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Numerical approximations of parabolic differential functional equations with the initial boundary conditions of the Neumann type

by Roman Ciarski (Gdańsk)

Abstract. The aim of this paper is to present a numerical approximation for quasilinear parabolic differential functional equations with initial boundary conditions of the Neumann type. The convergence result is proved for a difference scheme with the property that the difference operators approximating mixed derivatives depend on the local properties of the coefficients of the differential equation. A numerical example is given.

1. Introduction. We will denote by $C(U,V)$ the class of all continuous functions $w\colon U \to V$ with U and V being any metric spaces. Let $M_{n\times n}$ be the set of $n\times n$ matrices with real elements. For $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ and $X\in M_{n\times n}$, $X=[X_{kj}]_{j,k=1}^n$, we put

$$\|x\| = |x_1|+\dots+|x_n|,\quad \|X\| = \max\Big\{\sum_{j=1}^n |x_{kj}|\colon 1\le k\le n\Big\}.$$

Writing a vectorial inequality we mean that the same inequality holds for the corresponding components. Let $a>0$, $\mathbb{R}_+=[0,+\infty)$, and $b=(b_1,\dots,b_n)\in\mathbb{R}^n$ be given, where $b_k>0$ for $1\le k\le n$. Define

$$E=[0,a]\times(-b,b),\qquad E_0=\{0\}\times[-b,b],$$
$$\partial_0 E=([0,a]\times[-b,b])\setminus E,\qquad E^*=E_0\cup E\cup\partial_0 E.$$

Assume that

$$\varrho=[\varrho_{kj}]_{j,k=1}^n\colon E\times C(E^*,\mathbb{R})\to M_{n\times n},\quad f\colon E\times C(E^*,\mathbb{R})\times\mathbb{R}^n\to\mathbb{R}$$

are given functions of the variables (t,x,w) and (t,x,w,p) respectively.

We consider a quasilinear differential functional equation with Neumann initial boundary conditions

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$$(1)\quad \begin{cases} \partial_t z(t,x) = \sum_{k,j=1}^n \varrho_{kj}(t,x,z)\partial_{x_k x_j} z(t,x) + f(t,x,z,\partial_x z(t,x)), \\ z(t,x) = \varphi_0(t,x) \qquad \text{for } (t,x) \in E_0, \\ \partial_{x_j} z(t,x) = \varphi_j(t,x) \quad \text{for } (t,x) \in \partial_0 E \text{ and } x_j = b_j \text{ or } x_j = -b_j, \end{cases}$$

where $\varphi_0\colon E_0 \to \mathbb{R}$ and $\varphi_j\colon \partial_0 E \to \mathbb{R}, 1 \le j \le n$, are given and $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$.

The difference methods for nonlinear parabolic equations with Neumann boundary conditions were initiated in the papers by Malec [4]–[6] and Węglowski [8]. In [1], some general difference operators were introduced and their stability was investigated. The results of [4]–[6] and [8] do not apply to quasilinear equations. The difference scheme applied in this paper has the property that the difference operators approximating the mixed derivatives depend on the local properties of the function ϱ. We give sufficient conditions for convergence of the difference method for problem (1). The convergence is proved by consistency and stability arguments. We are interested in the numerical approximation of a classical solution of the above problem.

The norm of any $z \in C(E^*, \mathbb{R})$ is defined by

$$\|z\|_{E^*} = \max\{|z(t,x)|\colon (t,x) \in E^*\}.$$

We will need the norm

$$\|z\|_t = \max\{|z(\theta,x)|\colon 0 \le \theta \le t \text{ and } (\theta,x) \in E^*\}.$$

For $t \in [0,a]$ we write $H_t = [0,t] \times [-b,b]$. We assume that problem (1) is of Volterra type, that is, if $t \in [0,a]$ and $z, \overline{z} \in C(E^*, \mathbb{R})$ and $z(\theta) = \overline{z}(\theta)$ for $\theta \in H_t$ then $f(t,x,z,q) = f(t,x,\overline{z},q)$ and $\varrho_{kj}(t,x,z) = \varrho_{kj}(t,x,\overline{z})$ for $x \in [-b,b]$, $q \in \mathbb{R}^n$ and $j,k = 1,\ldots,n$.

2. Difference functional equations. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers respectively. For $x, \overline{x} \in \mathbb{R}^n$, $x = (x_1,\ldots,x_n)$, $\overline{x} = (\overline{x}_1,\ldots,\overline{x}_n)$, we put $x * \overline{x} = (x_1\overline{x}_1,\ldots,x_n\overline{x}_n)$. We define a mesh on the set E^* in the following way. Suppose that $h = (h_0,h') \in \mathbb{R}_+^{1+n}$, where $h' = (h_1,\ldots,h_n)$ are the steps of the mesh. Denote by $\varDelta$ the set of all $h = (h_0,h')$ such that there exist $N_b = (N_{b1},\ldots,N_{bn}) \in \mathbb{Z}^n$ with $N_b * h' = b$. For $h \in \varDelta$ we write $\|h\| = h_0 + h_1 + \cdots + h_n$ and $\|h'\| = h_1 + \cdots + h_n$. It is required that $\varDelta \ne \emptyset$ and that there exists a sequence $\{h^{(j)}\}$ in $\varDelta$ such that $\lim_{j\to\infty} \|h^{(j)}\| = 0$.

Nodal points are defined by:

$$t^{(i)} = ih_0, \qquad x^{(m)} = m * h = (m_1h_1,\ldots,m_nh_n) = (x_1^{(m_1)},\ldots,x_n^{(m_n)}),$$

where $(i,m) \in \mathbb{Z}^{1+n}$. Obviously there exists $N_a \in \mathbb{N}$ such that $N_a h_0 \le a$

$< (N_a + 1)h_0$. Let

$$R_h^{1+n} = \{(t^{(i)}, x^{(m)}) \colon (i, m) \in \mathbb{Z}^{1+n}\}$$

and

$$E_h = \bar{E} \cap R_h^{1+n}, \qquad \partial_0 E_h = \partial_0 E \cap R_h^{1+n},$$
$$E_{0 \cdot h} = E_0 \cap R_h^{1+n}, \qquad E_h^* = E_h \cup E_{0 \cdot h} \cup \partial_0 E_h.$$

For $z \colon E_h^* \to \mathbb{R}$ we write

$$z^{(i,m)} = z(t^{(i)}, x^{(m)}).$$

The norm of any $z \colon E_h^* \to \mathbb{R}$ is defined by

$$\|z\|_h = \max\{|z^{(i,m)}| \colon (t^{(i)}, x^{(m)}) \in E_h^*\}.$$

For any $t^{(i)}$ we will need the norm

$$\|z\|_{h \cdot i} = \max\{|z^{(r,m)}| \colon 0 \le r \le i \text{ and } (t^{(r)}, x^{(m)}) \in E_h^*\}.$$

