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Periodic solutions of a three-species periodic reaction-diffusion system

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Abstract. We study a periodic reaction-diffusion system of a competitive model with Dirichlet boundary conditions. By the method of upper and lower solutions and an argument similar to that of Ahmad and Lazer, we establish the existence of periodic solutions and also investigate the stability and global attractivity of positive periodic solutions under certain conditions.

1. Introduction. In this paper, we consider the following three-species periodic reaction-diffusion system:

$$(1.1)\qquad \begin{cases} \dfrac{\partial u_1}{\partial t} - d_1 \Delta u_1 = u_1(a_1 - b_{11}u_1 - b_{12}u_2 - b_{13}u_3 - e_1 u_1 u_2 u_3) & \text{in } \Omega \times \mathbb{R}^+, \\ \dfrac{\partial u_2}{\partial t} - d_2 \Delta u_2 = u_2(a_2 - b_{21}u_1 - b_{22}u_2 - b_{23}u_3 - e_2 u_1 u_2 u_3) & \text{in } \Omega \times \mathbb{R}^+, \\ \dfrac{\partial u_3}{\partial t} - d_3 \Delta u_3 = u_3(a_3 - b_{31}u_1 - b_{32}u_2 - b_{33}u_3 - e_3 u_1 u_2 u_3) & \text{in } \Omega \times \mathbb{R}^+, \end{cases}$$

with Dirichlet boundary conditions

$$(1.2)\qquad u_i(x,t) = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+,\ i = 1,2,3,$$

and the initial conditions

$$(1.3)\qquad u_i(x,0) = u_{i0}(x) \quad \text{on } \Omega,\ i = 1,2,3,$$

where $\Omega \subset \mathbb{R}^N$ ($N \ge 1$) is a bounded domain with $C^{2+\alpha}$-smooth ($0 < \alpha < 1$) boundary $\partial\Omega$, $d_i = d_i(t) \in C_T(\overline{Q}_T)$ ($i = 1,2,3$) are strictly positive smooth functions, $a_i = a_i(x,t), b_{ij} = b_{ij}(x,t), e_i = e_i(x,t) \in C_T(\overline{Q}_T)$ ($i,j = 1,2,3$) are positive smooth functions on $\overline{\Omega} \times \mathbb{R}^+$. Here $C_T(\overline{Q}_T)$ denotes the set of

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functions which are continuous in $\overline{Q}_T = \overline{\Omega} \times (0, T)$ and periodic in t with period T.

In population dynamics, the system (1.1) can be used to model the situation where a bounded region in $\mathbb{R}^N$ is inhabited by three species which interact with each other and migrate from regions of high to low population densities. In (1.1), u_1, u_2 and u_3 stand for the population densities of three competing species; d_i $(i = 1, 2, 3)$ is the diffusion coefficient of the ith species, a_i $(i = 1, 2, 3)$ is the intrinsic growth rate of the ith species; b_{ii} $(i = 1, 2, 3)$ is the rate of intra-specific competition of the ith species; b_{ij} $(i \neq j,\ i, j = 1, 2, 3)$ is the rate of inter-specific competition of the ith and jth species; e_i $(i = 1, 2, 3)$ is the rate of toxic inhibition of the ith species by the other species. The time dependence of the coefficients reflects the fact that the time periodic variations of the habitat are taken into account.

In this paper, we study the existence of periodic solutions and the stability and global attractivity of positive periodic solutions under certain conditions. The paper is organized as follows. In Section 2, we present the background and related work. In Section 3, we introduce some necessary preliminaries. In Section 4, we investigate the existence of positive periodic solutions and the asymptotic global attractivity of positive periodic solutions. In Section 5, we give numerical illustrations.

2. Notation. Recently, various mathematical models have been proposed in the study of population dynamics. Assuming that one species produces a substance toxic to the other but only when the other is present, Maynard Smith [M] considered the following modified two-species Lotka–Volterra competitive system and studied its stability properties:

$$\tag{2.1} \begin{cases} \dfrac{du_1(t)}{dt} = u_1(t)(a_1 - b_1 u_1(t) - c_1 u_2(t) - e_1 u_1(t) u_2(t)), \\ \dfrac{du_2(t)}{dt} = u_2(t)(a_2 - b_2 u_2(t) - c_2 u_1(t) - e_2 u_1(t) u_2(t)). \end{cases}$$

This model has attracted great interest of mathematicians. For example, [C] proposed sufficient conditions for the existence of positive periodic solutions and [SC] studied when the periodic solutions oscillate. However, it is not enough to consider the dynamics with respect to time t. The distribution of the species in space should also be considered. In view of the natural tendency of each species to diffuse to areas of smaller population concentration, researchers were led to the following reaction-diffusion system:

$$\tag{2.2} \begin{cases} \dfrac{\partial u_1}{\partial t} - d_1 \Delta u_1 = u_1(a_1 - b_1 u_1 - b_1 u_2 - b_1 u_3 - e_1 u_1 u_2), \\ \dfrac{\partial u_2}{\partial t} - d_2 \Delta u_2 = u_2(a_2 - b_2 u_1 - b_2 u_2 - b_2 u_3 - e_2 u_1 u_2). \end{cases}$$

For this system, a lot of results have been produced. For details, see [TL1, TL2, TZ] and the references therein.

