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Simple connection matrices

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Abstract. We introduce simple connection matrices. We prove the existence of simple connection matrices for filtered differential vector spaces and Morse decompositions of compact metric spaces.

Introduction. The idea of the connection matrix is due to Charles Conley, but its existence in the case of flows was first established by Robert Franzosa [6]. Later Robbin and Salamon [9] extended the connection matrix theory to the setting of discrete dynamical systems. Connection matrices can be seen as algebraic representations of the dynamical system. They express the relationship between certain (co)homology groups. Connection matrices appear in a wide variety of situations. In this paper we study the simplest possible version of these algebraic tools and for that reason we will call them simple connection matrices. We introduce simple connection matrices for filtered differential vector spaces. A filtered differential vector space is a finite filtration of a given vector space together with an endomorphism d such that $d^2 = 0$ and d preserves the filtration. A simple connection matrix is a subspace of the filtered differential vector space which provides information on some homology groups associated with the filtered differential vector space. We prove the existence of such connection matrices.

It is natural to try to relate this purely algebraic theory to topological dynamics. An understanding of the above relation is one of the goals of the Conley index theory. The standard references here are [4]–[8]. In this paper we consider flows on compact metric spaces. We have restricted ourselves to the case of flows to avoid additional complications. Our purpose is to investigate simple connection matrices for Morse decompositions.

The organization of the paper is as follows. Section 1 presents some preliminaries from the Conley index theory for (continuous-time) flows. In Sec-

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tion 2 we introduce filtered differential vector spaces and prove the existence of simple connection matrices. The proof was motivated by [9]. In Section 3 we indicate how the algebraic techniques from the previous section may be applied to dynamical systems. For this purpose we introduce the notion of the simple connection matrix for a Morse decomposition and examine some elementary properties of this concept. Section 4 contains an example which illustrates how simple connection matrices may be computed and represented. For more references on the material presented here, see [1]–[3].

1. Preliminaries. We recall briefly some standard definitions from the Conley index theory. The contents of this section will not be needed until Section 3.

Throughout this paper X denotes a compact metric space and φ denotes a *flow* on X, i.e. a continuous map $\varphi : \mathbb{R} \times X \to X$ satisfying

$$\varphi(0,x) = x, \qquad \varphi(t,\varphi(s,x)) = \varphi(t+s,x).$$

If $I \subset \mathbb{R}$ and $A \subset X$, then $\varphi(I,A) := \{\varphi(t,x) \mid t \in I \text{ and } x \in A\}$. For a given subset $N \subset X$ the set $\operatorname{Inv}(N) := \{x \in X \mid \varphi(\mathbb{R},x) \subset N\}$ is called the *invariant part* of N. We say that $S \subset X$ is *invariant* if $\operatorname{Inv}(S) = S$.

Recall that given a set $Y \subset X$ the *positive limit set* of Y is given by

$$\omega^+(Y) := \bigcap_{t>0} \operatorname{cl}(\varphi([t,\infty),Y))$$

and the *negative limit set* of Y is

$$\omega^-(Y) := \bigcap_{t<0} \operatorname{cl}(\varphi((-\infty,t],Y)).$$

A compact set $N \subset X$ is called an *isolating neighborhood* if $\operatorname{Inv}(N) \subset \operatorname{int}(N)$. A compact invariant set $S \subset X$ is called an *isolated invariant set* if $S = \operatorname{Inv}(N)$ for some isolating neighborhood N. A subset $A \subset L$ is called *positively invariant* in L if given $x \in A$ and $\varphi([0,t],x) \subset L$, we have $\varphi([0,t],x) \subset A$. A subset A of L is called an *exit set* for L if given $x \in L$ such that $\varphi([0,\infty),x) \not\subset L$, there exists $t \geq 0$ such that $\varphi([0,t],x) \subset L$ and $\varphi(t,x) \in A$.

Let S be an isolated invariant set. A pair (N^1,N^0) of compact sets is called an *index pair* for S if:

(i) $S = \operatorname{Inv}(\operatorname{cl}(N^1 \setminus N^0)) \subset \operatorname{int}(N^1 \setminus N^0)$,
(ii) N^0 is positively invariant in N^1,
(iii) N^0 is an exit set for N^1.

