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A Green's function for θ-incomplete polynomials

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Abstract. Let K be any subset of $\mathbb{C}^N$. We define a pluricomplex Green's function $V_{K,\theta}$ for θ-incomplete polynomials. We establish properties of $V_{K,\theta}$ analogous to those of the weighted pluricomplex Green's function. When K is a regular compact subset of $\mathbb{R}^N$, we show that every continuous function that can be approximated uniformly on K by θ-incomplete polynomials, must vanish on $K\setminus\operatorname{supp}(dd^cV_{K,\theta})^N$. We prove a version of Siciak's theorem and a comparison theorem for θ-incomplete polynomials. We compute $\operatorname{supp}(dd^cV_{K,\theta})^N$ when K is a compact section.

1. Introduction

Definition 1.1. For $0 < \theta < 1$, the set $\pi_{n,\theta}$ shall denote the collection of all polynomials P of the form $P(z) = \sum_{|\alpha|=\lceil n\theta\rceil}^{n} c_\alpha z^\alpha$. Here $z \in \mathbb{C}^N$ and $\lceil x\rceil$ denotes the least integer greater than or equal to x. If $P \in \pi_{n,\theta}$ for some $n \geq 0$ then we will refer to P as a *θ-incomplete polynomial.*

That is, a θ-incomplete polynomial is a polynomial that has no terms of degree smaller than θ times the degree of the polynomial. For the collection of all polynomials of degree at most n we will simply write π_n. θ-incomplete polynomials of several variables have previously been defined in [2] as polynomials of the form $P(z) = \sum_{|\alpha|=\lfloor n\theta\rfloor}^{n} c_\alpha z^\alpha$ where $\lfloor x\rfloor$ denotes the integer part of x, which yields a slightly different class of polynomials. However, the results in [2] can be verified for the new definition with essentially the same proofs. The new definition is convenient for the purpose of this paper because we would like the function $n^{-1}\log|P|$ to be in L_θ when $P \in \pi_{n,\theta}$. Also, under the new definition the class of θ-incomplete polynomials is closed under multiplication.

By the Weierstrass approximation theorem we know that every continuous function on the closed interval $[0,1]$ can be approximated uniformly by polynomials. The following result of Lorentz, Saff, von Golitschek and

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Kuijlaars ([7], [9], [4], [6]) gives a version of the Weierstrass theorem for θ-incomplete polynomials.

THEOREM 1.2. *For the interval* $[0,1] \subset \mathbb{C}$ *and* $f \in C[0,1]$ *there exists* $P_n \in \pi_{n,\theta}$ *with* $P_n \to f$ *uniformly as* $n \to \infty$ *if and only if* $f \equiv 0$ *on* $[0,\theta^2]$.

In other words, on the interval $[0,1]$, uniform limits of θ-incomplete polynomials (in the above sense) are precisely the continuous functions that vanish on the subinterval $[0,\theta^2]$. Given a compact set $K \subset \mathbb{R}^N$, let $C(K)$ be the Banach algebra of all continuous functions on K with the supremum norm. Let $C_\theta(K)$ be the subalgebra consisting of all functions $f \in C(K)$ admitting uniform approximation $P_n \to f$ as $n \to \infty$ on K by a sequence of θ-incomplete polynomials $P_n \in \pi_{n,\theta}$. Let $Z_\theta = \{x \in K : f(x) = 0 \text{ for all } f \in C_\theta(K)\}$. The following result is a consequence of the Stone–Weierstrass theorem.

THEOREM 1.3 ([2]). *For every compact set* $K \subset \mathbb{R}^N \subset \mathbb{C}^N$ *we have*

$$C_\theta(K) = \{f \in C(K) : f \equiv 0 \text{ on } Z_\theta\}.$$

For certain sets $K \subset \mathbb{R}^N$ that generalize the interval $[0,1]$, the set Z_θ can be determined explicitly to be $\theta^2 K$ (see [2]).

We make the following definitions, analogous to the Lelong classes L and L^+ and the Siciak extremal function V_K (see [10]).

DEFINITION 1.4. For $\theta \in \mathbb{R}$ and $K \subset \mathbb{C}^N$, we let

$$\begin{aligned} L_\theta &:= \{u \in L : u(z) \le \theta \log|z| + C_u \text{ on } B(0,1)\}, \\ L_\theta^+ &:= \{u \in L_\theta : \max\{\theta \log|z|, \log|z|\} + C_u \le u(z) \text{ for all } z \in \mathbb{C}^N\}, \\ V_{K,\theta}(z) &:= \sup\{u(z) : u \le 0 \text{ on } K,\ u \in L_\theta\}. \end{aligned}$$

Here the constant C_u depends on the function u. Observe that if $P \in \pi_{n,\theta}$ then $n^{-1}\log|P| \in L_\theta$. The main purpose of this paper will be to establish the basic properties of $V_{K,\theta}$ analogous to those of V_K. In particular we prove a version of Siciak's theorem for θ-incomplete polynomials. Then we establish:

THEOREM 1.5. *If* $K \subset \mathbb{R}^N$ *is a regular compact set*, *then* $K \setminus S_\theta \subset Z_\theta$.

Here $S_\theta = \operatorname{supp}(dd^c V_{K,\theta})^N$. We will conclude by using this result to compute S_θ for certain subsets of $\mathbb{R}^n$. The second section of the paper will review some basic facts from pluripotential theory and weighted pluripotential theory. Many of the results in the third section are similar to results about V_K which can be found in [5]. Finally, I would like to thank the referee, who suggested the general statement of Theorem 4.13 and whose comments have improved the overall coherence and quality of this exposition.

