

$$\sigma(d) = \lambda(d) = \overline{\lambda}(1/f) = \sigma(f) = \sigma. \tag{2.14}$$

Since

$$\begin{aligned} a_k(f')^k + a_{k-1}(f')^{k-1}f + \cdots + a_0 f^k \\ = \frac{a_k q_1^k(z) + a_{k-1}q_1^{k-1}(z)q(z)d(z) + \cdots + a_0 q^k(z)d^k(z)}{p_1^k(z)}, \end{aligned}$$

we see that if z_0 is a pole of f', then $d(z_0) = 0$, but $q_1(z_0) \neq 0$. So, z_0 is not a zero of $a_k q_1^k(z) + a_{k-1}q_1^{k-1}(z)q(z)d(z) + \cdots + a_0 q^k(z)d^k(z)$. Hence $a_k q_1^k(z) + a_{k-1}q_1^{k-1}(z)q(z)d(z) + \cdots + a_0 q^k(z)d^k(z)$ and $p_1(z)$ are irreducible. By (2.14), we get

$$\begin{aligned} \lambda(a_k(f')^k + a_{k-1}(f')^{k-1} + \cdots + a_0 f) &= \lambda(a_k q_1^k + a_{k-1}q_1^{k-1}qd + \cdots + a_0 q^k d^k) \\ &= \sigma(a_k q_1^k + a_{k-1}q_1^{k-1}qd + \cdots + a_0 q^k d^k) \\ &\geq \sigma(d) = \sigma(f). \end{aligned}$$

Lemma 2.5 is thus proved.

3. Proof of Theorem 1.1. By Lemma 2.2, $g(z)$ is transcendental. By Lemma 2.1, there exists an ε-set E such that

$$g(z) = (c_1 + \cdots + c_k)f'(z)(1 + o(1)) \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E. \tag{3.1}$$

Set

$$H = \{|z| : z \in E,\ g(z) = 0 \text{ or } f'(z) = 0\}.$$

Then H has finite linear measure. By (3.1), for $|z| = r \notin H$, we obtain

$$|g(z) - (c_1 + \cdots + c_k)f'(z)| = |o(f'(z))| < |g(z)| + |(c_1 + \cdots + c_k)f'(z)|.$$

Applying Cauchy's argument principle, for $|z| = r \notin H$, we have

$$n(r, 1/g) - n(r, g) = n(r, 1/f') - n(r, f').$$

Since f is a transcendental entire function and $\sigma(f) < 1$, we have

$$\lambda(g) = \lambda(f') = \sigma(f') = \sigma(f) = \sigma. \tag{3.2}$$

Next, we prove that $\lambda(G) = \sigma(G) = \sigma(f) = \sigma$. Suppose that z_j is a zero of $g(z)$. If $f(z_j) \neq 0$, then z_j must be a zero of $G(z)$. If $f(z_j) = 0$, then $f(z_j + c_1) + \cdots + f(z_j + c_k) = 0$. By the hypotheses, there exist at most finitely many such points. Hence

$$n(r, 1/G) = n(r, 1/g) + O(1). \tag{3.3}$$

By (3.2) and (3.3), $\lambda(G) = \sigma(G) = \sigma(f) = \sigma$. Theorem 1.1 is thus proved.

4. Proof of Theorem 1.2. By Lemma 2.3 and the fact that f is transcendental, $g_k(z)$ is a transcendental entire function. Thus,

$$\sigma(g_k) \le \sigma(f). \tag{4.1}$$

Using the same method as in the proof of Lemma 2.3, for $|z| = r \notin [0,1] \cup E_1 \cup E_2$ and $|f(z)| = M(r,f)$, with E, E_1 and E_2 defined as in the proof of Lemma 2.3, we have

$$c_1 \cdots c_k \left(\frac{\upsilon(r)}{z}\right)^k (1+o(1)) + \cdots + (c_1 + \cdots + c_k)\frac{\upsilon(r)}{r} = \frac{g_k(z)}{M(r,f)^k}. \tag{4.2}$$

Together with $\sigma(f) < 1$, as $r \to \infty$ we get

$$\left(\frac{\upsilon(r)}{r}\right)^j = o\left(\frac{\upsilon(r)}{r}\right) \quad (j = 2, \dots, k). \tag{4.3}$$

Now (4.2) and (4.3) imply that

$$C\frac{\upsilon(r)}{r}(1+o(1)) \le \frac{|g_k(z)|}{M(r,f)^k}, \tag{4.4}$$

where C is a constant. By (4.4), we get

$$\sigma(g_k) \ge \sigma(f). \tag{4.5}$$

By (4.1) and (4.5), we get $\sigma(g_k) = \sigma(f)$, so $\lambda(g_k) = \sigma(g_k) = \sigma(f)$.