Let

$$E'_h = \{(t^{(i)}, x^{(m)}) \in E_h \colon 0 \le i \le N_a - 1\}$$

and denote by $\mathfrak{F}(E_h^*, \mathbb{R})$ the set of all functions $w \colon E_h^* \to \mathbb{R}$. Suppose that

$$\varrho_h = [\varrho_{h \cdot kj}]_{j,k=1}^n \colon E'_h \times \mathfrak{F}(E_h^*, \mathbb{R}) \to M_{n \times n}, \qquad f_h \colon E'_h \times \mathfrak{F}(E_h^*, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R},$$
$$\varphi_{0 \cdot h} \colon E_{0 \cdot h} \to \mathbb{R}, \qquad \varphi_{j \cdot h} \colon \partial_0 E_h \to \mathbb{R}, \qquad j = 1, \ldots, n,$$

are given functions. We will approximate solutions of problem (1) by means of solutions of a difference equation with initial boundary condition of Neumann type. To do that, for every $(t^{(i)}, x^{(m)}) \in \partial_0 E_h$ we define

$$\begin{aligned} \mathcal{A}^{(m)} = \{\alpha = (\alpha_1, \ldots, \alpha_n) \colon & \alpha_j \in \{0, 1\} \text{ if } x_j^{(m_j)} = b_j, \\ & \alpha_j \in \{0, -1\} \text{ if } x_j^{(m_j)} = -b_j, \\ & \alpha_j = 0 \text{ if } -b_j < x_j^{(m_j)} < b_j, \\ & \text{and } \|\alpha\| = 1 \text{ or } \|\alpha\| = 2,\ 1 \le j \le n\}, \end{aligned}$$

where $\|\alpha\| = |\alpha_1| + \cdots + |\alpha_n|$, and

$$\begin{aligned} \partial E_h^{+1} &= \{(t^{(i)}, x^{(m+\alpha)}) \colon 0 \le i \le N_a, (t^{(i)}, x^{(m)}) \in \partial_0 E_h \text{ and } \alpha \in \mathcal{A}^{(m)}\}, \\ E_h^{+1} &= \partial E_h^{+1} \cup E_h. \end{aligned}$$

Now we consider the difference problem

$$\delta_0 z^{(i,m)} = \sum_{k,j=1}^n \varrho_{h \cdot kj}(t^{(i)}, x^{(m)}, z) \delta_{kj}^{(2)} z^{(i,m)} + f_h(t^{(i)}, x^{(m)}, z, \delta z^{(i,m)}) \tag{2}$$

with Neumann boundary conditions

$$z^{(i,m)} = \varphi_{0\cdot h}^{(i,m)} \quad \text{on } E_{0\cdot h}, \tag{3}$$

$$z_h^{(i,m+\alpha)} - z_h^{(i,m-\alpha)} = 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j\cdot h}^{(i,m)} \quad \text{on } \partial_0 E_h \text{ for } \alpha \in \mathcal{A}^{(m)}. \tag{4}$$

Let us notice that $(t^{(i)}, x^{(m+\alpha)}) \in \partial E_h^{+1}$ and $(t^{(i)}, x^{(m-\alpha)}) \in E_h$.

Let $e_j \in \mathbb{R}^n$ be the standard unit vectors. The difference operators δ_0, $\delta = (\delta_1, \ldots, \delta_n)$, δ_α and $\delta^{(2)} = [\delta_{kj}]_{j,k=1}^n$ are defined in the following way:

$$\delta_0 z^{(i,m)} = \frac{1}{h_0}\,[z^{(i+1,m)} - z^{(i,m)}], \tag{5}$$

$$\delta_j z^{(i,m)} = \frac{1}{2h_j}\,[z^{(i,m+e_j)} - z^{(i,m-e_j)}], \quad 1 \le j \le n, \tag{6}$$

$$\delta_{kk}^{(2)} z^{(i,m)} = \delta_k^+ \delta_k^- z^{(i,m)}, \tag{7}$$

$$\delta_{kj}^{(2)} z^{(i,m)} = \frac{1}{2}\,[\delta_k^+ \delta_j^+ z^{(i,m)} + \delta_k^- \delta_j^- z^{(i,m)}] \quad \text{if } \varrho_{h\cdot kj}(t^{(i)}, x^{(m)}, z) \ge 0, \tag{8}$$

$$\delta_{kj}^{(2)} z^{(i,m)} = \frac{1}{2}\,[\delta_k^+ \delta_j^- z^{(i,m)} + \delta_k^- \delta_j^+ z^{(i,m)}] \quad \text{if } \varrho_{h\cdot kj}(t^{(i)}, x^{(m)}, z) < 0, \tag{9}$$

where

$$\delta_k^+ z^{(i,m)} = \frac{1}{h_k}\,[z^{(i,m+e_k)} - z^{(i,m)}], \qquad \delta_k^- z^{(i,m)} = \frac{1}{h_k}\,[z^{(i,m)} - z^{(i,m-e_k)}].$$

There exists exactly one solution u_h: $E^* \to \mathbb{R}$ of problem (2)–(4). Let the operator F_h be defined by

$$F_h[z]^{(i,m)} = \sum_{k,j=1}^{n} \varrho_{h\cdot kj}(t^{(i)}, x^{(m)}, z)\delta_{kj} z^{(i,m)} + f_h(t^{(i)}, x^{(m)}, z, \delta z^{(i,m)}). \tag{10}$$

Our purpose is to examine the relation between the solution u_h of (2)–(4) and a function v_h: $E_h^{+1} \to \mathbb{R}$ satisfying the condition

$$|\delta_0 v_h^{(i,m)} - F_h[v_h]^{(i,m)}| \le \gamma(h) \qquad \text{on } E_h', \tag{11}$$

$$|v_h^{(i,m)} - \varphi_{0\cdot h}^{(i,m)}| \le \gamma_0(h) \qquad \text{on } E_{0\cdot h}, \tag{12}$$

$$\left| v_h^{(i,m+\alpha)} - v_h^{(i,m-\alpha)} - 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j\cdot h}^{(i,m)} \right| \le C_\varphi \|h'\|^3 \quad \text{on } \partial_0 E_h, \tag{13}$$

where

$$\gamma, \gamma_0 \colon \Delta \to \mathbb{R}_+, \quad \lim_{h\to 0} \gamma_0(h) = 0, \quad \lim_{h\to 0} \gamma(h) = 0, \quad C_\varphi \in \mathbb{R}_+, \quad \alpha \in \mathcal{A}^{(m)}.$$

The function v_h satisfying the above relations is considered to be an approximate solution of problem (2)–(4).