2. Background material. We let $\mathbb{C}^N$ denote complex N-space and we will write $z = (z_1, \ldots, z_N)$ where $z_i \in \mathbb{C}$ for elements of $\mathbb{C}^N$. An N-*multi-*

index is an N-tuple of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_N)$. We will use the multi-index notation z^α to denote the monomial $z_1^{\alpha_1} \cdots z_N^{\alpha_N}$. The degree of this monomial is $|\alpha| = \alpha_1 + \cdots + \alpha_N$. The open unit ball in $\mathbb{C}^N$ is denoted by $B(0,1)$. The Lelong class, the class of logarithmically homogeneous plurisubharmonic functions and the pluricomplex Green function for K are defined respectively as (see [10])

$$L := \{u \in \mathrm{PSH}(\mathbb{C}^N) : u(z) \le \log|z| + C_u \text{ on } \mathbb{C}^N \setminus B(0,1)\},$$
$$H := \{u \in \mathrm{PSH}(\mathbb{C}^N) : u(\lambda z) = u(z) + \log|\lambda| \text{ for all } \lambda \in \mathbb{C} \text{ and } z \in \mathbb{C}^N\},$$
$$V_K(z) := \sup\{u(z) : u \le 0 \text{ on } K,\ u \in L\}.$$

For every function f on an open set $U \in \mathbb{C}^N$, the upper semicontinuous regularization of f is defined as $f^*(z) := \limsup_{w\to z} f(w)$. A set P is called *pluripolar* if for every $x \in P$ there is a neighbourhood U of x and $v \in \mathrm{PSH}(U)$ with $v = -\infty$ on $U \cap P$. A set P is called *L-polar* if $P \subset \{x \in \mathbb{C}^N : v = -\infty\}$ for some $v \in L$. It is known from [5] that a set P is pluripolar if and only if it is L-polar. A property is said to hold *quasi-everywhere* (q.e.) if it holds everywhere except possibly on a pluripolar set. A subset $K \subset \mathbb{C}^N$ is said to be *non-pluripolar* in a neighbourhood of any of its points if for every $x \in K$ and for every neighbourhood U of x the intersection $K \cap U$ is non-pluripolar. Let $\|f\|_K^* := \inf\{\|f\|_{K\setminus P} : P$ a pluripolar subset of $K\}$. Note that if f is a continuous function on K and K is non-pluripolar in a neighbourhood of any of its points then $\|f\|_K^* = \|f\|_K$. A compact set $K \subset \mathbb{C}^N$ is said to be *regular* if $V_K^* = 0$ on K. We will need the following comparison theorem.

THEOREM 2.1 ([5]). *Let $G \subset \mathbb{C}^N$ be a bounded open set. Suppose that u and v are bounded plurisubharmonic functions on G such that*

$$\liminf_{z\to w,\, z\in G} (u(z) - v(z)) \ge 0 \quad \text{for all } w \in \partial G.$$

Then

$$\int_{\{u<v\}} (dd^c v)^N \le \int_{\{u<v\}} (dd^c u)^N.$$

We will also need the following result of Siciak [10].

THEOREM 2.2. *If $u \in L$ then there exists a sequence $n_j \nearrow \infty$, polynomials $P_{k,j}$ with $\deg P_{k,j} \le n_j$ and integers t_j such that $n_j^{-1} \max_{1\le k\le t_j} \log|P_{k,j}(z)|$ decreases to $u(z)$ as $j \to \infty$.*

We state the following definitions and results from weighted pluripotential theory for reference, as they are similar to the definitions and results in this paper. A good introduction to weighted pluripotential theory is given in Appendix B of [8].

DEFINITION 2.3. If $K \subset \mathbb{C}^N$ is a closed set and w is a non-negative real-valued function on K then w is called a *weight function*. A weight function

w is called *admissible* if

(i) w is upper semicontinuous,
(ii) the set of points in K where w is strictly greater than zero is non-pluripolar,
(iii) if K is unbounded then $\|z\|w(z) \to 0$ as $\|z\| \to \infty$ in K.

Define $Q(z) = -\log w(z)$. Then the *weighted pluricomplex Green function* of K with respect to Q is

$$V_{K,Q}(z) := \sup\{u(z) : u \le Q \text{ on } K,\ u \in L\}.$$

Let $V^*_{K,Q}(z)$ denote its upper semicontinuous regularization. Let $\mu_w := (dd^c V^*_{K,Q})^N$ and $S_w := \operatorname{supp} \mu_w$.

The following result on weighted approximation appears in [2].

THEOREM 2.4. *Suppose that $K \subset \mathbb{R}^N \subset \mathbb{C}^N$ is closed, and that w is a continuous admissible weight on K. Then there exists a closed set $Z_w \subset K$ such that there exists $P_n(z) \in \pi_n$ with $w^n P_n(z) \to f$ uniformly on K as $n \to \infty$ if and only if $f \in C(K)$ and $f \equiv 0$ on Z_w.*

The next theorem relates the sets S_w and Z_w.

THEOREM 2.5. *If $K \subset \mathbb{R}^N \subset \mathbb{C}^N$ is closed and non-pluripolar in a neighbourhood of any of its points, and if w is a continuous admissible weight on K, then $K \setminus S_w \subset Z_w$.*

DEFINITION 2.6. If K is a closed set and w is an admissible weight on K then

$$S^*_w := \{z \in K : V^*_{K,Q}(z) \ge Q(z)\},$$
$$\Phi_{K,Q}(z) := \sup\{|P_n(z)|^{1/n} : \|w^n P_n\|_K \le 1,\ P_n \in \pi_n,\ n \ge 1\},$$
$$\Psi_{K,Q}(z) := \sup\{|P_n(z)|^{1/n} : \|w^n P_n\|^*_K \le 1,\ P_n \in \pi_n,\ n \ge 1\}.$$

THEOREM 2.7 ([8]). *If $K \subset \mathbb{C}^N$ is a closed set and w is an admissible weight on K then $S_w \subset S^*_w$ and S_w is non-pluripolar.*

If $P \in \pi_n$ and $|w^n P| \le M$ q.e. on S_w then $|P| \le M e^{nV^*_{K,Q}}$ on $\mathbb{C}^N$.

If $P \in \pi_n$ then $\|w^n P\|^*_K = \|w^n P\|^*_{S_w}$.

If $S \subset K$ is closed and $\|w^n P\|^*_S = \|w^n P\|^*_K$ for all $P \in \pi_n$ then $S_w \subset S$.

Finally, $V_{K,Q} = \log \Phi_{K,Q}$ and $V^*_{K,Q} = (\log \Psi_{K,Q})^*$.

Let Γ be a compact subset of a hyperplane $L \subset \mathbb{R}^N \setminus \{0\}$. The set $K := \{tx : 0 \le t \le 1,\ x \in \Gamma\}$ is called a *compact section* if it is non-pluripolar in a neighbourhood of any of its points. If the hyperplane is given by the equation $\sum_{j=1}^N c_j x_j = d$ then the associated linear form for K is defined as $l(x) = d^{-1} \sum_{j=1}^N c_j x_j$. For a compact section and an appropriate weight w, the zero set Z_w and the set S_w are known explicitly.