The *homological Conley index* of S is defined by

$$CH_*(S) := H_*(N^1/N^0,[N^0]) \approx H_*(N^1,N^0),$$

where (N^1, N^0) is any index pair for S and H_* stands for the singular homology with field coefficients. Unfortunately, it is not true that for any index pair $H_*(N^1/N^0, [N^0]) \approx H_*(N^1, N^0)$. However, this isomorphism holds for regular index pairs (see Salamon [10, Sec. 5] for the definition and more details). For that reason we will assume that we are working with regular index pairs and index filtrations. Another way to overcome this problem is to use the Alexander–Spanier cohomology functor instead of the usual homology.

We give two more definitions from the Conley index theory. Let S be an isolated invariant set.

Definition 1.1. A collection $\{M_i\}_{i=1}^n$ of mutually disjoint compact invariant subsets of S is a *Morse decomposition* of S if for every $x \in S\setminus\bigcup_{i=1}^n M_i$ there are indices $i < j$ such that $\omega^+(x) \subset M_i$ and $\omega^-(x) \subset M_j$.

The sets M_i are called *Morse sets.* Moreover, we define *generalized Morse sets* for $i \le j$:

$$M_{ji} := \Big\{x \in S \,\Big|\, \omega^+(x) \cup \omega^-(x) \subset \bigcup_{k=i}^{j} M_k\Big\}.$$

In particular, $M_{jj} = M_j$. It is easy to check that all M_{ji} are isolated invariant sets.

Definition 1.2. An *index filtration* for the Morse decomposition $\{M_i\}_{i=1}^n$ is a collection $\{N^i\}_{i=0}^n$ of compact sets such that

(1) $N^0 \subset N^1 \subset \dots \subset N^n$,
(2) for any $i \le j$, (N^j, N^{i-1}) is an index pair for M_{ji}.

Let us formulate the natural

Theorem 1.3. *For any given Morse decomposition there exists an index filtration.*

This was proved by Salamon [10].

The simplest nontrivial case of a Morse decomposition of an isolated invariant set S is one consisting of two elements $\{M_1, M_2\}$. It is called an *attractor-repeller pair* in S. The *set of connecting orbits* from M_2 to M_1 in S is

$$C(M_2, M_1; S) := \{x \in S \mid \omega^+(x) \subset M_1,\, \omega^-(x) \subset M_2\}.$$

An index filtration for an attractor-repeller pair $\{M_1, M_2\}$ is reduced to an *index triple* $N^0 \subset N^1 \subset N^2$, where

- (N^2, N^0) is an index pair for S,
- (N^2, N^1) is an index pair for M_2,
- (N^1, N^0) is an index pair for M_1.

Let ∂ denote the boundary map in a long exact homology sequence:

$$\cdots \to H_k(N^1, N^0) \to H_k(N^2, N^0) \to H_k(N^2, N^1) \xrightarrow{\partial} H_{k-1}(N^1, N^0) \to \cdots$$

The importance of the boundary map ∂ is reflected in the following

THEOREM 1.4. *If $\partial \neq 0$, then $C(M_2, M_1; S) \neq \emptyset$.*

Proof. If $C(M_2, M_1; S) = \emptyset$, then $S = M_1 \cup M_2$, so

$$\begin{aligned} H_*(N^2, N^0) &= CH_*(S) \approx CH_*(M_1) \oplus CH_*(M_2) \\ &= H_*(N^1, N^0) \oplus H_*(N^2, N^1) \end{aligned}$$

and consequently $\partial = 0$. ∎

2. Simple connection matrices. Recall that a *filtration* of a vector space A is a sequence $\{A^i\}_{i=0}^n$ of subspaces of A such that

$$0 = A^0 \subset A^1 \subset \cdots \subset A^n = A.$$

A pair consisting of a vector space and its filtration is called a *filtered vector space*. Similarly, a *graded vector space* is a vector space A with a sequence $\{A_i\}_{i=1}^n$ of subspaces of A such that

$$A = \bigoplus_{i=1}^{n} A_i.$$

Moreover, a grading $\{A_i\}_{i=1}^n$ of A is called a *splitting* for the filtration $\{A^i\}_{i=0}^n$ of A if

$$A^i = \bigoplus_{k=1}^{i} A_k$$

for any $1 \le i \le n$. Of course, the splitting is not unique.