Next, we prove that $\lambda(G_k) = \sigma(G_k) = \sigma(f) = \sigma$.

Since $G_k(z) = g_k(z)/f^k(z)$ and f is an entire function, we know that if z_0 is a zero of $g_k(z)$ but not a zero of $G_k(z)$, then z_0 must be a zero of $f(z)$. Thus, there exists at least one c_j $(j = 1, \dots, k)$ such that $z_0 + c_j$ is a zero of $f(z_0 + c_j)$. Now suppose that (i) holds: at most finitely many zeros z_l, z_m of $f(z)$ satisfy $z_l - z_m = c_j$ $(j = 1, \dots, k)$. Hence, $f(z)$ has only finitely many such zeros z_0. If z_0 is a zero of $G_k(z)$, then z_0 is also a zero of $g_k(z)$, so that

$$n(r, 1/G_k) = n(r, 1/g_k) + o(1).$$

Now assume that (ii) holds. Then there exist α $(0 < \alpha < l-1)$ and N (> 0) such that when $j > N$, $\alpha|z_j| > c > \max\{c_1, \dots, c_k\}$ and $|z_{j+1}| - |z_j| > \alpha|z_j| > c$. Thus (i) holds. It is easy to get

$$n(r, 1/G_k) = n(r, 1/g_k) + o(1).$$

This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3. Let E be an ε-set which contains all zeros and poles of $g(z)$, $f(z)$, $f(z + c_j)$ $(j = 1, \dots, k)$, f', and define

$$E_R = \{r : z \in E,\ |z| = r < R\}, \qquad R \in (1, \infty),$$
$$E_\infty = \{r : z \in E,\ |z| = r < \infty\}.$$

Then by the properties of ε-sets and $\sigma(f) < 1$, we see that E_∞ has finite linear measure and E_R has linear measure $o(1)R/2$.

By Lemma 2.4, there exist R arbitrarily large and σ_1 ($\sigma < \sigma_1 < 1$) with

$$T(32R, f') < R^{\sigma_1}, \tag{5.1}$$

and there exists a set $J_R \subseteq [R/2, R] \setminus E_R$ of linear measure $(1 - o(1))R/2$ such that for $|z| = r \in J_R$,

$$f(z + c_1) + \cdots + f(z + c_k) - kf(z) = (c_1 + \cdots + c_k)f'(z)(1 + o(1)). \tag{5.2}$$

Let

$$F_R = \{r \in [R/2, R] : n(r, f) = n(r - (|c_1| + \cdots + |c_k|), f)\}. \tag{5.3}$$

Then F_R has linear measure

$$m(F_R) \geq (1 - o(1))R/2. \tag{5.4}$$

To see this, note that there are at most $o(R)$ points $p_k \in [R/3, R]$ at which $n(t, f)$ is discontinuous, by (5.1), and if $r \in [R/2, R]$ is such that $n(r, f) > n(r - (|c_1| + \cdots + |c_k|), f)$, then $r \in [p_k, p_k + 1]$ for some k.

From (5.1)–(5.4) and $J_R \cap E_R = \emptyset$, we see that there exists $r \in F_R \cap J_R$ such that $g(z)$, $f(z)$, $f(z+c_j)$ ($j = 1, \ldots, k$), f' have no zeros and poles with $|z| = r$.

Without loss of generality, for all poles b_j of $f(z)$, we may assume that $b_j + k_1c_i + k_2c_l$ ($k_d = 0, \pm 1$, $d = 1, 2$; $i, l \in \{1, \ldots, k\}$) are not poles.