THEOREM 2.8 ([2]). *If $K \subset \mathbb{R}^N \subset \mathbb{C}^N$ is a compact section and $w(x) = l(x)^{\theta/(1-\theta)}$, where $l(x)$ is the linear form associated with K and $0 < \theta < 1$, then $Z_w = \theta^2 K$ and $S_w = \overline{K \setminus \theta^2 K}$.*

3. Basic properties of $V_{K,\theta}$. We will begin by making a simple observation regarding $V_{K,\theta}$:

$$\begin{aligned} V_{K,\theta}(z) &= \sup\{u(z) : u \leq 0 \text{ on } K,\ u \in L_\theta\} \\ &= \theta \log|z| \\ &\quad + (1-\theta)\sup\left\{\frac{1}{1-\theta}(u(z) - \theta\log|z|) : u \leq 0 \text{ on } K, u \in L_\theta\right\} \\ &\geq \theta\log|z| + (1-\theta)\sup\left\{v(z) : v \in L, v(z) \leq \frac{-\theta}{1-\theta}\log|z| \text{ on } K\right\} \\ &= \theta\log|z| + (1-\theta)V_{K,Q}(z) \end{aligned}$$

where $Q = -\log w$ and $w = |z|^{\theta/(1-\theta)}$. The inequality can be seen by taking $u(z) = (1-\theta)v(z) + \theta\log|z|$ for a given v. When $N = 1$ we can take $v = (1-\theta)^{-1}(u(z) - \theta\log|z|)$ for a given u, because in one complex variable, the function $u(z) - \theta\log|z|$ is a subharmonic function with a removable singularity at the origin. However, for $N > 1$, this function is not necessarily plurisubharmonic. We summarize the above discussion in the following theorem.

THEOREM 3.1. *If $K \subset \mathbb{C}^N$ is a compact set then $V_{K,\theta}(z) \geq \theta\log|z| + (1-\theta)V_{K,Q}(z)$ where $Q = -\log w$ and $w = |z|^{\theta/(1-\theta)}$. If $N = 1$, that is, if $K \subset \mathbb{C}$, then equality holds.*

Hence in one complex variable, the function $V_{K,\theta}$ is essentially given by the weighted pluricomplex Green's function for the weight $w = |z|^{\theta/(1-\theta)}$.

For $\theta \leq 0$, we have $L = L_\theta$ because in this case the additional condition is redundant.

PROPOSITION 3.2. *$L_1 = H$.*

Proof. First suppose that $u \in H$. Then $u(0) = -\infty$ and for any $z \in \mathbb{C}^N \setminus \{0\}$ we can write $u(z) = u(z/|z|) + \log|z| \leq \|u\|_{\partial B(0,1)} + \log|z|$ on $\mathbb{C}^N$. This shows that $u \in L_1$. Conversely, suppose that $u \in L_1$. Then on any $\mathbb{C}$-line through the origin the function $v(z) = u(z) - \log|z|$ is subharmonic on $\mathbb{C} \setminus \{0\}$ and is bounded from above. From the removable singularity theorem for subharmonic functions it follows that v extends (on this line) to an entire subharmonic function that is bounded above. Hence v is constant on lines through the origin. The last statement is equivalent to saying that $u \in H$. ∎

PROPOSITION 3.3. *For $\theta > 1$, we have $L_\theta = \{-\infty\}$.*

Proof. Let $u \in L_\theta$. As in the above proof, the extension of the function v to any line through the origin must be constant. But clearly, $v(0) = -\infty$. So, $v \equiv -\infty$. Consequently, $u \equiv -\infty$. ∎

Note that when $\theta_1 \leq \theta_2$ we have $L_{\theta_2} \subset L_{\theta_1}$ and consequently $V_{K,\theta_2} \leq V_{K,\theta_1}$.

EXAMPLE 3.4. For the unit ball, $V_{\overline{B(0,1)},\theta}(z) = \max\{\theta \log|z|, \log|z|\}$.

A set P is called *L_θ-polar* if it is contained in the set $\{u \equiv -\infty\}$ for some $u \in L_\theta$.

LEMMA 3.5. *For $0 < \theta < 1$ and $P \subset \mathbb{C}^N$, P is L_θ-polar if and only if P is L-polar.*

Proof. Suppose that P is L-polar. Take $u \equiv -\infty$ on P, $u \in L$. Then the function $v(z) = \theta \log|z| + (1-\theta)u(z) \in L_\theta$ and $v \equiv -\infty$ on P. The converse is immediate. ∎

LEMMA 3.6. *If $E \subset \mathbb{C}^N$ is pluripolar then $V^*_{E,\theta} \equiv \infty$.*

Proof. Take $w \in L_\theta$, $w = -\infty$ on E. Then $w + n \leq V_{E,\theta}$ for all n. It follows that $V_{E,\theta} = \infty$ except possibly on the pluripolar set $\{w = -\infty\}$. So $V^*_{E,\theta} \equiv \infty$. ∎

LEMMA 3.7. *If $E \subset \mathbb{C}^N$ is bounded and non-pluripolar then $V^*_{E,\theta} \in L^+_\theta$.*

Proof. Observe that $V^*_{E,\theta} \leq V^*_E$. Because E is non-pluripolar we have $V^*_E \in L$. Consequently, $V^*_{E,\theta} \in L$. To show that $V^*_E \in L_\theta$ observe that $V^*_{E,\theta} \leq M$ on $B(0,1)$ for some constant M. Let u be in the defining class of $V_{E,\theta}$. Then $\theta^{-1}(u - M)$ is a non-positive plurisubharmonic function on $B(0,1)$ with a logarithmic pole at the origin. Letting $g_{B(0,1)}(z,0)$ denote the Green function [5] for the unit ball with logarithmic pole at the origin, namely $\log|z|$, we conclude that $\theta^{-1}(u - M) \leq g_{B(0,1)}(z,0)$ on $B(0,1)$. It follows that $u(z) \leq \theta \log|z| + M$ on $B(0,1)$. Taking the supremum over all such u we conclude that $V^*_{E,\theta} \leq \theta \log|z| + M$ on $B(0,1)$. Thus $V^*_{E,\theta} \in L_\theta$. Letting $|E| := \sup_{z\in E}|z|$ we conclude that $\max\{\theta \log(|z|/|E|), \log(|z|/|E|)\}$ is a candidate for $V_{E,\theta}$. It follows that $V^*_{E,\theta} \in L^+_\theta$. ∎

LEMMA 3.8. *If $P \subset \mathbb{C}^N$ is pluripolar and $E \subset \mathbb{C}^N$ is bounded then $V^*_{E\setminus P,\theta} = V^*_{E,\theta}$.*