DEFINITION 2.1. A *filtered differential vector space* is a filtered vector space A together with an endomorphism d such that $d^2 = 0$ and d preserves the filtration, i.e. $dA^i \subset A^i$.

If $\{A_i\}_{i=1}^n$ is a splitting of a filtered differential vector space, then the condition that d is filtration preserving is equivalent to the fact that the components of d in the direct sum decomposition $d_{p,q} : A_q \to A_p$ satisfy $d_{p,q} = 0$ for each $p > q$, which means that the differential d written in the form of a usual matrix is triangular.

A *homomorphism of filtered differential vector spaces* is any homomorphism of vector spaces $h : A \to \widehat{A}$ such that $\widehat{d}h = hd$ and h preserves the filtration, i.e. $hA^i \subset \widehat{A}^i$. It is easy to see that filtered differential vector spaces and their homomorphisms form a category. Of course, if $h : A \to \widehat{A}$ is a homomorphism of filtered differential vector spaces, then h induces homology homomorphisms

$$h : H(A^i/A^j) \to H(\widehat{A}^i/\widehat{A}^j)$$

for any $j \le i$. Furthermore, it follows from the five lemma that if $h : H(A) \to H(\widehat{A})$ is an isomorphism, then so are all $h : H(A^i/A^j) \to H(\widehat{A}^i/\widehat{A}^j)$.

It is high time to introduce the crucial definition of this paper.

DEFINITION 2.2. A filtered differential vector space C is called a *simple connection matrix* if $dC^i \subset C^{i-1}$.

The requirement in the above definition constitutes a significant strengthening of the condition that d preserves the filtration. Here are some elementary properties of simple connection matrices. For example, if $\{C_i\}_{i=1}^n$ is a splitting of the filtered differential vector space C, then C is a simple connection matrix if and only if the components of the differential d in the direct sum decomposition satisfy $d_{p,q} = 0$ for each $p \ge q$, which means that d treated as a matrix is strictly triangular. Another consequence of the last definition is stated below.

PROPOSITION 2.3. *If C is a simple connection matrix, then* $H(C^i/C^{i-1}) = C^i/C^{i-1}$ *for each* $1 \le i \le n$.

Proof. Let $d : C^i/C^{i-1} \to C^i/C^{i-1}$ be the differential on the quotient. Since $dC^i \subset C^{i-1}$, we have $\ker d = C^i/C^{i-1}$ and $\operatorname{Im} d = 0$. Therefore

$$H(C^i/C^{i-1}) = \ker d/\operatorname{Im} d = C^i/C^{i-1}. \ \blacksquare$$

Let us formulate the main result of this section. This lemma states that any filtered differential vector space decomposes as $A = B \oplus C \oplus dB$, where C is a simple connection matrix and the components C_i of C may be identified with the relative homology $H(A^i/A^{i-1})$. Observe that assertions (1) and (4) below are nothing but the statements that $\{A_i\}_{i=1}^n$ is a splitting of A and that $C = \bigoplus_{i=1}^n C_i$ is a simple connection matrix respectively.

LEMMA 2.4 (Decomposition Lemma). *For any filtered differential vector space A there are graded vector spaces $\{A_i\}_{i=1}^n$, $\{B_i\}_{i=1}^n$, $\{C_i\}_{i=1}^n$ such that for each $1 \le i \le n$:*

(1) $A^i = A^{i-1} \oplus A_i$,
(2) $A_i = B_i \oplus C_i \oplus dB_i$,
(3) *the differential d maps B_i isomorphically onto dB_i, i.e.* $B_i \cap d^{-1}0 = 0$,
(4) $dC_i \subset C^{i-1}$, *where* $C^{i-1} = \bigoplus_{k=1}^{i-1} C_k$.

Proof. Assume by induction that vector spaces A_k, B_k, C_k exist for $k \le i$ and satisfy conditions (1)–(4). The proof will be completed by constructing spaces A_{i+1}, B_{i+1}, C_{i+1} satisfying (1)–(4).