ASSUMPTION $\mathrm{H}[\varrho_h, f_h]$. The functions $\varrho_h\colon E'_h \times \mathfrak{F}(E^*_h, \mathbb{R}) \to M_{n\times n}$ and $f_h\colon E'_h \times \mathfrak{F}(E^*_h, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ satisfy the following conditions:

(i) there exists $L \in \mathbb{R}_+$ such that

$$\|\varrho_h(t^{(i)}, x^{(m)}, w) - \varrho_h(t^{(i)}, x^{(m)}, \overline{w})\| \le L\|w - \overline{w}\|_{h\cdot i},$$
$$|f_h(t^{(i)}, x^{(m)}, w, p) - f_h(t^{(i)}, x^{(m)}, \overline{w}, p)| \le L\|w - \overline{w}\|_{h\cdot i},$$

(ii) the derivative $\partial_p f_h = (\partial_{p_1} f_h, \ldots, \partial_{p_n} f_h)$ exists on $E'_h \times \mathfrak{F}(E^*_h, \mathbb{R}) \times \mathbb{R}^n$ and $\partial_p f_h(t, x, w, \cdot) \in C(\mathbb{R}^n, \mathbb{R}^n)$.

THEOREM 1. *Suppose that Assumption* $\mathrm{H}[\varrho_h, f_h]$ *holds and*

(i) $h \in \Delta$ *and*

$$1 - 2h_0 \sum_{k=1}^{n} \frac{1}{h_k^2}\, \varrho_{h\cdot kk}(Q) + h_0 \sum_{\substack{k,j=1\\ j\ne k}}^{n} \frac{1}{h_k h_j}\, |\varrho_{h\cdot kj}(Q)| \ge 0, \tag{14}$$

$$\frac{1}{h_k}\varrho_{h\cdot kk}(Q) - \sum_{\substack{j=1\\ j\ne k}}^{n} \frac{1}{h_j}\, |\varrho_{h\cdot kj}(Q)| - \frac{1}{2}\, |\partial_{p_k} f_h(P)| \ge 0, \qquad 1 \le k \le n, \tag{15}$$

where $Q = (t, x, w) \in E'_h \times \mathfrak{F}(E^*_h, \mathbb{R})$ *and* $P = (t, x, w, p) \in E'_h \times \mathfrak{F}(E^*_h, \mathbb{R}) \times \mathbb{R}^n$,

(ii) $u_h\colon E_h^{+1} \to \mathbb{R}$ *is the solution of problem* (2)–(4),

(iii) $v_h\colon E_h^{+1} \to \mathbb{R}$ *satisfies relations* (11)–(13),

(iv) *there exists* $c_0 \in \mathbb{R}_+$ *such that*

$$|\delta^{(2)}_{kj} v_h^{(i,m)}| \le c_0 \qquad \text{on } E_h \text{ for } 1 \le k, j \le n,$$

(v) *there exists* $\widetilde{C} \in \mathbb{R}_+$ *such that* $\|h'\|^2 \le \widetilde{C} h_0$.

Under these assumptions we have

$$|u_h^{(i,m)} - v_h^{(i,m)}| \le \gamma_0(h) e^{\widetilde{L}a} + \beta(h)\, \frac{e^{\widetilde{L}a} - 1}{\widetilde{L}} \qquad \text{on } E_h \tag{16}$$

if $L > 0$, *and*

$$|u_h^{(i,m)} - v_h^{(i,m)}| \le \gamma_0(h) + a\gamma(h) \qquad \text{on } E_h \tag{17}$$

if $L = 0$, *where* $\widetilde{L} = L(1 + nc_0)$ *and*

$$\beta\colon \Delta \to \mathbb{R}_+, \qquad \lim_{h\to 0} \beta(h) = 0.$$

Proof. Let $\Gamma\colon E'_h \to \mathbb{R}$, $\Gamma_{0\cdot h}\colon E_{0\cdot h} \to \mathbb{R}$ and $\Gamma_{\partial\cdot h}\colon \partial_0 E_h \to \mathbb{R}$ be defined by

$$\begin{aligned}
\delta_0 v_h^{(i,m)} &= F_h[v_h]^{(i,m)} + \Gamma_h^{(i,m)} && \text{on } E'_h,\\
v_h^{(i,m)} &= \varphi_{0\cdot h}^{(i,m)} + \Gamma_{0\cdot h}^{(i,m)} && \text{on } E_{0\cdot h},\\
v_h^{(i,m+\alpha)} - v_h^{(i,m-\alpha)} &= 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j\cdot h}^{(i,m)} + \Gamma_{\partial\cdot h}^{(i,m)} && \text{on } \partial_0 E_h \text{ for } \alpha \in \mathcal{A}^{(m)}.
\end{aligned}$$

Then

$$|\Gamma_h^{(i,m)}| \le \gamma(h) \qquad \text{on } E'_h \text{ with } \lim_{h\to 0}\gamma(h)=0,$$

$$|\Gamma_{0\cdot h}^{(i,m)}| \le \gamma_0(h) \qquad \text{on } E_{0\cdot h} \text{ with } \lim_{h\to 0}\gamma_0(h)=0,$$

$$|\Gamma_{\partial\cdot h}^{(i,m)}| \le C_\varphi \|h'\|^3 \quad \text{on } \partial_0 E_h.$$

The function $\varepsilon_h = u_h - v_h$ satisfies the difference functional equation

$$\begin{aligned}(18)\qquad \delta_0\varepsilon_h^{(i,m)} = {} & \sum_{k,j=1}^{n} \varrho_{h\cdot kj}(t^{(i)}, x^{(m)}, u_h)\delta_{kj}^{(2)}\varepsilon_h^{(i,m)} \\ & + f_h(t^{(i)}, x^{(m)}, v_h, \delta u_h{}^{(i,m)}) - f_h(t^{(i)}, x^{(m)}, v_h, \delta v_h{}^{(i,m)}) + \Lambda_h^{(i,m)},\end{aligned}$$

where

$$\begin{aligned}(19)\qquad \Lambda_h^{(i,m)} = {} & \sum_{k,j=1}^{n} [\varrho_{h\cdot kj}(t^{(i)}, x^{(m)}, u_h) - \varrho_{h\cdot kj}(t^{(i)}, x^{(m)}, v_h)]\delta_{kj}^{(2)} v_h^{(i,m)} \\ & + f_h(t^{(i)}, x^{(m)}, u_h, \delta u_h{}^{(i,m)}) - f_h(t^{(i)}, x^{(m)}, v_h, \delta u_h{}^{(i,m)}) - \Gamma_h^{(i,m)}\end{aligned}$$

on E'_h and

$$(20)\qquad \varepsilon_h^{(i,m+\alpha)} = \varepsilon_h^{(i,m-\alpha)} + \Gamma_{\partial\cdot h}^{(i,m)}$$

on $\partial_0 E'_h$. Let us deal with ε_h on E'_h first. Write

$$I_+^{(i,m)} = \{(k,j)\colon 1 \le k,j \le n,\ k \ne j,\ \varrho_{h\cdot kj}(t^{(i)}, x^{(m)}, u_h) \ge 0\},$$

$$I_-^{(i,m)} = \{(k,j)\colon 1 \le k,j \le n,\ k \ne j,\ \varrho_{h\cdot kj}(t^{(i)}, x^{(m)}, u_h) < 0\}.$$