Proof. If P is pluripolar then P is L-polar and hence L_θ-polar. Take $u \in L_\theta$ with $u \leq 0$ on $E \setminus P$. Take $v \in L_\theta$ with $v \leq 0$ on $E \setminus P$ and $v \equiv -\infty$ on P. Then

$$(1-\varepsilon)u + \varepsilon v \leq V_{E,\theta} \leq V^*_{E,\theta}.$$

Letting ε go to zero we conclude that $u \le V^*_{E,\theta}$ q.e., hence everywhere. Therefore, $V^*_{E\setminus P,\theta} \le V^*_{E,\theta}$. The reverse inequality is immediate so the result follows. ∎

LEMMA 3.9. *If K_j are compact sets such that $K_j \searrow K$ then $V_{K_j,\theta} \nearrow V_{K,\theta}$.*

Proof. Let $u \in L_\theta$, $u \le 0$ on K and $\varepsilon > 0$. Then the set $\{u < \varepsilon\}$ is a neighbourhood of K, so there exists j_0 such that whenever $j \ge j_0$ we have $K_j \subset \{u < \varepsilon\}$. In other words, $u - \varepsilon \le 0$ on K_j for all $j \ge j_0$. So, $u - \varepsilon \le V_{K_j,\theta}$ for all $j \ge j_0$. It follows that $u - \varepsilon \le \lim_{j\to\infty} V_{K_j,\theta} \le V_{K,\theta}$. The result follows by taking the supremum over all such functions u. ∎

LEMMA 3.10. *Let $E \subset \mathbb{C}^N$ be bounded and $0 < \theta < 1$. Then $V_{E,\theta+\varepsilon} \nearrow V_{E,\theta}$ as $\varepsilon \searrow 0$.*

Proof. Suppose that $u \in L_\theta$, that $u \le 0$ on E and that $\varepsilon > 0$ is so small that $\theta + \varepsilon < 1$. Let

$$v_\varepsilon(z) := \frac{1-\theta-\varepsilon}{1-\theta}\, u(z) + \frac{\varepsilon}{1-\theta} \log\frac{|z|}{|E|}.$$

Then $v_\varepsilon \le 0$ on E. Furthermore,

$$\begin{aligned} v_\varepsilon(z) &\le \frac{1-\theta-\varepsilon}{1-\theta}\,(\max\{\theta\log|z|, \log|z|\} + C_u) + \frac{\varepsilon}{1-\theta}\log\frac{|z|}{|E|} \\ &= \max\{(\theta+\varepsilon)\log|z|, \log|z|\} + \frac{1-\theta-\varepsilon}{1-\theta}\,C_u - \frac{\varepsilon}{1-\theta}\log|E|. \end{aligned}$$

So, $v_\varepsilon(z) \le V_{E,\theta+\varepsilon}(z)$. Letting $\varepsilon \searrow 0$, we get $u(z) \le \liminf_{\varepsilon\to0} V_{E,\theta+\varepsilon} \le \limsup_{\varepsilon\to0} V_{E,\theta+\varepsilon} \le V_{E,\theta}$. The result follows by taking the supremum over all such functions u. ∎

LEMMA 3.11. *If $K \subset \mathbb{C}^N$ is a regular compact set then $V^*_{K,\theta} = V_{K,\theta}$.*

Proof. Since K is regular it is non-pluripolar, hence $V^*_{K,\theta} \in L_\theta$ by Lemma 3.7. Also $V^*_{K,\theta} \le V^*_K = 0$ on K. Hence $V^*_{K,\theta}$ is in the family defining $V_{K,\theta}$. Consequently, $V^*_{K,\theta} \le V_{K,\theta}$. ∎

We have established that if E is non-pluripolar and bounded then $V^*_{E,\theta}$ is in the class L^+_θ. Consequently, $(dd^c V^*_{E,\theta})^N$ is well defined [5].

THEOREM 3.12. *If $E \subset \mathbb{C}^N$ is bounded and non-pluripolar then $(dd^c V^*_{E,\theta})^N \equiv 0$ on $\mathbb{C}^N \setminus \overline{E} \cup \{0\}$.*

Proof. Let $G \subset \mathbb{C}^N \setminus \overline{E} \cup \{0\}$ be a bounded open set. Let u be a plurisubharmonic function on G majorized by $V^*_{E,\theta}$ on ∂G. Then the function $v(z) := \max\{u(z), V^*_{E,\theta}(z)\}$ on G and $:= V^*_{E,\theta}(z)$ elsewhere on $\mathbb{C}^N$ has $v \in L_\theta$ and $v \le 0$ q.e. on E. So, v is non-positive on E except possibly on a pluripolar set F. Take $w \in L_\theta$ with $w = -\infty$ on F and w non-positive on E.

Then $(1+\varepsilon)^{-1}(v+\varepsilon w)$ is a candidate for $V_{E,\theta}$. So, $(1+\varepsilon)^{-1}(v+\varepsilon w) \le V^*_{E,\theta}$. Letting $\varepsilon \to 0$ we conclude that $v \le V^*_{E,\theta}$ q.e. in G, hence everywhere in G. So, $V^*_{E,\theta}$ is a maximal plurisubharmonic function on $\mathbb{C}^N \setminus \overline{E} \cup \{0\}$. But on this set, $V^*_{E,\theta}$ is locally bounded. Hence, $(dd^c V^*_{E,\theta})^N \equiv 0$ on $\mathbb{C}^N \setminus \overline{E} \cup \{0\}$. ∎

THEOREM 3.13. *If $E \subset \mathbb{C}^N$ is non-pluripolar and bounded then for some $\varepsilon > 0$ sufficiently small, we have $(dd^c V^*_{E,\theta})^N \equiv 0$ on $B(0,\varepsilon) \setminus \{0\}$.*

Proof. The set E is non-pluripolar, so $E \setminus B(0,\delta)$ is non-pluripolar for δ sufficiently small. Fix such a δ. Then $V^*_{E\setminus B(0,\delta),\theta}$ has a pole at the origin. This means that there exists ε such that $0 < \varepsilon < \delta$ with $V^*_{E\setminus B(0,\delta),\theta} \le 0$ on $B(0,\varepsilon)$. Now fix such an ε. Then $V^*_{E\setminus B(0,\varepsilon),\theta} \le 0$ on $B(0,\varepsilon)$. Hence, $V^*_{E\setminus B(0,\varepsilon),\theta} \le 0$ q.e. on E. By the same argument used in the preceding proof we conclude that $V^*_{E\setminus B(0,\varepsilon),\theta} \le V^*_{E,\theta}$ q.e., hence everywhere. Thus equality holds. So, on $B(0,\varepsilon) \setminus \{0\}$, we have $(dd^c V^*_{E,\theta})^N = (dd^c V^*_{E\setminus B(0,\varepsilon)}, \theta)^N \equiv 0$. ∎