In recent years, periodic parabolic systems have been extensively studied: see e.g. [AL, BH, FC, JW, ML, P, Z]. Mathematically, there are essential difficulties in passing from single-species to multiple-species competitive models, and less is known about the dynamical behavior of competitive systems though the single species model is well understood. Moreover, the periodicity of parameters is realistic and important when the effect of environmental factors is taken into account. In an important work, Ahmad and Lazer [AL] studied the periodic diffusion system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - d_1 \Delta u_1 = u_1(a_1 - b_1 u_1 - b_1 u_2), \\ \dfrac{\partial u_2}{\partial t} - d_2 \Delta u_2 = u_2(a_2 - b_2 u_1 - b_2 u_2) \end{cases} \tag{2.3}$$

with homogeneous Neumann boundary conditions. By the method of upper and lower solutions, Ahmad and Lazer discussed upper and lower bounds for coexistence states and the limiting behavior of solutions. Prompted by the work of Ahmad and Lazer, we consider the three-species periodic diffusion system (1.1). We give conditions for the existence of positive periodic solutions and the global attractivity of positive periodic solutions. Some of our results extend the existing results for the above two-species Lotka–Volterra competitive systems.

3. Preliminaries. In this section, we define solutions of the problem (1.1)–(1.3) and give some lemmas needed in the proofs of our main results.

For $0 < \alpha < 1$, we denote

$$E = C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times \mathbb{R}^+),$$
$$F = \{\omega \in E : \omega|_{\partial\Omega\times\mathbb{R}^+} = 0,\ \omega(x,t) = \omega(x,t+T) \text{ in } \Omega \times \mathbb{R}^+\}.$$

DEFINITION 3.1. A vector function (u_1, u_2, u_3) is said to be a *classical solution* of the problem (1.1)–(1.3) if (u_1, u_2, u_3) maps $\Omega \times \mathbb{R}^+$ to E^3 and satisfies (1.1)–(1.3); (u_1, u_2, u_3) is said to be a *classical T-periodic solution* of (1.1)–(1.2) if (u_1, u_2, u_3) maps $\Omega \times \mathbb{R}^+$ to F^3 and satisfies (1.1)–(1.2).

LEMMA 3.1 ([BH]). *Let $\lambda(d_i, a_i)$ be the principal eigenvalue of the periodic eigenvalue problem*

$$\begin{cases} \dfrac{\partial \varphi_i}{\partial t} - d_i \Delta \varphi_i - a_i \varphi_i = \lambda_i \varphi_i & \textit{in } \Omega \times \mathbb{R}^+, \\ \varphi_i = 0 & \textit{on } \partial\Omega \times \mathbb{R}^+, \\ \varphi_i(x,0) = \varphi_i(x,T) & \textit{on } \Omega, i = 1,2,3. \end{cases} \tag{3.1}$$

Then $\lambda_{i1} \equiv \lambda(d_i, a_i)$ *is real and its corresponding eigenfunction* φ_i *does not change sign in* $\Omega \times \mathbb{R}^+$. *We always choose* φ_i *positive and normalize it so that* $\max \varphi_i = 1$ *in* $\Omega \times \mathbb{R}^+$.

In the following, for a given bounded function $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$, we denote $f_M = \sup\{f(x,t) : (x,t) \in \overline{\Omega} \times \mathbb{R}^+\}$, $f_L = \inf\{f(x,t) : (x,t) \in \overline{\Omega} \times \mathbb{R}^+\}$.

Consider the logistic problem

$$\frac{\partial u}{\partial t} - d\Delta u = u(a - bu) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{3.2}$$

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{3.3}$$

$$u(x,0) = u_0(x) \quad \text{on } \Omega, \tag{3.4}$$

where $d = d(x,t)$ is a strictly positive smooth T-periodic function, and $a = a(x,t)$ and $b = b(x,t)$ are positive smooth T-periodic functions on $\Omega \times \mathbb{R}^+$. For the above problem, we have the following lemma [AL, P, Z].

LEMMA 3.2 ([AL]). (1) *If* $\lambda_1(d,a) < 0$, *then the problem* (3.2)–(3.3) *admits exactly one positive classical* T*-periodic solution* $w[d,a,b] \equiv w[a,b]$ *satisfying*

$$0 < w[a,b] \le (a/b)_M \quad \textit{in } \Omega \times \mathbb{R}^+. \tag{3.5}$$

Furthermore, if $u(x,t)$ *is a solution of* (3.2)–(3.4), *then*

$$\lim_{t\to\infty} |w[a,b] - u(x,t)| = 0$$

uniformly for $x \in \overline{\Omega}$, *where* $u|_{\partial\Omega\times\mathbb{R}^+} = 0$, $u(x,0) \ge 0$ $(\not\equiv 0)$ *in* Ω, *and* $w[d,a,b] \equiv w[a,b]$ *denotes the solution of the problem* (3.2)–(3.3) *with the coefficients* $d = d(x,t)$, $a = a(x,t)$ *and* $b = b(x,t)$.