From (5) and the mean value theorem, we can rewrite (18) as

$$\begin{aligned}\varepsilon_h^{(i+1,m)} = \varepsilon_h^{(i,m)} & + h_0 \sum_{k=1}^{n} \varrho_{h\cdot kk}(t^{(i)}, x^{(m)}, u_h)\delta_{kk}^{(2)}\varepsilon_h^{(i,m)} \\ & + h_0 \sum_{(k,j)\in I_+^{(i,m)}} \varrho_{h\cdot kj}(t^{(i)}, x^{(m)}, u_h)\delta_{kj}^{(2)}\varepsilon_h^{(i,m)} \\ & + h_0 \sum_{(k,j)\in I_-^{(i,m)}} \varrho_{h\cdot kj}(t^{(i)}, x^{(m)}, u_h)\delta_{kj}^{(2)}\varepsilon_h^{(i,m)} \\ & + h_0 \sum_{k=1}^{n} \partial_{p_k} f_h(P)\, \frac{\varepsilon_h^{(i,m+e_k)} - \varepsilon_h^{(i,m-e_k)}}{2h_k} + h_0\Lambda_h^{(i,m)}.\end{aligned}$$

By (7)–(9) and regrouping terms, the function ε_h satisfies on E'_h the recursive equation

$$\begin{aligned}\varepsilon_h^{(i+1,m)} &= \varepsilon_h^{(i,m)}\boldsymbol{A}^{(i,m)} + h_0\sum_{k=1}^{n}\varepsilon_h^{(i,m+e_k)}\boldsymbol{B}_k^{(i,m)} + h_0\sum_{k=1}^{n}\varepsilon_h^{(i,m-e_k)}\boldsymbol{C}_k^{(i,m)} \\ &\quad + h_0\sum_{(k,j)\in I_+^{(i,m)}}[\varepsilon_h^{(i,m+e_k+e_j)}+\varepsilon_h^{(i,m-e_k-e_j)}]\boldsymbol{D}_{k,j}^{(i,m)} \\ &\quad + h_0\sum_{(k,j)\in I_-^{(i,m)}}[\varepsilon_h^{(i,m+e_k-e_j)}+\varepsilon_h^{(i,m-e_k+e_j)}]\boldsymbol{D}_{k,j}^{(i,m)} + h_0\Lambda_h^{(i,m)},\end{aligned} \tag{21}$$

where

$$\begin{aligned}\boldsymbol{A}^{(i,m)} &= 1 - 2h_0\sum_{k=1}^{n}\frac{1}{h_k^2}\,\varrho_{h\cdot kk}(t^{(i)},x^{(m)},u_h) \\ &\qquad + h_0\sum_{\substack{k,j=1\\ j\neq k}}^{n}\frac{1}{h_kh_j}\,|\varrho_{h\cdot kj}(t^{(i)},x^{(m)},u_h)|, \\ \boldsymbol{B}_k^{(i,m)} &= \frac{1}{h_k^2}\,\varrho_{h\cdot kk}(t^{(i)},x^{(m)},u_h) \\ &\qquad - \sum_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{h_kh_j}\,|\varrho_{h\cdot kj}(t^{(i)},x^{(m)},u_h)| + \frac{1}{2h_k}\,\partial_{p_k}f_h(P), \\ \boldsymbol{C}_k^{(i,m)} &= \frac{1}{h_k^2}\,\varrho_{h\cdot kk}(t^{(i)},x^{(m)},u_h) \\ &\qquad - \sum_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{h_kh_j}\,|\varrho_{h\cdot kj}(t^{(i)},x^{(m)},u_h)| - \frac{1}{2h_k}\,\partial_{p_k}f_h(P), \\ \boldsymbol{D}_{k,j}^{(i,m)} &= \frac{1}{2h_kh_j}\,|\varrho_{h\cdot kj}(t^{(i)},x^{(m)},u_h)|.\end{aligned}$$

Let ω_h and $\widetilde{\omega}_h$ be given by

$$\omega_h^{(i)} = \max\{|\varepsilon_h^{(r,m)}|\colon (t^{(r)},x^{(m)})\in E_h^*\cap([-\tau_0,t^{(i)}]\times\mathbb{R}^n)\}, \quad 0\le i\le N_a, \tag{22}$$

$$\widetilde{\omega}_h^{(i)} = \max\{|\varepsilon_h^{(r,m)}|\colon (t^{(r)},x^{(m)})\in E_h^{+1}\cap([-\tau_0,t^{(i)}]\times\mathbb{R}^n)\}, \quad 0\le i\le N_a. \tag{23}$$

With this notation Λ_h (see (19)) can be estimated as follows:

$$|\Lambda_h^{(i,m)}| \le \widetilde{\omega}_h^{(i)}L(1+nc_0)+\gamma(h) \quad \text{on } E_h'. \tag{24}$$

We conclude from (12), (20), (21) and (24) that the functions ω_h and $\widetilde{\omega}_h$ satisfy the recursive inequalities

$$\omega_h^{(i+1)} \le \widetilde{\omega}_h^{(i)}(1+\widetilde{L}h_0) + h_0\gamma(h), \tag{25}$$

$$\widetilde{\omega}_h^{(i)} \le \omega_h^{(i)} + \sqrt{h_0}h_0C_\varphi\widetilde{C}, \tag{26}$$

for $0 \le i \le N_a - 1$ and

$$\omega_h^{(0)} \le \gamma_0(h). \tag{27}$$

Consider the difference equations

$$\eta_h^{(i+1)} = \widetilde{\eta}_h^{(i)}(1+\widetilde{L}h_0) + h_0\gamma(h),$$
$$\widetilde{\eta}_h^{(i)} = \eta_h^{(i)} + \sqrt{h_0}h_0C_\varphi\widetilde{C},$$

for $0 \le i \le N_a - 1$ with the initial condition

$$\eta^{(0)} = \gamma_0(h),$$

and its solutions

$$\eta_h^{(0)} = \gamma_0(h),$$
$$\widetilde{\eta}_h^{(0)} = \gamma_0(h) + \sqrt{h_0}\,h_0C_\varphi\widetilde{C},$$
$$\eta_h^{(i)} = \gamma_0(h)(1+\widetilde{L}h_0)^i + h_0[(1+\widetilde{L}h_0)\sqrt{h_0}\,C_\varphi\widetilde{C} + \gamma(h)]\sum_{j=0}^{i-1}(1+\widetilde{L}h_0)^j,$$

for $1 \le i \le N_a$. Thus

$$\eta_h^{(i)} \le \gamma_0(h)e^{\widetilde{L}a} + [(1+\widetilde{L}h_0)\sqrt{h_0}\,C_\varphi\widetilde{C} + \gamma(h)]\,\frac{e^{\widetilde{L}a}-1}{\widetilde{L}}.$$

It follows from (25)–(27) that

$$\omega_h^{(i)} \le \eta_h^{(i)} \quad \text{for } 0 \le i \le N_a.$$

This gives (16), (17) and Theorem 1 is proved. ∎

3. Difference method for the mixed problem. We will need an interpolating operator T_h: $\mathfrak{F}(E_h^*, \mathbb{R}) \to C(E, \mathbb{R})$. Let

$$S_+ = \{\xi = (\xi_1, \ldots, \xi_n) \colon \xi_j = \{0, 1\} \text{ for } 0 \le j \le n\}$$