From the condition of Theorem 1.3, there exists r_0, independent of R and r, such that if $f(z)$ has a pole of multiplicity m at z_0 and $r_0 \leq |z_0| \leq r - (|c_1| + \cdots + |c_k|)$, then $f(z_0) = \infty$, $f(z_0 \pm c_j) \neq \infty$, thus from

$$\begin{aligned} g(z) &= f(z + c_1) + f(z + c_2) + \cdots + f(z + c_k) - kf(z), \\ g(z - c_j) &= f(z + c_1 - c_j) + \cdots + f(z + c_{j+1} - c_j) \\ &\quad + f(z) + \cdots + f(z + c_k - c_j) - kf(z - c_j) \quad (j = 1, \ldots, k), \end{aligned}$$

we know that $g(z)$ has poles at z_0, $z_0 - c_j$ ($j = 1, \ldots, k$), each with multiplicity m. So

$$n(r, g) \geq (k + 1)n(r - (|c_1| + \cdots + |c_k|), f) = (k + 1)n(r, f).$$

By (5.2) and $g(z)$, $f(z)$, $f(z + c_j)$ ($j = 1, \ldots, k$), f' have no zeros and poles with $|z| = r \in F_R \cap J_R$. Applying Cauchy's argument principle, we obtain

$$\begin{aligned} n(r, 1/g) &= n(r, 1/f') - n(r, f') + n(r, g) \\ &\geq n(r, 1/f') - n(r, f') + (k + 1)n(r, f) + O(1) \\ &\geq n(r, 1/f') + kn(r, f) + O(1). \end{aligned} \tag{5.5}$$

If $\lambda(f') < \sigma(f') = \sigma(f)$, then $\lambda(1/f') = \sigma(f') = \sigma(f)$, so that $\lambda(1/f) = \sigma(f)$. Hence $\lambda(g) = \sigma(g) = \sigma(f)$. If $\lambda(1/f) < \sigma(f)$, then $\lambda(1/f') < \sigma(f)$, so that $\lambda(f') = \sigma(f)$. Hence $\lambda(g) = \sigma(g) = \sigma(f)$.

Finally, using the same method as in the proof of Theorem 1.2, we can prove that if $f(z)$ has at most finitely many zeros a_j which satisfy $f(a_j+c_1) + \cdots + f(a_j+c_k) = 0$, then $G(z)$ has infinitely many zeros and $\lambda(G) = \sigma(G) = \sigma(f)$. Theorem 1.3 is proved.

6. Proof of Theorem 1.4. Set

$$F(z) = f'[c_1\cdots c_k(f')^{k-1} + A_{k-1}(f')^{k-2}f + \cdots + (c_1+\cdots+c_k)f^{k-1}]. \tag{6.1}$$

By Lemma 2.3, $g_k(z)$ is transcendental. As in the proof of Theorem 1.3, as $|z|\to\infty$ and $|z| = r \in J_R$, we obtain

$$g_k(z) = F(z)(1+o(1))$$

and

$$n(r,1/g_k) = n(r,1/F) - n(r,F) + n(r,g_k) \quad \text{for } |z| = r \in (F_R \cap J_R)\setminus E_R, \tag{6.2}$$

where F_R, J_R, E and E_R are defined as in the proof of Theorem 1.3; E contains all zeros and poles of g_k, F, f, $f(z+c_j)$ $(j=1,\dots,k)$ and f'.

Under the assumptions of Theorem 1.4, there exists r_0, independent of R and r, such that if $f(z)$ has a pole of multiplicity m at z_0 and $r_0 \le |z_0| \le r - (|c_1|+\cdots+|c_k|)$, then by the hypotheses and the expression of $g_k(z)$, $g_k(z-c_j)$ $(j=1,\dots,k)$, we know that $g_k(z)$ has poles at z_0, z_0-c_j $(j=1,\dots,k)$ of multiplicity km, m, respectively. Hence

$$n(r,g_k) \ge 2kn(r-(|c_1|+\cdots+|c_k|),f) + O(1) = 2kn(r,f) + O(1). \tag{6.3}$$

Since $F(z)$ has a pole of multiplicity $km+k$ at z_0, we have

$$n(r,F) = kn(r,f) + k\overline{n}(r,f). \tag{6.4}$$

By (6.1),

$$n(r,1/F) = n(r,1/f') + n\left(r, \frac{1}{c_1\cdots c_k(f')^{k-1}+\cdots+(c_1+\cdots+c_k)f^{k-1}}\right). \tag{6.5}$$