(2) *If* $\lambda_1(d,a) \ge 0$, *then the problem* (3.2)–(3.3) *has no nontrivial* T*-periodic solutions. If* $\lambda_1(d,a) \ge 0$ *and* $u(x,t)$ *is a solution of* (3.2)–(3.4), *then* $\lim_{t\to\infty} u(x,t) = 0$.

The following lemmas play an essential role in investigating the global asymptotic behavior of the solutions of the problem (1.1)–(1.3).

LEMMA 3.3 ([YL]). *If* $u, v \in C^1(\overline{\Omega})$, $u|_{\partial\Omega} = v|_{\partial\Omega} = 0$, $v(x) > 0$ *for* $x \in \Omega$ *and* $\frac{\partial v}{\partial n}|_{\partial\Omega} < 0$, *where* $\frac{\partial}{\partial n}$ *represents the outward normal derivative on* $\partial\Omega$, *then there exists a positive constant* K *such that* $u(x) \le Kv(x)$ *for all* $x \in \Omega$.

LEMMA 3.4 ([FC]). *If* $\lambda(d,a) < 0$, $a \le A$, *and* $0 < B \le b$, *then* $w[a,b] \le w[A,B]$.

LEMMA 3.5. *Assume that* (u_1, u_2, u_3) *is a classical positive* T*-periodic solution of* (1.1)–(1.2) *and*

$$-\lambda_{11} > b_{12M}\left(\frac{a_2}{b_{22}}\right)_M + b_{13M}\left(\frac{a_3}{b_{33}}\right)_M, \tag{3.6}$$

$$-\lambda_{21} > b_{21M}\left(\frac{a_1}{b_{11}}\right)_M + b_{23M}\left(\frac{a_3}{b_{33}}\right)_M, \tag{3.7}$$

$$-\lambda_{31} > b_{31M}\left(\frac{a_1}{b_{11}}\right)_M + b_{32M}\left(\frac{a_2}{b_{22}}\right)_M. \tag{3.8}$$

Then in $\Omega \times \mathbb{R}$, we have

$$\varphi_i \le u_i \le w[d_i, a_i, b_{ii}] \equiv w_i, \qquad i = 1, 2, 3. \tag{3.9}$$

Proof. According to Lemma 3.2 and (3.6)–(3.8), if $\lambda_{i1} < 0$ $(i = 1, 2, 3)$, then there exist w_i $(i = 1, 2, 3)$ satisfying (3.2)–(3.4) with coefficient functions a_i, b_i. So we have

$$\begin{aligned}
&\frac{\partial w_1}{\partial t} - d_1 \Delta w_1 \ge w_1(a_1 - b_{11}w_1 - b_{12}w_2 - b_{13}w_3 - e_1w_1w_2w_3) \text{ in } \Omega \times \mathbb{R}^+,\\
&\frac{\partial w_2}{\partial t} - d_2 \Delta w_2 \ge w_2(a_2 - b_{21}w_1 - b_{22}w_2 - b_{23}w_3 - e_2w_1w_2w_3) \text{ in } \Omega \times \mathbb{R}^+,\\
&\frac{\partial w_3}{\partial t} - d_3 \Delta w_3 \ge w_3(a_3 - b_{31}w_1 - b_{32}w_2 - b_{33}w_3 - e_3w_1w_2w_3) \text{ in } \Omega \times \mathbb{R}^+,\\
&w_i(x,t) = 0 \ge 0 \qquad\qquad\qquad \text{on } \partial\Omega \times \mathbb{R}^+,\ i = 1, 2, 3,\\
&w_i(x,0) = w_i(x,T) \ge w_i(x,T) \quad \text{on } \Omega,\ i = 1, 2, 3.
\end{aligned}$$

Thus (w_1, w_2, w_3) is an upper solution of (1.1)–(1.2) with the initial conditions $u_i(x,0) = u_i(x,T)$ $(i = 1, 2, 3)$. Obviously, u_1, u_2, u_3 is a lower solution. So by the comparison theorem for periodic parabolic equations [SC] and Lemma 3.2, we have $u_i(x,t) \le w_i$ $(i = 1, 2, 3)$. Furthermore, $(\varphi_1, \varphi_2, \varphi_3)$ is a lower solution of (1.1)–(1.2) with the initial conditions $u_i(x,0) = u_i(x,T)$ $(i = 1, 2, 3)$. So we have $u_i(x,t) \ge \varphi_i (i = 1, 2, 3)$. ∎

LEMMA 3.6 ([YL, Theorem 5.3]). *Assume that $U_i \ge 0$, $V_i \ge 0$ $(i = 1, 2, 3)$ are smooth functions on $\overline{\Omega} \times \mathbb{R}^+$ and (U_1, V_2, V_3) satisfies*

$$\begin{aligned}
&\frac{\partial U_1}{\partial t} - d_1 \Delta U_1 \ge U_1(a_1 - b_{11}U_1 - b_{12}V_2 - b_{13}V_3 - e_1U_1V_2V_3) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{on } \Omega \times \mathbb{R}^+,\\
&\frac{\partial V_2}{\partial t} - d_2 \Delta V_2 \le V_2(a_2 - b_{21}U_1 - b_{22}V_2 - b_{23}U_3 - e_2U_1V_2U_3) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{on } \Omega \times \mathbb{R}^+,\\
&\frac{\partial V_3}{\partial t} - d_3 \Delta V_3 \le V_3(a_3 - b_{31}U_1 - b_{32}U_2 - b_{33}V_3 - e_3U_1U_2V_3) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{on } \Omega \times \mathbb{R}^+,
\end{aligned} \tag{3.10}$$