Let $z \in \mathfrak{F}(E_h^*, \mathbb{R})$. For every $(t, x) \in E$ there is $(t^{(i)}, x^{(m)}) \in E_h$ such that $(t^{(i+1)}, x^{(m+1)}) \in E'_h$, where $m+1 = (m_1+1, \ldots, m_n+1)$ and $t^{(i)} \le t \le t^{(i+1)}$, $x^{(m)} \le x \le x^{(m+1)}$. Set

$$(T_h z)(t,x) = \frac{t-t^{(i)}}{h_0} \sum_{\xi \in S_+} z^{(i+1,m+\xi)} \left(\frac{x-x^{(m)}}{h'}\right)^{\xi} \left(1 - \frac{x-x^{(m)}}{h'}\right)^{1-\xi}$$
$$+ \left(1 - \frac{t-t^{(i)}}{h_0}\right) \sum_{\xi \in S_+} z^{(i,m+\xi)} \left(\frac{x-x^{(m)}}{h'}\right)^{\xi} \left(1 - \frac{x-x^{(m)}}{h'}\right)^{1-\xi},$$

where

$$\left(\frac{x-x^{(m)}}{h'}\right)^{\xi} = \prod_{j=1}^{n} \left(\frac{x_j - x_j^{(m_j)}}{h_j}\right)^{\xi_j},$$
$$\left(1 - \frac{x-x^{(m)}}{h'}\right)^{1-\xi} = \prod_{j=1}^{n} \left(1 - \frac{x_j - x_j^{(m_j)}}{h_j}\right)^{1-\xi_j}.$$

In the above formulas we adopt the convention that $0^0 = 1$. For $h_0 N_a < t \le a$ we put

$$(T_h z)(t,x) = (T_h z)(h_0 N_a, x).$$

LEMMA 2. *Suppose that*

(i) $z(t,\cdot)\colon [-b,b] \to \mathbb{R}$ *is of class* C^2 *for* $t \in [0,a]$ *and* $z_h = z|_{E_h^*}$,

(ii) $\widetilde{d}_2 \in \mathbb{R}_+$ *is such that on* E^*,

(28) $$|\partial_{x_j x_k} z(t,x)| \le \widetilde{d}_2 \quad \textit{for } j,k = 1,\ldots,n,$$

(iii) *there exists* $\widetilde{c} \in \mathbb{R}_+$ *such that* $h_0 < \widetilde{c}\|h'\|^2$,

(iv) *there is* $L \in \mathbb{R}_+$ *such that*

(29) $$|z(t,x) - z(\overline{t},x)| \le L|t - \overline{t}|.$$

Then

(30) $$\|T_h z_h - z\|_{E^*} \le C_0 \|h'\|^2,$$

where $C_0 = \widetilde{d}_2 + 2L\widetilde{c}$ *and* $\|h'\| = h_1 + \cdots + h_n$.

Proof. Let $(t,x) \in E$ and (i,m) be such that $t^{(i)} \le t \le t^{(i+1)}$ and $x^{(m)} \le x \le x^{(m+1)}$. Then

(31) $$(T_h z)(t,x) - z(t,x)$$
$$= \sum_{\xi \in S_+} z^{(i,m+\xi)} \left(\frac{x-x^{(m)}}{h'}\right)^{\xi} \left(1 - \frac{x-x^{(m)}}{h'}\right)^{1-\xi} - z(t^{(i)},x)$$
$$+ \left(\frac{t-t^{(i)}}{h_0}\right) \sum_{\xi \in S_+} [z^{(i+1,m+\xi)} - z^{(i,m+\xi)}] \left(\frac{x-x^{(m)}}{h'}\right)^{\xi} \left(1 - \frac{x-x^{(m)}}{h'}\right)^{1-\xi}$$
$$+ z(t^{(i)},x) - z(t,x).$$

It is easy to prove by induction that

$$(32)\qquad \sum_{\xi\in S_+}\left(\frac{x-x^{(m)}}{h'}\right)^{\xi}\left(1-\frac{x-x^{(m)}}{h'}\right)^{1-\xi}=1$$

and

$$(33)\qquad \sum_{\xi\in S_+}\left(\frac{x-x^{(m)}}{h'}\right)^{\xi}\left(1-\frac{x-x^{(m)}}{h'}\right)^{1-\xi}\xi_k h_k = x_k - x_k^{(m_k)}.$$

Thus, it is easily seen that

$$(34)\qquad \left|\left(\frac{t-t^{(i)}}{h_0}\right)\sum_{\xi\in S_+}[z^{(i+1,m+\xi)}-z^{(i,m+\xi)}]\left(\frac{x-x^{(m)}}{h'}\right)^{\xi}\left(1-\frac{x-x^{(m)}}{h'}\right)^{1-\xi}\right| \le L\widetilde{c}\|h'\|^2$$

and

$$(35)\qquad |z(t^{(i)},x)-z(t,x)|\le L\widetilde{c}\|h'\|^2.$$

Finally, by the Taylor formula, (28), (32) and (33) we obtain

$$(36)\qquad \left|\sum_{\xi\in S_+} z^{(i,m+\xi)}\left(\frac{x-x^{(m)}}{h'}\right)^{\xi}\left(1-\frac{x-x^{(m)}}{h'}\right)^{1-\xi}-z(t^{(i)},x)\right|\le \widetilde{d_2}\|h'\|^2.$$

From (31) and (34)–(36) we get

$$|(T_h z)(t,x)-z(t,x)|\le \widetilde{d_2}\|h'\|^2+2L\widetilde{c}\|h'\|^2,$$

which implies assertion (30). ∎

Assumption H$[\varrho, f]$. Suppose that

(i) ϱ: $E'\times C(E,\mathbb{R})\to M_{n\times n}$ and f: $E\times C(E,\mathbb{R})\times\mathbb{R}^n\to\mathbb{R}$ are continuous,

(ii) there exists $L\in\mathbb{R}_+$ such that

$$\|\varrho(t,x,w)-\varrho(t,x,\overline{w})\|\le L\|w-\overline{w}\|_t,$$
$$|f(t,x,w,p)-f(t,x,\overline{w},p)|\le L\|w-\overline{w}\|_t,$$

(iii) $\partial_p f=(\partial_{p_1}f,\dots,\partial_{p_n}f)$ exists on $E\times C(E,\mathbb{R})\times\mathbb{R}^n$ and $\partial_p f(t,x,w,\cdot)\in C(\mathbb{R}^n,\mathbb{R}^n)$.