THEOREM 3.14. *If $E \subset \mathbb{C}^N$ is non-pluripolar and bounded then in some neighbourhood of the origin, $(dd^c V^*_{E,\theta})^N \equiv (2\pi\theta)^N \delta_0$ where δ_0 is the Dirac delta measure.*

Proof. By Theorem 3.9 in [3] or Theorem 6.3.5 in [5] we know that $(dd^c V^*_{E,\theta})^N = (2\pi\theta)^N \delta_0$ at the origin. By the previous theorem we know that $(dd^c V^*_{E,\theta})^N = 0$ elsewhere near the origin. ∎

The following theorem and proof are exact counterparts of Lemma 6.5 in [1].

THEOREM 3.15. *If $u \in L_\theta$, $v \in {L_\theta}^+$, and $u \le v$ holds $(dd^c v)^N$-almost everywhere, then $u \le v$.*

Proof. Without loss of generality, we can make the following two assumptions:

(1) $v(z) \ge \theta \log|z| + ((1-\theta)/2)\log 3$ on $B(0,1)$.
(2) $v(z) \ge \frac{1}{2}\log(2+|z|^2)$ on $\mathbb{C}^N \setminus B(0,1)$.

Also, because $\max\{\theta\log|z|, \log|z|\} + C \le v(z)$ on $\mathbb{C}^N$, we can replace u with $\max\{u(z), \theta\log|z| + C, \log|z| + C\}$. Now suppose that $u > v$ at some point in $\mathbb{C}^N$ (note that this point is not the origin).

Select $\varepsilon, \delta, \eta$ in such a way that $\varepsilon\theta < \eta\theta < \eta\theta + \delta(1-\theta) < \varepsilon$, $\delta < \varepsilon$ and

$$S_{\varepsilon,\delta,\eta} := \left\{ z \in \mathbb{C}^N : u(z) + \eta\theta\log|z| + \delta\,\frac{1-\theta}{2}\log(2+|z|^2) > (1+\varepsilon)v(z) \right\}$$

is non-empty. Then this set must have positive Lebesgue measure. For $|z|$ large enough, we have

$$u(z) + \eta\theta\log|z| + \delta\,\frac{1-\theta}{2}\log(2+|z|^2) < (1+\varepsilon)\frac{1}{2}\log(2+|z|^2) \le (1+\varepsilon)v(z).$$

For $|z|$ small enough, we have

$$u(z) + \eta\theta \log|z| + \delta \frac{1-\theta}{2} \log(2 + |z|^2) < (1+\varepsilon)\left[\theta \log|z| + \frac{1-\theta}{2} \log 3\right] \le (1+\varepsilon)v(z).$$

It follows that $S_{\varepsilon,\delta,\eta}$ is a bounded set that is also bounded away from zero. In other words, $S_{\varepsilon,\delta,\eta} \subset A$ for some set A of the form $A = \{z \in \mathbb{C}^N : r < |z| < R\}$. The functions $u(z) + \eta\theta\log|z| + \delta((1-\theta)/2)\log(2 + |z|^2)$ and $(1+\varepsilon)v(z)$ are bounded plurisubharmonic functions on A, and we have verified that on the boundary of A the condition necessary to apply Theorem 2.1 is satisfied. So,

$$\begin{aligned} 0 &< \int\limits_{S_{\varepsilon,\delta,\eta}} \left(dd^c\left(\delta \frac{1-\theta}{2}\log(2+|z|^2)\right)\right)^N \\ &\le \int\limits_{S_{\varepsilon,\delta,\eta}} \left(dd^c\left(u(z) + \eta\theta\log|z| + \delta\frac{1-\theta}{2}\log(2+|z|^2)\right)\right)^N \\ &\le \int\limits_{S_{\varepsilon,\delta,\eta}} (1+\varepsilon)^N (dd^c v)^N. \end{aligned}$$

On $S_{\varepsilon,\delta,\eta}$, we have

$$\begin{aligned} (1+\varepsilon)v(z) &< u(z) + \eta\theta\log|z| + \delta\frac{1-\theta}{2}\log(2+|z|^2) \\ &\le v(z) + \eta\theta\log|z| + \delta\frac{1-\theta}{2}\log(2+|z|^2) \end{aligned}$$

$(dd^c v)^N$-almost everywhere. This implies that on $S_{\varepsilon,\delta,\eta}$, we have

$$v(z) < \frac{\eta\theta}{\varepsilon}\log|z| + \frac{\delta}{\varepsilon}\,\frac{1-\theta}{2}\log(2+|z|^2)$$

$(dd^c v)^N$-almost everywhere. But on the unit ball,

$$v(z) \ge \theta\log|z| + \frac{1-\theta}{2}\log 3 \ge \frac{\eta\theta}{\varepsilon}\log|z| + \frac{\delta}{\varepsilon}\,\frac{1-\theta}{2}\log(2+|z|^2),$$

while on the complement of the unit ball we have

$$v(z) \ge \frac{1}{2}\log(2+|z|^2) \ge \frac{\eta\theta}{\varepsilon}\log|z| + \frac{\delta}{\varepsilon}\,\frac{1-\theta}{2}\log(2+|z|^2).$$

Consequently, $S_{\varepsilon,\delta,\eta}$ is a set of $(dd^c v)^N$ measure zero.

This contradicts the integral inequality derived earlier. Thus, $u \le v$ on $\mathbb{C}^N$. ■

DEFINITION 3.16. Let $K \subset \mathbb{C}^N$ be non-pluripolar and bounded. Let $\mu_\theta := (dd^c V^*_{K,\theta})^N$, $S_\theta := \operatorname{supp}\mu_\theta$ and $S^*_\theta := \{z \in K : V^*_{K,\theta} \ge 0\}$.