(V_1, U_2, U_3) satisfies the corresponding reversed inequalities, $U_i = V_i = 0$ on $\partial\Omega \times \mathbb{R}^+$ and $V_i(x,0) \le u_{i0}(x) \le U_i(x,0)$ on $\overline{\Omega}$. Then for any positive smooth initial value $u_{i0} \ge 0$ $(\not\equiv 0)$ $(i = 1, 2, 3)$, the problem (1.1)–(1.3)

admits a classical solution (u_1, u_2, u_3) *such that*

$$V_i \le u_i \le U_i \quad \text{in } \Omega \times \mathbb{R}^+, \tag{3.11}$$

where (U_1, U_2, U_3) *(resp.* (V_1, V_2, V_3)*) is an upper (resp. a lower) solution of the problem* (1.1)–(1.3).

4. Main results. In this section, we show the main results of this paper.

THEOREM 4.1. *Assume that* (3.6)–(3.8) *hold. Then the problem* (1.1)–(1.2) *admits two positive* T*-periodic solutions* (u_{1*}, u_2^*, u_3^*) *and* (u_1^*, u_{2*}, u_{3*}) *such that for any positive* T*-periodic solution* (u_{1T}, u_{2T}, u_{3T}) *of* (1.1)–(1.2),

$$u_{i*} \le u_{iT} \le u_i^* \quad \text{on } \overline{\Omega} \times \mathbb{R}^+,\ i = 1, 2, 3. \tag{4.1}$$

Moreover, if (u_1, u_2, u_3) *is a solution of the system* (1.1)–(1.3) *and*

$$u_{i0}(x) \ge 0 \ (\not\equiv 0) \text{ in } \Omega \quad \text{and} \quad u_{i0}|_{\partial\Omega} = 0, \qquad i = 1, 2, 3, \tag{4.2}$$

then for any $\varepsilon > 0$, *there exists a corresponding* $t_\varepsilon > 0$ *such that*

$$u_{i*}(x,t) - \varepsilon\varphi_i < u_i(x,t) < u_i^*(x,t) + \varepsilon\varphi_i, \qquad i = 1, 2, 3, \tag{4.3}$$

for all $x \in \overline{\Omega}$ *and* $t > t_\varepsilon$.

Proof. Since (3.1), (3.6), (3.7), (3.8) hold, we can see that there exist $\delta > 0$ and $r \ge 1$ such that

$$\begin{aligned}
-\lambda_{11} &\ge b_{11}\delta\varphi_1 + b_{12}rw_2 + b_{13}rw_3 + e_1\delta rr\varphi_1 w_2 w_3,\\
-\lambda_{21} &\ge b_{21}rw_1 + b_{22}\delta\varphi_2 + b_{23}rw_3 + e_2\delta rrw_1\varphi_2 w_3,\\
-\lambda_{31} &\ge b_{31}rw_1 + b_{32}rw_2 + b_{33}\delta\varphi_3 + e_3\delta rrw_1 w_2\varphi_3,
\end{aligned} \tag{4.4}$$

where φ_i is the principal eigenfunction of (3.1) with $\max_{\overline{\Omega}\times\mathbb{R}^+} \varphi_i(x,t) = 1$ and w_i have been introduced in (3.9).

First, we consider the solution of the problem (1.1)–(1.3) with the following initial conditions:

$$u_{10}(x) = rw[d_1, a_1, b_{11}]|_{t=0}, \qquad u_{20}(x) = \delta\varphi_2(x, 0), \qquad u_{30}(x) = \delta\varphi_3(x, 0).$$

From (4.4), we see that

$$(\delta\varphi_1(x,t), \delta\varphi_2(x,t), \delta\varphi_3(x,t)) \quad (\text{resp. } (rw_1(x,t), rw_2(x,t), rw_3(x,t)))$$

is a lower (resp. upper) solution of (1.1)–(1.3). By Lemma 3.6, there exists a unique global classical solution (u_1, u_2, u_3) of (1.1)–(1.3) such that

$$\delta\varphi_i(x,t) \le u_i(x,t) \le rw_i(x,t) \quad \text{on } \overline{\Omega} \times \mathbb{R}^+,\ i = 1, 2, 3.$$

Let $u_{i1}(x,t) = u_i(x, t+T)$ $(i = 1, 2, 3)$. Then we have

$$\begin{aligned}
\delta\varphi_1(x,0) &\le u_{11}(x,0) = u_1(x,T) \le rw_1|_{t=0} = u_1(x,0),\\
\delta\varphi_2(x,0) &= u_2(x,0) \le u_2(x,T) = u_{21}(x,0) \le rw_2|_{t=0},\\
\delta\varphi_3(x,0) &= u_3(x,0) \le u_3(x,T) = u_{31}(x,0) \le rw_3|_{t=0}.
\end{aligned}$$