Now we will approximate the solution of the functional differential problem (1) by the solution of the difference problem

$$(37)\qquad \delta_0 z^{(i,m)} = \sum_{k,j=1}^{n}\varrho_{h\cdot kj}(t^{(i)},x^{(m)},T_h z)\delta_{kj}^{(2)}z^{(i,m)}+f_h(t^{(i)},x^{(m)},T_h z,\delta z^{(i,m)})$$

with the initial Neumann boundary conditions

$$z^{(i,m)} = \varphi_0^{(i,m)} \quad \text{on } E_{0\cdot h}, \tag{38}$$

$$z_h^{(i,m+\alpha)} - z_h^{(i,m-\alpha)} = 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j\cdot h}^{(i,m)} \tag{39}$$
$$\text{for } (t^{(i)}, x^{(m)}) \in \partial_0 E_h \text{ and } \alpha \in \mathcal{A}^{(m)}.$$

Let Ω be an open set such that $E^* \subset \Omega$.

THEOREM 3. *Suppose that assumption* $\mathrm{H}[\varrho, f]$ *holds and*

(i) $h \in \Delta$ *and*

$$1 - 2h_0 \sum_{k=1}^{n} \frac{1}{h_k^2}\, \varrho_{kk}(t,x,w) + h_0 \sum_{\substack{k,j=1\\ j\neq k}}^{n} \frac{1}{h_k h_j}\, |\varrho_{kj}(t,x,w)| \geq 0,$$
$$\frac{1}{h_k}\, \varrho_{kk}(t,x,w) - \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{h_j}\, |\varrho_{kj}(t,x,w)| - \frac{1}{2}\, |\partial_{p_k} f(t,x,w,p)| \geq 0,$$
$$1 \leq k \leq n,$$

(ii) *there exists* $c_\star \in \mathbb{R}_+$ *such that* $h_k \leq c_\star h_j$ *for* $1 \leq k, j \leq n$,

(iii) *there exist* $\widetilde{C}, \widetilde{c} \in \mathbb{R}_+$ *such that* $\widetilde{C}^{-1}\|h'\|^2 \leq h_0 \leq \widetilde{c}\|h'\|^2$,

(iv) u_h: $E_h^{+1} \to \mathbb{R}$ *is a solution of* (37)–(39),

(v) v: $\Omega \to \mathbb{R}$ *is a solution of* (1) *on* E^* *and* $v_h = v|_{E_h^*}$, $\varphi_{j\cdot h} = \varphi_j|_{\partial_0 E_h}$, $1 \leq j \leq n$,

(vi) *there exists* γ_0: $\Delta \to \mathbb{R}_+$ *such that*

$$|\varphi_0^{(i,m)} - \varphi_{0\cdot h}^{(i,m)}| \leq \gamma_0(h) \quad \text{on } E_{0\cdot h}, \qquad \lim_{h\to 0} \gamma_0(h) = 0, \tag{40}$$

(vii) $v(\cdot, x)$ *is of class* C^1 *and* $v(t,\cdot)$ *is of class* C^3.

Then there is η: $\Delta \to \mathbb{R}_+$ *such that on* E_h,

$$|u_h^{(i,m)} - v_h^{(i,m)}| \leq \eta(h), \qquad \lim_{h\to 0} \eta(h) = 0. \tag{41}$$

Proof. We apply Theorem 1. Write

$$\psi_h^{(i,m)} = \delta_0 v_h^{(i,m)} - \sum_{k,j=1}^{n} \varrho_{kj}(t^{(i)}, x^{(m)}, T_h v_h)\delta_{kj}^{(2)} v_h^{(i,m)}$$
$$- f(t^{(i)}, x^{(m)}, T_h v_h, \delta v_h{}^{(i,m)}), \tag{42}$$
$$\xi_h^{(i,m)} = v_h^{(i,m+\alpha)} - v_h^{(i,m-\alpha)} v - 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j\cdot h}^{(i,m)}.$$

We see at once that on E'_h,

$$\begin{aligned}\psi_h^{(i,m)} &= \delta_0 v_h^{(i,m)} - \partial_t v(t^{(i)}, x^{(m)}) \\ &\quad + \sum_{k,j=1}^{n} [\varrho_{kj}(t^{(i)}, x^{(m)}, v) - \varrho_{kj}(t^{(i)}, x^{(m)}, T_h v_h)] \delta_{kj}^{(2)} v_h^{(i,m)} \\ &\quad + \sum_{k,j=1}^{n} \varrho_{kj}(t^{(i)}, x^{(m)}, v) [\partial_{x_k x_j}^{(2)} v^{(i,m)} - \delta_{kj}^{(2)} v_h^{(i,m)}] \\ &\quad + f(t^{(i)}, x^{(m)}, v, \partial_x v(t^{(i)}, x^{(m)})) - f(t^{(i)}, x^{(m)}, T_h v_h, \partial_x v(t^{(i)}, x^{(m)})) \\ &\quad + f(t^{(i)}, x^{(m)}, T_h v_h, \partial_x v(t^{(i)}, x^{(m)})) - f(t^{(i)}, x^{(m)}, T_h v_h, \delta v_h^{(i,m)}).\end{aligned}$$

It is easily seen that there is $\gamma\colon \varDelta \to \mathbb{R}_+$ such that

$$|\psi_h^{(i,m)}| \le \gamma(h \quad \text{on } E'_h, \qquad \lim_{h\to 0} \gamma(h) = 0.$$

Now on ∂E_h we have

$$\xi_h^{(i,m)} = v_h^{(i,m+\alpha)} - v_h^{(i,m-\alpha)} - 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j\cdot h}^{(i,m)}.$$

By the Taylor formula we get

$$\begin{aligned} v^{(i,m+\alpha)} &- v^{(i,m-\alpha)} - 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j\cdot h}^{(i,m)} \\ &= v^{(i,m)} + \sum_{j=1}^{n} \alpha_j h_j \partial_{x_j} v^{(i,m)} + \frac{1}{2}\sum_{k,j=1}^{n} \alpha_j \alpha_k h_j h_k \partial_{x_j x_k}^{(2)} v^{(i,m)} \\ &\quad + \frac{1}{6}\sum_{j,k,l=1}^{n} \alpha_j \alpha_k \alpha_l h_j h_k h_l \partial_{x_j x_k x_l}^{(3)} v(P) \\ &\quad - \Big[v^{(i,m)} - \sum_{j=1}^{n} \alpha_j h_j \partial_{x_j} v^{(i,m)} + \frac{1}{2}\sum_{k,j=1}^{n} \alpha_j \alpha_k h_j h_k \partial_{x_j x_k}^{(2)} v^{(i,m)} \\ &\quad - \frac{1}{6}\sum_{j,k,l=1}^{n} \alpha_j \alpha_k \alpha_l h_j h_k h_l \partial_{x_j x_k x_l}^{(3)} v(Q) \Big] - 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j\cdot h}^{(i,m)} \\ &= \frac{1}{6}\sum_{j,k,l=1}^{n} \alpha_j \alpha_k \alpha_l h_j h_k h_l \big[\partial_{x_j x_k x_l}^{(3)} v(P) - \partial_{x_j x_k x_l}^{(3)} v(Q) \big] \end{aligned}$$

and finally

$$|\xi_h^{(i,m)}| \le \widetilde{d}_3 \Big| \sum_{j,k,l=1}^{n} \alpha_j \alpha_k \alpha_l h_j h_k h_l \Big| \le \widetilde{d}_3 \|h'\|^3,$$

where $\widetilde{d}_3 \in \mathbb{R}_+$. Thus, all the assumptions of Theorem 1 are satisfied and assertion (41) follows from (16), (17). ■

Now we give an error estimate for the method (37)–(39).