By (6.2)–(6.5), we get

$$n(r,1/g_k) \ge n(r,1/f') + kn(r,f) - k\overline{n}(r,f) + n\left(r, \frac{1}{c_1\cdots c_k(f')^{k-1}+\cdots+(c_1+\cdots+c_k)f^{k-1}}\right) + O(1). \tag{6.6}$$

If $\overline{\lambda}(1/f) < \lambda(1/f)$, then by (6.5) and (6.6), we have

$$n(r,1/g_k) \ge n(r,1/f') + n(r,f) + O(1). \tag{6.7}$$

As in the proof of Theorem 1.3, we can deduce $\lambda(g_k) = \sigma(g_k) = \sigma(f)$.

If $\overline{\lambda}(1/f) = \lambda(1/f)$, then by (6.3), we have

$$n(r, 1/g_k) \geq n(r, 1/f') + n\left(r, \frac{1}{c_1 \cdots c_k (f')^{k-1} + \cdots + (c_1 + \cdots + c_k) f^{k-1}}\right) + O(1). \tag{6.8}$$

By Lemma 2.5 and (6.8), $\lambda(g_k) = \sigma(g_k) = \sigma(f)$.

Finally, similarly to the proof of Theorem 1.3, we can prove that $\lambda(G_k) = \sigma(f)$.

Acknowledgements. This research was partly supported by the NNSF of China (No. 10771121), the NSF of Shangdong Province, China (No. Z2008A01) and the NSFC-RFBR.

References

[1] M. Ablowitz, R. G. Halburd and B. Herbst, *On the extension of the Painlevé property to difference equations*, Nonlinearity 13 (2000), 889–905.

[2] W. Bergweiler and A. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoamer. 11 (1995), 355–373.

[3] W. Bergweiler and J. K. Langley, *Zeros of differences of meromorphic functions*, Math. Proc. Cambridge Philos. Soc. 142 (2007), 133–147.

[4] Z . X. Chen and K. H. Shon, *Estimates for the zeros of differences of meromorphic functions*, Sci. China Ser. A 52 (2009), 2447–2458.

[5] Y. M. Chiang and S. J. Feng, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*, Ramanujan J. 16 (2008), 105–129.

[6] J. Clunie, A. Eremenko and J. Rossi, *On equilibrium points of logarithmic and Newtonian potentials*, J. London Math. Soc. 47 (1993), 309–320.

[7] A. Eremenko, J. K. Langley and J. Rossi, *On the zeros of meromorphic functions of the form $\sum_{k=1}^{\infty} \frac{a_k}{z-z_k}$*, J. Anal. Math. 62 (1994), 271–286.

[8] R. G. Halburd and R. Korhonen, *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*, J. Math. Anal. Appl. 314 (2006), 477–487.

[9] —, —, *Nevanlinna theory for the difference operator*, Ann. Acad. Sci. Fenn. Math. 94 (2006), 463–478.

[10] W. Hayman, *Slowly growing integral and subharmonic functions*, Comment. Math. Helv. 34 (1960), 75–84.

[11] —, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[12] —, *The local growth of power series: a survey of the Wiman–Valiron method*, Canad. Math. Bull. 17 (1974), 317–358.

[13] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo and K. Tohge, *Complex difference equations of Malmquist type*, Comput. Methods Funct. Theory 1 (2001), 27–39.

[14] J. D. Hinchliffe, *The Bergweiler–Eremenko theorem for finite lower order*, Results Math. 43 (2003), 121–128.

[15] K. Ishizaki and N. Yanagihara, *Wiman–Valiron method for difference equations*, Nagoya Math. J. 175 (2004), 75–102.

[16] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, de Gruyter, Berlin, 1993.

[17] I. Laine and C. C. Yang, *Clunie theorems for difference and q-difference polynomials*, J. London Math. Soc. 76 (2007), 556–566.

[18] C. C. Yang and H. X. Yi, *Uniqueness of Meromorphic Functions*, Kluwer, Dordrecht, 2003.

Yong Liu, HongXun Yi
School of Mathematics
Shandong University
250100 Jinan, Shandong, P.R. China
E-mail: liuyongsdu@yahoo.cn
hxyi@sdu.edu.cn

Received 15.1.2010
and in final form 16.3.2010 (2146)