Set $(U_1, U_2, U_3) = (u_1, u_{21}, u_{31})$, $(V_1, V_2, V_3) = (u_{11}, u_2, u_3)$. Then (U_1, U_2, U_3) and (V_1, V_2, V_3) are respectively upper and lower solutions of (1.1)–(1.3). By Lemma 3.6, we have

$$u_{11}(x,t) \le u_1(x,t), \quad u_2(x,t) \le u_{21}(x,t), \quad u_3(x,t) \le u_{31}(x,t).$$

For each integer n, define $u_{in}(x,t) = u_i(x, t+nT)$ $(i = 1,2,3)$. Then a similar argument shows that

$$\begin{aligned} \delta\varphi_1(x,t) &\le u_{1n}(x,t) \le u_{1,n-1}(x,t) \le rw_1, \\ \delta\varphi_2(x,t) &\le u_{2,n-1}(x,t) \le u_{2n}(x,t) \le rw_2, \\ \delta\varphi_3(x,t) &\le u_{3,n-1}(x,t) \le u_{3n}(x,t) \le rw_3. \end{aligned} \tag{4.5}$$

So there exist functions u_1^*, u_{2*}, u_{3*} on $\overline{\Omega} \times \mathbb{R}^+$ such that

$$\lim_{n\to\infty} (u_{1n}, u_{2n}, u_{3n})(x,t) = (u_1^*, u_{2*}, u_{3*})(x,t), \tag{4.6}$$

$$\delta\varphi_1 \le u_1^* \le rw_1, \quad \delta\varphi_2 \le u_{2*} \le rw_2, \quad \delta\varphi_3 \le u_{3*} \le rw_3. \tag{4.7}$$

By an argument similar to that in [AL], we can show that u_1^*, u_{2*}, u_{3*} is a classical positive T-periodic solution of the problem (1.1)–(1.2) with

$$\begin{aligned} &\lim_{t\to\infty} (u_1(x,t) - u_1^*(x,t)) = 0, \\ &\lim_{t\to\infty} (u_2(x,t) - u_{2*}(x,t)) = 0, \\ &\lim_{t\to\infty} (u_3(x,t) - u_{3*}(x,t)) = 0, \end{aligned} \tag{4.8}$$

uniformly for x in $\overline{\Omega}$.

Second, we consider (1.1)–(1.3) with the initial conditions

$$u_{10}(x) = \delta\varphi_2(x,0), \quad u_{20}(x) = rw[d_2, a_2, b_{22}]|_{t=0},$$
$$u_{30}(x) = rw[d_3, a_3, b_{33}]|_{t=0}.$$

Since (rw_1, rw_2, rw_3) and $(\delta\varphi_1, \delta\varphi_2, \delta\varphi_3)$ are respectively upper and lower solutions of (1.1)–(1.3), by Lemma 3.6 there exists a unique global solution (v_1, v_2, v_3) of (1.1)–(1.3) such that

$$\delta\varphi_i(x,t) \le v_i(x,t) \le rw_i(x,t) \quad \text{on } \overline{\Omega} \times \mathbb{R}^+,\ i = 1,2,3.$$

A similar argument shows that there exist smooth functions u_{1*}, u_2^*, u_3^* defined on $\overline{\Omega} \times \mathbb{R}^+$ such that (u_{1*}, u_2^*, u_3^*) is a T-periodic solution of the problem (1.1)–(1.2), and

$$\begin{aligned} &\lim_{t\to\infty} (v_1(x,t) - u_{1*}(x,t)) = 0, \\ &\lim_{t\to\infty} (v_2(x,t) - u_2^*(x,t)) = 0, \\ &\lim_{t\to\infty} (v_3(x,t) - u_3^*(x,t)) = 0, \end{aligned} \tag{4.9}$$

uniformly for x in $\overline{\Omega}$.

Set $(U_1,U_2,U_3)=(u_1,v_2,v_3)$ and $(V_1,V_2,V_3)=(v_1,u_2,u_3)$. As $u_1(x,0)\ge v_1(x,0)$, $u_2(x,0)\le v_2(x,0)$ and $u_3(x,0)\le v_3(x,0)$, we see that (U_1,U_2,U_3) and (V_1,V_2,V_3) are respectively upper and lower solutions of (1.1)–(1.3). By Lemma 3.6, on $\overline{\Omega}\times\mathbb{R}^+$ we have

$$u_1(x,t)\ge v_1(x,t),\quad u_2(x,t)\le v_2(x,t),\quad u_3(x,t)\le v_3(x,t). \tag{4.10}$$

From (4.8) and (4.9), we have

$$u_{i*}(x,t)\le u_i^*(x,t),\qquad i=1,2,3,\qquad \text{for all } (x,t)\in\overline{\Omega}\times\mathbb{R}^+. \tag{4.11}$$

Let (u_1,u_2,u_3) be a classical positive T-periodic solution of (1.1)–(1.2). By (3.6)–(3.8) and Lemma 3.5, we have

$$u_{i*}(x,t)\le u_i(x,t)\le u_i^*(x,t)\quad \text{on } \overline{\Omega}\times\mathbb{R}^+,\ i=1,2,3. \tag{4.12}$$

Summing up, for all sufficiently small values of δ and r, we obtain the same T-periodic solutions (u_{1*},u_2^*,u_3^*) and (u_1^*,u_{2*},u_{3*}).