THEOREM 4. *Suppose that assumption* $\mathrm{H}[\varrho, f]$ *holds and*

(i) $h \in \Delta$ *and*

$$1 - 2h_0 \sum_{k=1}^{n} \frac{1}{h_k^2}\, \varrho_{kk}(t,x,w) + h_0 \sum_{\substack{k,j=1\\ j\neq k}}^{n} \frac{1}{h_k h_j}\, |\varrho_{kj}(t,x,w)| \geq 0,$$

$$\frac{1}{h_k}\, \varrho_{kk}(t,x,w) - \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{h_j}\, |\varrho_{kj}(t,x,w)| - \frac{1}{2}\, |\partial_{p_k} f(t,x,w,p)| \geq 0, \qquad 1 \leq k \leq n,$$

(ii) *there exists* $c_\star \in \mathbb{R}_+$ *such that* $h_k \leq c_\star h_j$ *for* $1 \leq k, j \leq n$,

(iii) *there exist* $\widetilde{C}, \widetilde{c} \in \mathbb{R}_+$ *such that* $\widetilde{C}^{-1}\|h'\|^2 \leq h_0 \leq \widetilde{c}\|h'\|^2$,

(iv) $u_h\colon E_h^{+1} \to \mathbb{R}$ *is a solution of* (37)–(39),

(v) $v\colon \Omega \to \mathbb{R}$ *is a solution of* (1) *on* E^* *and* $v_h = v|_{E_h^*}$, $\varphi_{j\cdot h} = \varphi_j|_{\partial_0 E_h}$, $1 \leq j \leq n$,

(vi) *there exists* $\gamma_0\colon \Delta \to \mathbb{R}_+$ *such that*

$$(43) \qquad |\varphi_0^{(i,m)} - \varphi_{0\cdot h}^{(i,m)}| \leq \gamma_0(h) \quad \text{on } E_{0\cdot h}, \qquad \lim_{h\to 0} \gamma_0(h) = 0,$$

(vii) $v|_\Omega$ *is of class* C^4,

(viii) *there exists* $\widetilde{d} \in \mathbb{R}_+$ *such that* $\|\partial_p f(t,x,w,p)\| \leq \widetilde{d}$,

(ix) *there exist* $\widetilde{d}_2, \widetilde{d}_4 \in \mathbb{R}_+$ *such that on* Ω,

$$|\partial_{tt} v(t,x)|, |\partial_{x_j x_k} v(t,x)| \leq \widetilde{d}_2, \qquad |\partial^{(4)}_{x_i x_j x_k x_l} v(t,x)| \leq \widetilde{d}_4, \qquad 1 \leq i,j,k,l \leq n.$$

Then there is $A \in \mathbb{R}_+$ *such that on* E_h,

$$(44) \qquad |u_h^{(i,m)} - v_h^{(i,m)}| \leq \gamma_0(h) e^{\widetilde{L}a} + A\|h'\|^2\, \frac{e^{\widetilde{L}a} - 1}{\widetilde{L}}$$

if $L > 0$, *and*

$$(45) \qquad |u_h^{(i,m)} - v_h^{(i,m)}| \leq \gamma_0(h) + aA\|h'\|^2$$

if $L = 0$, *where* $\widetilde{L} = L(1 + nd_2)$.

Proof. We apply Theorem 1. Write

$$(46) \qquad \psi_h^{(i,m)} = \delta_0 v_h^{(i,m)} - \sum_{k,j=1}^{n} \varrho_{kj}(t^{(i)}, x^{(m)}, T_h v_h) \delta_{kj}^{(2)} v_h^{(i,m)} - f(t^{(i)}, x^{(m)}, T_h v_h, \delta {v_h}^{(i,m)}).$$

As in the proof of Theorem 1 we can rewrite (46) as

$$\begin{aligned}\psi_h^{(i,m)} &= \delta_0 v_h^{(i,m)} - \partial_t v(t^{(i)}, x^{(m)}) \\ &\quad + \sum_{k,j=1}^{n} [\varrho_{kj}(t^{(i)}, x^{(m)}, v) - \varrho_{kj}(t^{(i)}, x^{(m)}, T_h v_h)]\delta_{kj}^{(2)} v_h^{(i,m)} \\ &\quad + \sum_{k,j=1}^{n} \varrho_{kj}(t^{(i)}, x^{(m)}, v)[\partial_{x_k x_j}^{(2)} v^{(i,m)} - \delta_{kj}^{(2)} v_h^{(i,m)}] \\ &\quad + f(t^{(i)}, x^{(m)}, v, \partial_x v(t^{(i)}, x^{(m)})) - f(t^{(i)}, x^{(m)}, T_h v_h, \partial_x v(t^{(i)}, x^{(m)})) \\ &\quad + f(t^{(i)}, x^{(m)}, T_h v_h, \partial_x v(t^{(i)}, x^{(m)})) - f(t^{(i)}, x^{(m)}, T_h v_h, \delta v_h^{(i,m)}).\end{aligned}$$

There are $\widetilde{r}, d_2 \in \mathbb{R}_+$ such that

$$\begin{aligned}|\delta_{kj}^{(2)} v^{(i,m)}| &\le d_2 \qquad \text{on } E_h,\ 1 \le k, j \le n, \\ |\varrho_{kj}(t, x, v)| &\le \widetilde{r} \qquad \text{on } E \times \mathbb{R}^n,\ 1 \le k, j \le n.\end{aligned}$$

From the Taylor formula we obtain

$$|\partial_t v(t^{(i)}, x^{(m)}) - \delta_0 v_h^{(i,m)}| \le \frac{h_0}{2}\,\widetilde{d}_2 \le \frac{1}{2}\,\widetilde{d}_2 \widetilde{c} \|h'\|^2$$

and

$$|\partial_{x_k x_j} v(t^{(i)}, x^{(m)}) - \delta_{kj} v_h^{(i,m)}| \le \widetilde{d}_4 \|h'\|^2 \left(\frac{7}{12} + \frac{1}{6}\,c_\star\right) \qquad \text{for } 1 \le k, j \le n.$$

The above estimates and Lemma 2 yield

$$\begin{aligned}|\psi_h^{(i,m)}| \le \frac{1}{2}\,\widetilde{d}_2 \widetilde{c} \|h'\|^2 + L\widetilde{d}_2 d_2 C_0 \|h'\|^2 + \widetilde{r}\widetilde{d}_4 n^2 \|h'\|^2 \left(\frac{7}{12} + \frac{1}{6}\,c_\star\right) \\ + L d_2 C_0 \|h'\|^2 + \frac{1}{6}\,\widetilde{d}\widetilde{d}_2 \|h'\|^2 \qquad \text{on } E_h'.\end{aligned}$$