LEMMA 3.17. *If $K \subset \mathbb{C}^N$ is a non-pluripolar compact set then $S_\theta \setminus \{0\} \subset S_\theta^*$, and the set $S_\theta \setminus \{0\}$ is non-pluripolar in a neighbourhood of any of its points.*

Proof. We will prove the first assertion by contradiction so suppose that $x_0 \notin S_\theta^*$ and $x_0 \in S_\theta \setminus \{0\}$. Then $V_{K,\theta}^*(x_0) < 0$. Next, $V_{K,\theta}^* = V_{K\setminus\{x_0\},\theta}^*$. Let $v = \lim_{\varepsilon\to 0} V_{K\setminus B(x_0,\varepsilon),\theta}^*$. It is easy to see that L_θ is closed under limits of decreasing sequences, so $v \in L_\theta$. Furthermore, $v \le 0$ q.e. on $K \setminus \{x_0\}$. So by the standard argument, $V_{K\setminus\{x_0\},\theta}^* \ge v$. For each $\varepsilon > 0$, we have $V_{K\setminus B(x_0,\varepsilon),\theta}^* \ge V_{K,\theta}^*$. Hence $v \ge V_{K,\theta}^*$. So, $v = V_{K,\theta}^*$. So $V_{K\setminus B(0,\varepsilon),\theta}^*(x_0) < 0$ for some $\varepsilon > 0$. It follows that $V_{K\setminus B(x_0,\varepsilon),\theta}^* < 0$ on some $B(x_0,\delta)$ where $0 < \delta < \varepsilon$. Then $V_{K\setminus B(x_0,\delta),\theta}^* < 0$ on $B(x_0,\delta)$. We conclude that $V_{K,\theta}^* \ge V_{K\setminus B(x_0,\delta),\theta}^*$. The opposite inequality is immediate so it follows that $V_{K,\theta}^* = V_{K\setminus B(x_0,\delta),\theta}^*$. Hence, $(dd^c V_{K,\theta}^*)^N \equiv 0$ on $B(x_0,\delta)$. Therefore, x_0 is not contained in S_θ. To see the second assertion, note that on $\mathbb{C}^N \setminus \{0\}$, $V_{K,\theta}^*$ is a locally bounded plurisubharmonic function. Hence, μ_θ places no mass on pluripolar subsets of $\mathbb{C}^N \setminus \{0\}$. By the definition of support of a measure, μ_θ must have positive mass in any neighbourhood of any point of S_θ. ∎

REMARK 3.18. Because $V_{K,\theta} = V_{K,\theta}^*$ q.e. on $\mathbb{C}^N$ it follows that $V_{K,\theta}^* \le 0$ q.e. on K and that $V_{K,\theta} = 0$ q.e. on S_θ^*.

4. Incomplete polynomials and approximation

THEOREM 4.1. *If $P(z) = \sum_{|\alpha|=\lceil n\theta\rceil}^{n} c_{\alpha,n} z^\alpha$ is a θ-incomplete polynomial and $|P(z)| \le M$ q.e. on S_θ then $|P(z)| \le M e^{nV_{K,\theta}^*}$ on $\mathbb{C}^N$.*

Proof. Observe that

$$\frac{1}{n}\log\frac{|P(z)|}{M} \le 0 \quad \text{ q.e. on } S_\theta.$$

Consequently,

$$\frac{1}{n}\log\frac{|P(z)|}{M} \le V_{K,\theta}^* \quad \text{ q.e. on } S_\theta.$$

By Theorem 3.15, this inequality must hold on all of $\mathbb{C}^N$. ∎

THEOREM 4.2. *If $P(z) = \sum_{|\alpha|=\lceil n\theta\rceil}^{n} c_{\alpha,n} z^\alpha$ is a θ-incomplete polynomial then $\|P\|_K^* = \|P\|_{S_\theta}^*$.*

Proof. We have $|P(z)| \le \|P\|_{S_\theta}^*$ q.e. on S_θ. Therefore, $|P(z)| \le \|P\|_{S_\theta}^* e^{nV_{K,\theta}^*}$ on $\mathbb{C}^N$. Consequently, $|P(z)| \le \|P\|_{S_\theta}^*$ q.e. on K. Hence, $\|P\|_K^* \le \|P\|_{S_\theta}^*$. So equality holds. ∎

DEFINITION 4.3. For $K \subset \mathbb{C}^N$ a compact set, let

(1) $$\Phi_{K,\theta}(z) := \sup\{|P_n(z)|^{1/n} : \|P_n\|_K \le 1,\ n \ge 1,\ P_n \in \pi_{n,\theta}\},$$

(2) $$\Psi_{K,\theta}(z) := \sup\{|P_n(z)|^{1/n} : \|P_n\|_K^* \le 1,\ n \ge 1,\ P_n \in \pi_{n,\theta}\}.$$

The cases $\theta = 0$ and $\theta = 1$ of the following theorem are proved in [10].

THEOREM 4.4. *For $K \subset \mathbb{C}^N$ compact, we have $V_{K,\theta} = \log \phi_{K,\theta}$.*

Proof. Let $\varepsilon > 0$. Suppose that $u \in L_{\theta+\varepsilon}$ and that $u \le 0$ on K. By Theorem 2.2 we have

$$u(z) = \lim_{j\to\infty} \frac{1}{n_j} \max_{1\le k\le t_j} \log |P_{k,j}(z)|,$$

where the sequence is decreasing and each $P_{k,j}$ is a polynomial of degree at most n_j.

Write $P_{k,j}(z) := \sum_{|\alpha|=0}^{n_j} c_{\alpha,k,j} z^\alpha$ and let $P'_{k,j}(z) := \sum_{|\alpha|=0}^{\lfloor n_j\theta\rfloor} c_{\alpha,k,j} z^\alpha$, where $\lfloor x\rfloor$ is the greatest integer that is less than or equal to x. That is, $P'_{k,j}(z)$ is the part of $P_{k,j}(z)$ consisting of monomials of degree smaller than or equal to $\lfloor n_j\theta\rfloor$. Now,

$$\begin{aligned}\|P'_{k,j}(z)\| &\le \binom{n_j+N}{N}\big(\max_{0\le|\alpha|\le\lfloor n_j\theta\rfloor} |c_{\alpha,k,j}|\big)\big(\max_{0\le|\alpha|\le n_j} |z_1|^{\alpha_1}\cdots|z_N|^{\alpha_N}\big)\\ &\le \binom{n_j+N}{N}\big(\max_{0\le|\alpha|\le\lfloor n_j\theta\rfloor} |c_{\alpha,k,j}|\big)(\max\{1,|z_1|,\dots,|z_N|\})^{n_j}.\end{aligned}$$