Consider (1.1)–(1.3) with the initial conditions $(u_{10}(x),u_{20}(x),u_{30}(x))$, where $u_{i0}(x)$ $(i=1,2,3)$ are nontrivial nonnegative smooth functions on $\overline{\Omega}$. Let p_i $(i=1,2,3)$ be the global solutions of the problem

$$\begin{cases} \dfrac{\partial p_i}{\partial t}-d_i\Delta p_i=p_i(a_i-b_{ii}p_i) & \text{in } \Omega\times\mathbb{R}^+,\\ p_i=0 & \text{on } \partial\Omega\times\mathbb{R}^+,\\ p_i(x,0)=p_{i0}(x) & \text{on } \Omega. \end{cases} \tag{4.13}$$

By the parabolic maximum principle, $p_i(x,t)>0$ for $(x,t)\in\Omega\times\mathbb{R}^+$, so that (p_1,p_2,p_3) and $(0,0,0)$ are respectively upper and lower solutions of (1.1)–(1.3). Therefore, there exists a unique global classical solution (u_1,u_2,u_3) such that

$$0\le u_i\le p_i\quad \text{on } \overline{\Omega}\times\mathbb{R}^+,\ i=1,2,3.$$

Since

$$\begin{aligned}\frac{\partial u_1}{\partial t}-d_1\Delta u_1+u_1(-(a_1)_M+(b_{11})_Mp_{1M}+(b_{12})_Mp_{2M}&\\ +(b_{13})_Mp_{3M}+(e_1)_Mp_{1M}p_{2M}p_{3M})&\ge 0,\end{aligned}$$

by the extended parabolic minimum principle [PW] we have $u_1(x,mT)>0$ in Ω for each integer $m>0$, and $\partial u_1/\partial n<0$ on $\partial\Omega$. By Lemma 3.3, there exists a sufficiently small positive constant δ_0 such that $\delta_0\varphi_1(x,mT)\le u_1(x,mT)$, $\delta_0\varphi_2(x,mT)\le u_2(x,mT)$, $\delta_0\varphi_3(x,mT)\le u_3(x,mT)$ in Ω.

Now we show that for any $r>1$, there exists $M\in\mathbb{N}$ such that $p_1(x,mT)\le rw_1(x,0)$, $p_2(x,mT)\le rw_2(x,0)$, $p_3(x,mT)\le rw_3(x,0)$ on $\overline{\Omega}$ if $m>M$.

In fact, by Lemma 3.2 we have

$$p_1(x,mT)\to w_1(x,0)\quad \text{in } C^{2+\alpha}(\overline{\Omega})\ (m\to\infty), \tag{4.14}$$

so that

$$\partial p_1(x,mT)/\partial n \to \partial w_1(x,0)/\partial n \quad \text{in } C(\partial\Omega)\ (m\to\infty). \tag{4.15}$$

As $\partial w_1(x,0)/\partial n|_{\partial\Omega} < 0$ and $\partial p_1(x,mT)/\partial n|_{\partial\Omega} < 0$, for any $r > 1$ there exists $M_1 \in \mathbb{N}$ such that $\partial(rw_1(x,0) - p_1(x,mT))/\partial n|_{\partial\Omega} < 0$ if $m > M_1$. Since $rw_1(x,0) - p_1(x,mT) = 0$ on $\partial\Omega$, there exists a domain $\Omega_\varepsilon \subset\subset \Omega$ such that $rw_1(x,0) - p_1(x,mT) > 0$ for all $x \in \Omega_\varepsilon \cup \partial\Omega_\varepsilon$. So we just need to show $p_1(x,mT) \le rw_1(x,0)$ for $x \in \Omega_\varepsilon$. To arrive at a contradiction, assume that for each $n \in \mathbb{N}$ there exist $r_0 > 1$, k_n $(> n)$ and $x_{k_n} \in \Omega_\varepsilon$ such that

$$p_1(x,k_nT) > r_0 w_1(x,0). \tag{4.16}$$

Since there exists a subsequence $\{x_{k'_n}\} \subset \{x_{k_n}\}$ such that $x_{k'_n} \to x_0$, we have $w_1(x_0,0) \ge w_1(x_0,0)$. This is a contradiction. Hence, there exists $M_2 \in \mathbb{N}$ such that $p_1(x,mT) < rw_1(x,0)$ in Ω_ε if $m > M_2$. That is, for any $r > 1$ there exists $M \in \mathbb{N}$ such that $p_1(x,mT) \le rw_1(x,0)$ if $m > M$; and similarly for p_2, p_3.