The diagram in Figure 1 shows the main idea of the construction. The construction will be divided into three steps.

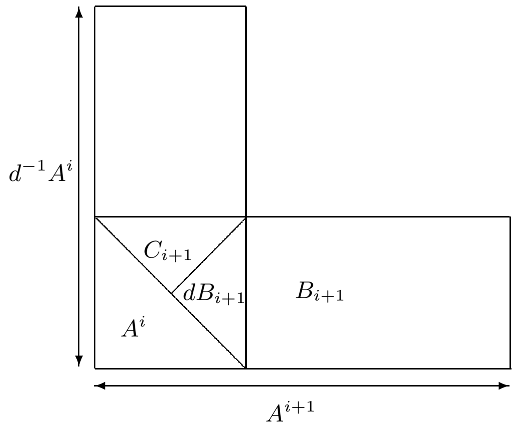

Fig. 1. The idea of the proof of Lemma 2.4

STEP 1. We first choose B_{i+1} to be any complement to $A^{i+1} \cap d^{-1}A^i$ in A^{i+1}, i.e.

$$A^{i+1} = B_{i+1} \oplus (A^{i+1} \cap d^{-1}A^i).$$

By the definition of B_{i+1}:

(i) $B_{i+1} \cap d^{-1}0 = 0$, i.e. $d|_{B_{i+1}}$ is a monomorphism,
(ii) $dB_{i+1} \subset dA^{i+1} \cap d^{-1}0 \subset A^{i+1} \cap d^{-1}A^i$,
(iii) $A^i \cap dB_{i+1} = 0$.

STEP 2. Then we choose C_{i+1} to be a complement to $A^i \oplus dB_{i+1}$ in $A^{i+1} \cap d^{-1}A^i$ satisfying

$$dC_{i+1} \subset C^i.$$

The existence of such a complement is equivalent to

$$A^{i+1} \cap d^{-1}A^i = A^i + (A^{i+1} \cap d^{-1}C^i).$$

It is obvious that the right-hand side is a subset of the left-hand side. The reverse inclusion may be deduced from the inclusion

$$A^{i+1} \cap d^{-1}A^i \subset B^i + (A^{i+1} \cap d^{-1}C^i),$$

which is justified below. Let $a \in A^{i+1} \cap d^{-1}A^i$. By the induction assumption, there are $b, e \in B^i$ and $c \in C^i$ such that

$$da = b + c + de.$$

Hence

$$dB^i \ni db = d(-c) \in dC^i \subset C^i.$$

Since $C^i \cap dB^i = 0$, we have $db = 0$, and consequently $b = 0$, because $d|_{B^i}$ is a monomorphism. Then $d(a - e) = c \in C^i$, i.e. $a - e \in A^{i+1} \cap d^{-1}C^i$. We thus get

$$a = e + (a - e) \in B^i + (A^{i+1} \cap d^{-1}C^i),$$

which completes the proof of the desired inclusion.

STEP 3. Finally, we define

$$A_{i+1} = B_{i+1} \oplus C_{i+1} \oplus dB_{i+1}.$$

It is easy to check that A_{i+1}, B_{i+1} and C_{i+1} satisfy conditions (1)–(4). ■

DEFINITION 2.5. A subspace C of the filtered differential vector space A is called a *simple connection matrix for* A if

(1) C is a simple connection matrix,
(2) the map $i : H(C) \to H(A)$ induced by the inclusion $C \subset A$ is an isomorphism.

It may be worth pointing out that in our category, C is a subspace of A if the inclusion $C \subset A$ preserves the filtration, i.e. $C^p \subset A^p$. Moreover, from (2) and the five lemma, we see at once that if C is a simple connection matrix for A, then $i : H(C^p/C^q) \to H(A^p/A^q)$ are isomorphisms for any $q \le p$. In particular,

$$H(A^p/A^{p-1}) = H(C^p/C^{p-1}) = C^p/C^{p-1} = C_p.$$

The following result is the most important consequence of the above decomposition lemma.