By Cauchy's estimate, for any $R < 1$ we have

$$\max_{0\le|\alpha|\le\lfloor n_j\theta\rfloor} |c_{\alpha,k,j}| \le \max_{0\le|\alpha|\le\lfloor n_j\theta\rfloor} \frac{\|P_{k,j}\|_{P(0,R)}}{R^{|\alpha|}} \le \frac{\|P_{k,j}\|_{P(0,R)}}{R^{\lfloor n_j\theta\rfloor}}.$$

Moreover, $u(z) \le (\theta+\varepsilon)\log|z| + M$ on $B(0,1)$. So, $u(z) \le (\theta+\varepsilon)\log N^{1/2}R + M$ on the Shilov boundary of the polydisc of radius R centered at the origin denoted by $\partial_S P(0,R) = \{(z_1,\dots,z_N) : |z_1| = \cdots = |z_N| = R\}$. By Dini's theorem there exists j_0 such that whenever $j \ge j_0$ we have

$$\frac{1}{n_j}\max_{1\le k\le t_j} \log|P_{k,j}(z)| \le (\theta+\varepsilon)\log N^{1/2}R + M + 1 \quad \text{on } \partial_S P(0,R).$$

Hence, $\|P_{k,j}\|_{P(0,R)}^{1/n_j} = \|P_{k,j}\|_{\partial_S P(0,R)}^{1/n_j} \le (N^{1/2}R)^{\theta+\varepsilon} e^{M+1}$. Therefore, for all $z \in \mathbb{C}^N$ we have

$$|P'_{k,j}(z)| \le \binom{n_j+N}{N} \frac{(\max\{1,|z_1|,\dots,|z_N|\})^{n_j}}{R^{\lfloor n_j\theta\rfloor}} ((N^{1/2}R)^{\theta+\varepsilon} e^{M+1})^{n_j}.$$

Hence,

$$\limsup_{j\to\infty} \max_{1\le k\le t_j} |P'_{k,j}(z)|^{1/n_j} \le \frac{N^{1/2}R^{\theta+\varepsilon} e^{M+1}(\max\{1,|z_1|,\dots,|z_N|\})}{R^\theta}.$$

Since this holds for any $1 > R > 0$ sufficiently small, we have

$$\limsup_{j\to\infty} \max_{1\le k\le t_j} |P'_{k,j}(z)|^{1/n_j} = 0 \quad \text{at every } z \in \mathbb{C}^N.$$

Applying the triangle inequality repeatedly, we get

$$\begin{aligned}
&\limsup_{j\to\infty} \max_{1\le k\le t_j} |P_{k,j}(z) - P'_{k,j}(z)|^{1/n_j} \\
&\quad \le \limsup_{j\to\infty} \max_{1\le k\le t_j} (|P_{k,j}| + |P'_{k,j}|)^{1/n_j} \\
&\quad \le \limsup_{j\to\infty} \max_{1\le k\le t_j} (|P_{k,j}(z)|^{1/n_j} + |P'_{k,j}(z)|^{1/n_j}) = \lim_{j\to\infty} \max_{1\le k\le t_j} |P_{k,j}(z)|^{1/n_j} \\
&\quad \le \liminf_{j\to\infty} \max_{1\le k\le t_j} (|P_{k,j}(z) - P'_{k,j}(z)|^{1/n_j} + |P'_{k,j}(z)|^{1/n_j}) \\
&\quad = \liminf_{j\to\infty} \max_{1\le k\le t_j} |P_{k,j}(z) - P'_{k,j}(z)|^{1/n_j}.
\end{aligned}$$

This shows that

$$u(z) = \lim_{j\to\infty} \frac{1}{n_j} \max_{1\le k\le t_j} \log |P_{k,j} - P'_{k,j}|.$$

We have expressed u as a pointwise limit using θ-incomplete polynomials. However, unlike the original sequence, this sequence is not necessarily decreasing. So instead of Dini's theorem we use Hartogs' lemma. The new sequence can readily be seen to be uniformly bounded on any ball centered at the origin just by applying Cauchy estimates and the fact that the original sequence is uniformly bounded on $P(0,1)$.

Hence, Hartogs' lemma gives us a j_1 such that whenever $j \ge j_1$ we have

$$\frac{1}{n_j} \max_{1\le k\le t_j} \log |P_{k,j} - P'_{k,j}| \le \varepsilon_1 \quad \text{on } K.$$

It follows that

$$\frac{1}{n_j} \max_{1\le k\le t_j} \log |P_{k,j} - P'_{k,j}| e^{-\varepsilon_1 n_j} \le \log \Phi_{K,\theta}.$$

Thus, $u(z) - \varepsilon_1 \le \log \Phi_{K,\theta}(z)$. So, $u(z) \le \log \Phi_{K,\theta}$. Taking the supremum over all such u, we infer that $V_{K,\theta+\varepsilon} \le \log \Phi_{K,\theta}$. Letting $\varepsilon \to 0$ we obtain $V_{K,\theta} \le \log \Phi_{K,\theta}$. The reverse inequality follows from the definition. ■

COROLLARY 4.5. *If $K \subset \mathbb{C}^N$ is a compact set and $0 < \theta < 1$ then the function $V_{K,\theta}$ is lower semicontinuous on $\mathbb{C}^N \setminus \{0\}$.*

Proof. We see that $\Phi_{K,\theta}$ is lower semicontinuous since it is a supremum over a family of continuous functions. The result follows from the theorem since the logarithm of a non-negative lower semicontinuous function is lower semicontinuous wherever the function is non-zero. Now, $V_{K,\theta} \ge \log(|z|/|K|)$ on $\mathbb{C}^N$ so $\Phi_{K,\theta}$ is non-zero on $\mathbb{C}^N \setminus \{0\}$. ■

COROLLARY 4.6. *If $K \subset \mathbb{C}^N$ is a regular compact set then $V_{K,\theta}$ is a plurisubharmonic function that is continuous on $\mathbb{C}^N \setminus \{0\}$.*

COROLLARY 4.7. *For $K \subset \mathbb{C}^N$ compact, we have $(\log \Psi_{K,\theta})^* = V^*_{K,\theta}$.*