From the above claims we see that for any $r > 1$ there exist a sufficiently large positive constant M and sufficiently small $\delta > 0$ such that if $m > M$, we have

$$\delta\varphi_i(x,0) \le u_i(x,mT) \le p_i(x,mT) \le rw_i(x,0), \qquad i = 1,2,3.$$

Let $(\underline{u}_1, \overline{u}_2, \overline{u}_3)$ and $(\overline{u}_1, \underline{u}_2, \underline{u}_3)$ be the unique solutions of (1.1)–(1.2) with

$$(\underline{u}_1(x,mT), \overline{u}_2(x,mT), \overline{u}_3(x,mT)) = (\delta\varphi_1(x,0), rw_2(x,0), rw_3(x,0)),$$
$$(\overline{u}_1(x,mT), \underline{u}_2(x,mT), \underline{u}_3(x,mT)) = (rw_1(x,0), \delta\varphi_2(x,0), \delta\varphi_3(x,0)).$$

By Lemma 3.6, we have

$$\delta\varphi_i(x,t) \le \underline{u}_i(x,t) \le u_i(x,t) \le \overline{u}_i(x,t) \le rw_i(x,t), \qquad i = 1,2,3, \tag{4.17}$$

on $\overline{\Omega} \times [mT,\infty)$, and

$$\lim_{t\to\infty}[\underline{u}_i(x,t) - u_{i*}(x,t)] = 0 = \lim_{t\to\infty}[\overline{u}_i(x,t) - u_i^*(x,t)], \qquad i = 1,2,3, \tag{4.18}$$

uniformly for x in $\overline{\Omega}$. Furthermore, (4.18) also holds in $C^{2,\alpha}(\Omega)$. Hence, for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$u_{i_*}(x,t) - \varepsilon\varphi_i < u_i(x,t) < u_i^*(x,t) + \varepsilon\varphi_i, \qquad i = 1,2,3, \tag{4.19}$$

on $\overline{\Omega} \times (t_\varepsilon,\infty)$. The proof is complete. ∎

Let λ_0 be the principal eigenvalue of $-\Delta$ under the homogeneous Dirichlet boundary condition. Obviously, if d_i and a_i are positive constants, $\lambda(d_i,a_i) = d_i\lambda_0 - a_i$. Consequently,

$$\lambda(d_i,a_i) \le (d_i)_M\lambda_0 - (a_i)_L, \qquad i = 1,2,3.$$

These arguments give the following corollary.

COROLLARY 4.1. *Assume that*

$$-(d_1)_M\lambda_0 + (a_1)_L > b_{12M}\left(\frac{a_2}{b_{22}}\right)_M + b_{13M}\left(\frac{a_3}{b_{33}}\right)_M, \tag{4.20}$$

$$-(d_2)_M\lambda_0 + (a_2)_L > b_{21M}\left(\frac{a_1}{b_{11}}\right)_M + b_{23M}\left(\frac{a_3}{b_{33}}\right)_M, \tag{4.21}$$

$$-(d_3)_M\lambda_0 + (a_3)_L > b_{31M}\left(\frac{a_1}{b_{11}}\right)_M + b_{32M}\left(\frac{a_2}{b_{22}}\right)_M. \tag{4.22}$$

Then the conclusions of Theorem 4.1 *also hold.*

Using a similar method to that in Theorem 4.1, we obtain the following conclusion.

THEOREM 4.2. *Assume that* $\lambda_{11} < 0$, $\lambda_{21} \geq 0$, $\lambda_{31} \geq 0$. *Then* (1.1)–(1.2) *has a unique* T*-periodic solution* $(u_1^*, 0, 0)$. *Moreover, if* (u_1, u_2, u_3) *is a solution of* (1.1)–(1.3), *and*

$$u_{i0}(x) \geq 0 \ (\not\equiv 0) \quad \text{in } \Omega \quad \text{and} \quad u_{i0}|_{\partial\Omega} = 0, \qquad i = 1, 2, 3,$$

then for any $\varepsilon > 0$ *there exists* $t_\varepsilon > 0$ *such that*

$$\begin{gathered} u_1^*(x,t) - \varepsilon\varphi_1 < u_1(x,t) < u_1^*(x,t) + \varepsilon\varphi_1, \\ 0 < u_2(x,t) < \varepsilon\varphi_2, \qquad 0 < u_3(x,t) < \varepsilon\varphi_3, \end{gathered}$$

for all $x \in \overline{\Omega}$ *and* $t > t_\varepsilon$.

COROLLARY 4.2. *Assume that* $\lambda_{11} \geq 0$, $\lambda_{21} < 0$, $\lambda_{31} < 0$. *Then* (1.1)–(1.2) *has a unique* T*-periodic solution* $(0, u_2^*, u_3^*)$. *Moreover, if* (u_1, u_2, u_3) *is a solution of* (1.1)–(1.3), *and*

$$u_{i0}(x) \geq 0 \ (\not\equiv 0) \quad \text{in } \Omega \quad \text{and} \quad u_{i0}|_{\partial\Omega} = 0, \qquad i = 1, 2, 3,$$

then for any $\varepsilon > 0$ *there exists* $t_\varepsilon > 0$ *such that*

$$\begin{gathered} 0 < u_1(x,t) < \varepsilon\varphi_1, \\ u_2^*(x,t) - \varepsilon\varphi_2 < u_2(x,t) < u_2^*(x,t) + \varepsilon\varphi_2, \\ u_3^*(x,t) - \varepsilon\varphi_3 < u_3(x,t) < u_3^*(x,t) + \varepsilon\varphi_3, \end{gathered}$$

for all $x \in \overline{\Omega}$ *and* $t > t_\varepsilon$.