THEOREM 2.6. *There is a simple connection matrix for any filtered differential vector space.*

Proof. Let A be a filtered differential vector space and $C = \bigoplus_{i=1}^n C_i$ be as in the assertion of the previous lemma, so C is a simple connection matrix. Since $A = B \oplus C \oplus dB$ and

$$H(A) = \frac{\ker d}{\operatorname{Im} d} = \frac{(\ker d|_C) \oplus dB}{(\operatorname{Im} d|_C) \oplus dB} = \frac{\ker d|_C}{\operatorname{Im} d|_C} = H(C),$$

we see that the inclusion $C \subset A$ induces an isomorphism $i : H(C) \to H(A)$ in homology, which completes the proof. ■

3. Applications to dynamical systems. Let $\emptyset = N^0 \subset N^1 \subset \cdots \subset N^n = N$ be a topological filtration. Let $C(N^k)$ be the vector space of singular chains in N^k and $i_k : C(N^k) \to C(N)$ be the homomorphism induced by the inclusion $N^k \subset N$.

DEFINITION 3.1. A *simple connection matrix for the topological filtration* $\{N^k\}_{i=0}^n$ is a simple connection matrix for the filtered differential vector space $(\{i_k(C(N^k))\}_{i=0}^n, d)$, where d is the boundary map on singular chains.

Let X be a compact metric space.

DEFINITION 3.2. A *simple connection matrix for a Morse decomposition* $\{M_i\}_{i=1}^n$ of X is a simple connection matrix for any index filtration for this Morse decomposition.

We can now formulate two results concerning simple connection matrices for Morse decompositions. The first one ensures their existence.

THEOREM 3.3. *For any Morse decomposition of a compact metric space X there exists a simple connection matrix.*

Proof. By Theorem 1.3, there is an index filtration such that $N^0 = \emptyset$ and $N^n = X$ for any Morse decomposition of X. Hence the construction of the simple connection matrix, as presented in this section, poses no problem. ∎

It is easily seen that the simple connection matrix expresses the relationship between local Conley indices of the Morse sets and the total Conley index of the whole isolated invariant set. A similar relationship is given by so-called Morse inequalities (see for instance [11]).

The next result states that some information on the Morse decomposition may be found using its simple connection matrix. Let C be a simple connection matrix for the Morse decomposition and let $\{C_i\}_{i=1}^n$ be a splitting of C. As previously, let $d_{p,q} : C_q \to C_p$ denote a component of the differential d.

THEOREM 3.4. *If $d_{p-1,p} \neq 0$, then $C(M_p, M_{p-1}; X) \neq \emptyset$.*

Proof. From the definition of the simple connection matrix for the Morse decomposition $\{M_i\}_{i=1}^n$, we obtain the following commutative diagram:

$$\begin{array}{ccc}
C_p & \xrightarrow{d_{p-1,p}} & C_{p-1} \\
\downarrow & & \downarrow \\
H_*(N^p, N^{p-1}) & \xrightarrow{\partial} & H_*(N^{p-1}, N^{p-2})
\end{array}$$

in which the vertical maps are the canonical isomorphisms. Observe that $N^{p-2} \subset N^{p-1} \subset N^p$ is an index triple for the attractor-repeller pair (M_{p-1}, M_p) in $M_{p,p-1}$ and $\partial \neq 0$, since $d_{p-1,p} \neq 0$. By Theorem 1.4 the set $C(M_p, M_{p-1}; X)$ is nonempty. ∎

4. An example. It is not surprising that the simple connection matrix may be represented geometrically in the plane using the so-called *Zeeman diagram* Δ (see Figure 4). This diagram was defined by E. C. Zeeman to study the information contained in filtered differential groups (see [12] for more details).

The Zeeman diagram Δ is the union of a collection of unit squares in the plane. The number of these squares depends only on the length of the filtration. The union of any subcollection of squares in Δ is called a *region* of Δ. For example, the region to the left of the vertical line labeled A^i represents the vector subspace A^i in the filtration. Similarly, the regions below the horizontal lines represent the vector spaces dA^i or $d^{-1}A^i$. Since the differential d is filtration preserving, some squares in the diagram represent trivial spaces.

In the original Zeeman diagram the regions represent the quotients of some groups associated with the filtered differential group, but in our diagram the regions represent the components in the direct sum decompositions of some vector spaces.