Thus on E_h',

$$|\psi_h^{(i,m)}| \le \left[\frac{1}{2}\,\widetilde{d}_2 \widetilde{c} + L d_2 C_0 (1 + \widetilde{d}_2) + \widetilde{r}\widetilde{d}_4 n^2 \left(\frac{7}{12} + \frac{1}{6} c_\star\right) + \frac{1}{6}\,\widetilde{d}\widetilde{d}_2\right] \|h'\|^2.$$

On ∂E_h we have

$$\left| v_h^{(i,m+\alpha)} - v_h^{(i,m-\alpha)} - 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j\cdot h}^{(i,m)} \right| \le \widetilde{d}_3 \|h'\|^3.$$

Thus all the assumptions of Theorem 1 are satisfied and assertions (44), (45) follow from (16), (17). ■

4. Numerical example. For $n = 2$ we put

$$E = [0, 1] \times [-1, 1] \times [-1, 1],$$

Let z be the unknown function of the variables (t, x, y) and consider the differential integral equation

$$\begin{aligned}(47)\qquad \partial_t z(t,x,y) &= 2(x^2+1)\partial_{xx} z(t,x,y) + 2(y^2+1)\partial_{yy} z(t,x,y) \\ &\quad + \partial_{xy} z(t,x,y) \int_{(x-1)/2}^{(x+1)/2} \int_{(y-1)/2}^{(y+1)/2} z(t,\eta,\xi)\, d\xi\, d\eta \\ &\quad + 23e^t x^3y^3 - \frac{9}{64}\, e^{2t}x^2y^2(x^3+x)(y^3+y) - 12e^t xy(x^2+y^2)\end{aligned}$$

with the initial boundary condition

$$(48)\qquad z(t,x,y) = x^3y^3 \quad \text{for } (t,x,y) \in E_0$$

and the Neumann boundary conditions

$$(49)\qquad \partial_x z(t,x,y) = 3e^t x^2y^3 \quad \text{for } (t,x,y) \in \partial_0 E \text{ and } x = 1 \text{ or } x = -1,$$

$$(50)\qquad \partial_y z(t,x,y) = 3e^t x^3y^2 \quad \text{for } (t,x,y) \in \partial_0 E \text{ and } y = 1 \text{ or } y = -1,$$

where

$$E_0 = \{0\} \times [-1,1] \times [-1,1],$$
$$\partial_0 E = (0,1] \times [([-1,1]\times[-1,1]) \setminus ((-1,1)\times(-1,1))].$$

For the above problem we apply the difference method (37)–(39).

The function $v(t,x,y) = e^t x^3y^3$ is a solution of (47)–(50). Let $u_h\colon E_h^* \to \mathbb{R}$ be the solution of the corresponding difference equations and $\varepsilon_h = u_h - v$. The values $\varepsilon_h(0.6, x^{(j)}, y^{(k)})$, $\varepsilon_h(0.7, x^{(j)}, y^{(k)})$, $\varepsilon_h(0.8, x^{(j)}, y^{(k)})$, $\varepsilon_h(0.9, x^{(j)}, y^{(k)})$ are listed in Table 1 for $h_0 = 0.00001$, $h_1 = 0.02$ and $h_2 = 0.02$.

Table 1

		$t^{(i)} = 0.6$	$t^{(i)} = 0.7$	$t^{(i)} = 0.8$	$t^{(i)} = 0.9$
$x^{(j)}$,	$y^{(k)}$	ε_h	ε_h	ε_h	ε_h
-0.5	-0.5	$6.090 \cdot 10^{-4}$	$8.905 \cdot 10^{-4}$	$1.271 \cdot 10^{-3}$	$1.789 \cdot 10^{-3}$
-0.5	0.0	$5.040 \cdot 10^{-4}$	$7.290 \cdot 10^{-4}$	$1.033 \cdot 10^{-3}$	$1.445 \cdot 10^{-3}$
-0.5	0.5	$5.558 \cdot 10^{-4}$	$7.838 \cdot 10^{-4}$	$1.094 \cdot 10^{-3}$	$1.519 \cdot 10^{-3}$
0.0	-0.5	$8.419 \cdot 10^{-4}$	$1.149 \cdot 10^{-3}$	$1.553 \cdot 10^{-3}$	$2.091 \cdot 10^{-3}$
0.0	0.0	$5.684 \cdot 10^{-4}$	$8.048 \cdot 10^{-4}$	$1.120 \cdot 10^{-3}$	$1.544 \cdot 10^{-3}$
0.0	0.5	$4.275 \cdot 10^{-4}$	$6.337 \cdot 10^{-4}$	$9.139 \cdot 10^{-4}$	$1.297 \cdot 10^{-3}$
0.5	-0.5	$1.425 \cdot 10^{-3}$	$1.864 \cdot 10^{-3}$	$2.436 \cdot 10^{-3}$	$3.188 \cdot 10^{-3}$
0.5	0.0	$7.067 \cdot 10^{-4}$	$9.815 \cdot 10^{-4}$	$1.347 \cdot 10^{-3}$	$1.835 \cdot 10^{-3}$
0.5	0.5	$3.678 \cdot 10^{-4}$	$5.905 \cdot 10^{-4}$	$8.982 \cdot 10^{-4}$	$1.326 \cdot 10^{-3}$

Let $\varepsilon_{\max}$ be the largest and $\varepsilon_{\text{mean}}$ the mean value of all ε_h for a given $t^{(i)}$. The values are listed in Table 2.

Table 2

$t^{(i)}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$
0.0	$1.157 \cdot 10^{-3}$	$6.815 \cdot 10^{-5}$
0.1	$1.465 \cdot 10^{-3}$	$1.127 \cdot 10^{-4}$
0.2	$1.957 \cdot 10^{-3}$	$1.843 \cdot 10^{-4}$
0.3	$2.471 \cdot 10^{-3}$	$2.705 \cdot 10^{-4}$
0.4	$3.075 \cdot 10^{-3}$	$3.903 \cdot 10^{-4}$
0.5	$3.816 \cdot 10^{-3}$	$5.549 \cdot 10^{-4}$
0.6	$4.738 \cdot 10^{-3}$	$7.784 \cdot 10^{-4}$
0.7	$5.896 \cdot 10^{-3}$	$1.080 \cdot 10^{-3}$
0.8	$7.364 \cdot 10^{-3}$	$1.485 \cdot 10^{-3}$
0.9	$9.240 \cdot 10^{-3}$	$2.032 \cdot 10^{-3}$
1.0	$1.166 \cdot 10^{-2}$	$2.777 \cdot 10^{-3}$

The computation was performed on a PC computer.

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Institute of Mathematics
University of Gdańsk
Wita Stwosza 57
80-952 Gdańsk, Poland
E-mail: rciarski@math.univ.gda.pl

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