Proof. By the definitions, $\Phi_{K,\theta} \le \Psi_{K,\theta}$. It follows that $V^*_{K,\theta} \le (\log \Psi_{K,\theta})^*$. For the reverse inequality, let P_n be any polynomial of degree at most n from the defining family for $\Psi_{K,\theta}$. Then, $n^{-1}\log|P_n| \le 0$ q.e. on K. So, $n^{-1}\log|P_n| \le V^*_{K,\theta}$ q.e. on S_θ. Hence, $n^{-1}\log|P_n| \le V^*_{K,\theta}$ on all of $\mathbb{C}^N$. Therefore, $\log \Psi_{K,\theta} \le V^*_{K,\theta}$. ∎

THEOREM 4.8. *If $S \subset K \subset \mathbb{C}^N$ are compact sets with the property that $\|P\|^*_S = \|P\|^*_K$ for all θ-incomplete polynomials P then $S_\theta \subset S \cup \{0\}$.*

Proof. From the previous corollary we conclude that $V^*_{S,\theta} = V^*_{K,\theta}$. So, $S_\theta = \operatorname{supp}(dd^c V^*_{K,\theta})^N = \operatorname{supp}(dd^c V^*_{S,\theta})^N \subset S \cup \{0\}$. ∎

THEOREM 4.9. *Suppose that K is non-pluripolar and bounded and that a sequence $\{P_n\} \in \pi_{n,\theta}$ converges uniformly on S_θ as $n \to \infty$. Then $|P_n(z_0)| \to 0$ as $n \to \infty$ for all $z_0 \in \{z \in \mathbb{C}^N : V^*_{K,\theta}(z) < 0\}$.*

Proof. Let $z_0 \in \{V^*_{K,\theta} < 0\}$. Then $|P_n(z_0)| \le \|P_n\|^*_{S_\theta} e^{nV^*_{K,\theta}(z_0)}$. Consequently, $|P_n(z_0)| \to 0$ as $n \to \infty$. ∎

The following result is a counterpart of Theorem 3.2 in [2]. Its proof will require the Stone–Weierstrass theorem, in the following form:

THEOREM 4.10. *Let X be a compact Hausdorff space and let $C(X) := \{f : X \to \mathbb{R},\ f \text{ continuous}\}$. Let $A \subset C(X)$ and $Z(A) := \{x \in X : f(x) = 0$ for all $f \in A\}$. Suppose further that A has the following four properties:*

(i) *closedness under addition and real scalar multiplication,*
(ii) *closedness under multiplication,*
(iii) *closedness under uniform limits,*
(iv) *for all $x_1 \ne x_2$ in $X \setminus Z(A)$ there exists $f \in A$ with $f(x_1) \ne f(x_2)$.*

Then $A = \{f \in C(X) : f \equiv 0 \text{ on } Z(A)\}$.

THEOREM 4.11. *Let $K \subset \mathbb{R}^N \subset \mathbb{C}^N$ be a compact set that is non-pluripolar in a neighbourhood of any of its points. If a sequence $\{P_n\}$ with $P_n \in \pi_{n,\theta}$ converges uniformly on S_θ as $n \to \infty$ then $\{P_n(w)\}$ converges to 0 for every $w \in K \setminus S_\theta$.*

Proof. Suppose that $P_n \in \pi_{n,\theta}$ with $P_n \to f_0$ uniformly on S_θ as $n \to \infty$ and let $w = (w_1, \ldots, w_N) \in K \setminus S_\theta$. Define $p(z) = p(z_1, \ldots, z_N) = (z_1^2 + \cdots + z_N^2)^\gamma((z_1 - w_1)^2 + \cdots + (z_N - w_N)^2)^\beta$ where γ and β are positive integers such that $\gamma/(\gamma+\beta) > \theta$. With this choice, p is a θ-incomplete polynomial. Furthermore, since $K \subset \mathbb{R}^N$, p only vanishes on K at w and at the origin.

Now, S_θ and

$$A = \{f \in C[S_\theta] : \exists Q_n \in \pi_{n,\theta} \text{ with } Q_n \to f \text{ uniformly as } n \to \infty \\ \text{and } Q_n(w) = 0\}$$

satisfy the hypotheses of the Stone–Weierstrass theorem.

Hence $p(z)f_0(z) \in A$ iff $p(z)f_0(z) \equiv 0$ on $Z(A)$ iff $f_0(z) \equiv 0$ on $Z(A)$ iff $f_0(z) \in A$.

The second equivalence holds since $Z(A) \subset S_\theta$ and $f_0(0) = 0$. Hence, $f_0 \in A$. It follows that there exist $Q_n \in \pi_{n,\theta}$ with $Q_n \to f_0$ on S_θ uniformly as $n \to \infty$, with $Q_n(w) = 0$. Thus $|P_n(w)| \le \|Q_n - P_n\|_K = \|Q_n - P_n\|_K^* = \|Q_n - P_n\|^*{}_{S_\theta} = \|Q_n - P_n\|_{S_\theta}$. Letting $n \to \infty$ we get $|P_n(w)| \to 0$. ∎

Corollary 4.12. *$K \setminus S_\theta \subset Z_\theta$.*

Proof. Any sequence $P_n \in \pi_{n,\theta}$ converging uniformly on K satisfies the hypotheses of the previous theorem. ∎

Theorem 4.13. *If $K \subset \mathbb{R}^N$ is an intersection of compact sections and is non-pluripolar in a neighbourhood of any of its points then*

$$S_\theta = \overline{K \setminus \theta^2 K} \cup \{0\}.$$

Proof. From [2] we have $Z_\theta = \theta^2 K$. Moreover the inclusion $K \setminus S_\theta \subset Z_\theta$ gives $K \setminus \theta^2 K \subset S_\theta$. We also know that $\{0\} \subset S_\theta$. Now if $\|P\|^*_{\overline{K\setminus\theta^2 K}} = \|P\|^*_K$ for all θ-incomplete polynomials P then by Theorem 4.8, $S_\theta \subset \overline{K \setminus \theta^2 K} \cup \{0\}$ and we are done. Otherwise we have $\|P\|^*_{\overline{K\setminus\theta^2 K}} < \|P\|^*_K$ for some θ-incomplete polynomial P. Consequently, $\|P\|_{\overline{K\setminus\theta^2 K}} < \|P\|_K$. In this case take $q \in \theta^2 K$ with $|P(q)| = \|P\|_K$. Because K is an intersection of compact sections, it is starlike with respect to the origin, so restricting to the line through the origin and q we get a one-variable θ-incomplete polynomial p with $|p(q)| > \|p\|_{[\theta^2,1]}$, where $q < \theta^2$. This contradicts the fact that $\|p\|_{[\theta^2,1]} = \|p\|_{[0,1]}$ for all θ-incomplete polynomials p. ∎

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