Since the diagram offers a graphic and intuitive approach to simple connection matrices, we will use it in the description of the simple connection matrix in our example. We emphasize that our goal is to present the form of the simple connection matrix for a well known dynamical system, and not to give any relevant applications of the theory.

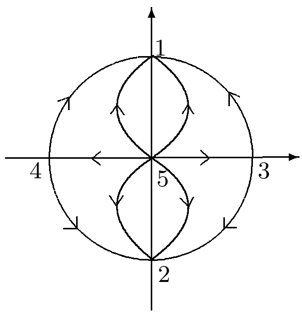

Fig. 2. The dynamics of φ

Let φ be a flow on the closed unit ball D^2 in $\mathbb{R}^2$ with the dynamics as in Figure 2. Assume that $X = D^2$ is an isolated invariant set and that

$$M_1 = (0,1),\ M_2 = (0,-1),\ M_3 = (1,0),\ M_4 = (-1,0),\ M_5 = (0,0)$$

form a Morse decomposition $\mathcal{M}$ of X. The index filtration $\{N^i\}_{i=0}^5$ for $\mathcal{M}$

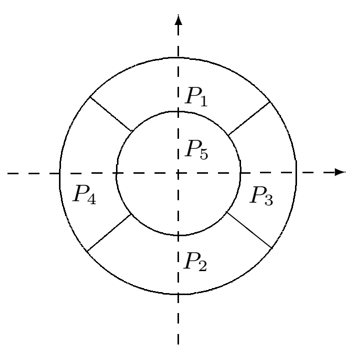

Fig. 3. The index filtration for $\mathcal{M}$

may be easily constructed from the sets P_i shown in Figure 3 using the formula $N^i = \bigcup_{k=0}^i P_k$. It is understood that $P_0 = \emptyset$. Moreover, a simple computation shows that the local Conley indices of the Morse sets are

$$CH_k(M_1) = CH_k(M_2) = \begin{cases} \mathbb{Q} & \text{if } k = 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$CH_k(M_3) = CH_k(M_4) = \begin{cases} \mathbb{Q} & \text{if } k = 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$CH_k(M_5) = \begin{cases} \mathbb{Q} & \text{if } k = 2, \\ 0 & \text{otherwise}, \end{cases}$$

and the total Conley index of the whole set is

$$CH_k(X) = \begin{cases} \mathbb{Q} & \text{if } k = 0, \\ 0 & \text{otherwise}. \end{cases}$$

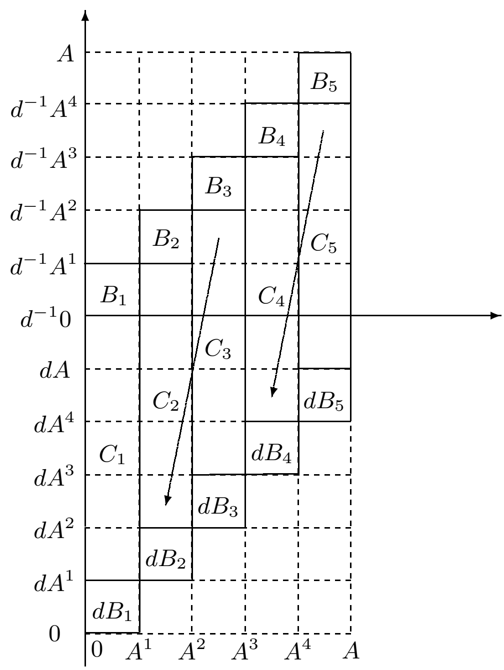

Fig. 4. The simple connection matrix for $\mathcal{M}$

Finally, an easy comparison of the above Conley indices shows that the Zeeman diagram Δ of the simple connection matrix for $\mathcal{M}$ is as in Figure 4. It is worth pointing out that our picture shows in fact not only the simple connection matrix $C = \bigoplus_{i=1}^{5} C_i$, but much more, namely, the full decomposition of the filtered differential vector space in the notation of 2.4. Observe that the arrows represent the only nonzero components of the differential d, i.e. the maps $d_{4,5}$ and $d_{2,3}$. The important point to note here is the form of the relation between the local indices represented as the columns $C_i = CH_*(M_i)$ and the total index $CH_*(X)$ represented as the row just below the x-axis.

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