5. Numerical illustrations. In this section, we present some numerical simulations to investigate the stability of the periodic solution. Take $\Omega = [0, \pi]$, $d_1 = 0.1$, $a_1 = 1 + \sin t$, $b_{11} = 0.08 + 0.08\cos t$, $b_{12} = 0.06 + 0.06\sin t$, $b_{13} = 0.004 + 0.004\sin t$, $e_1 = 0.00003 + 0.00003\cos t$, $u_{10} = 3.14157 + 3.14157\sin x$, $d_2 = 0.1$, $a_2 = 2 + 2\sin t$, $b_{21} = 0.05 + 0.05\cos t$, $b_{22} = 0.15 + 0.15\sin t$, $b_{23} = 0.002 + 0.002\sin t$, $e_2 = 0.00008 + 0.00008\sin t$, $u_{20} = 10.289 + 10.289\sin x$, $d_3 = 0.1$, $a_3 = 2 + 2\sin t$, $b_{31} = 0.045 + 0.045\cos t$, $b_{32} = 0.153 + 0.153\sin t$, $b_{33} = 0.002 + 0.002\sin t$, $e_3 = 0.00007 + 0.00007\sin t$, $u_{30} =$

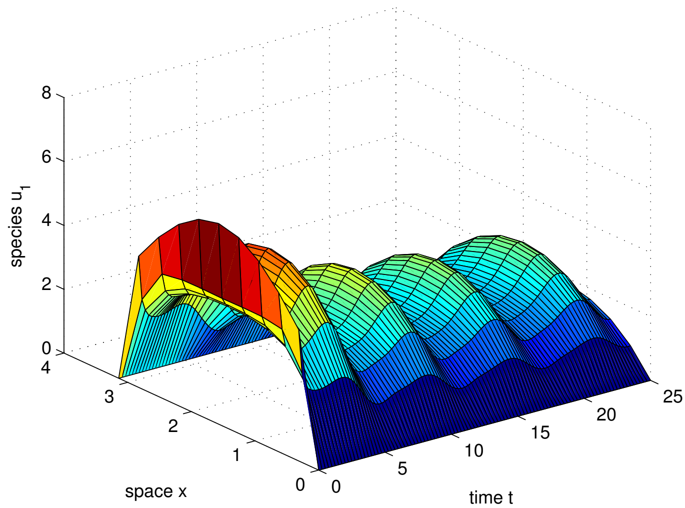

Fig. 1. Numerical simulation of u_1

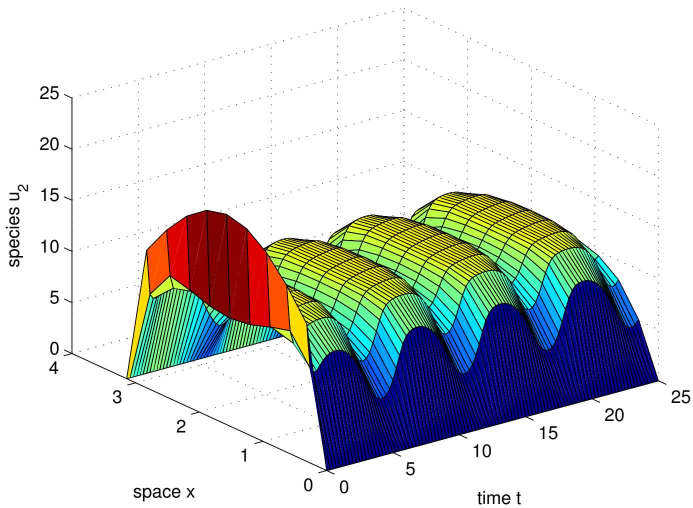

Fig. 2. Numerical simulation of u_2

$9.2601 + 9.2601 \sin x$. Obviously, the above coefficients satisfy the conditions of Theorem 4.1. By the forward difference scheme, we get numerical solutions of (1.1) (see Figures 1–3). From these figures we can see that the solution of

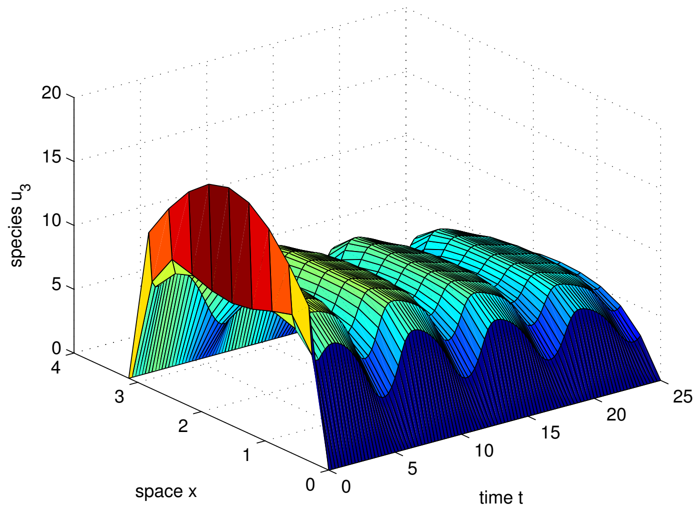

Fig. 3. Numerical simulation of u_3

the problem (1.1) will approach a stable periodic solution as $t \to \infty$, while the solution near the initial time may not be periodic.

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