ANNALES
POLONICI MATHEMATICI
84.2 (2004)

On decomposition of pairs of commuting isometries

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Abstract. A review of known decompositions of pairs of isometries is given. A new, finer decomposition and its properties are presented.

1. Introduction. Let H be a complex Hilbert space. Let H_0 be a subspace of H. Then P_{H_0} is the orthogonal projection on H_0. Denote by $L(H)$ the algebra of all bounded linear operators on H. Recall that an isometry $S \in L(H)$ is called a *unilateral shift* if there is a wandering subspace W which generates H (i.e. $S^n W \perp S^m W$ for any distinct $n, m \geq 0$ and $H = \bigoplus_{n\geq 0} S^n W$). Note that $W = \ker S^*$. The subspace H_0 *reduces* S (or is *reducing for* S) if H_0 and $H_0^\perp$ are invariant for S. Recall Wold's classical result [8]:

Theorem 1.1. *Let $S \in L(H)$ be an isometry. There is a unique decomposition*

$$H = H_{\mathrm{u}} \oplus H_{\mathrm{s}}$$

into orthogonal subspaces reducing S such that $S|_{H_{\mathrm{u}}}$ is a unitary operator and $S|_{H_{\mathrm{s}}}$ is a unilateral shift. Moreover

$$H_{\mathrm{u}} = \bigcap_{n\geq 0} S^n H, \qquad H_{\mathrm{s}} = \bigoplus_{n\geq 0} S^n(\ker S^*). \ \blacksquare$$

Let $S_1, S_2 \in L(H)$ be commuting isometries (we will keep this notation throughout the paper). We always call them a *pair of isometries*. A natural extension of Wold's result to a pair of commuting isometries would be a decomposition of the Hilbert space into four subspaces which reduce each of the operators S_1, S_2 either to a unitary operator or to a unilateral shift. Such a decomposition has been proved for pairs of doubly commuting operators by M. Słociński [6]. It does not exist if the isometries just commute. For a commuting semigroup of isometries I. Suciu [7] showed the existence of a decomposition into three parts: a maximal subspace where each oper-

2000 *Mathematics Subject Classification*: Primary 47B20; Secondary 47A15, 47A20.
Key words and phrases: commuting isometries, Wold decomposition.

ator is unitary, a *totally non-unitary subspace* and a *strange subspace*. This result shows that in the case of a pair of commuting isometries, the totally non-unitary subspace is a subspace where the operators are doubly commuting unilateral shifts. The case of commuting isometries has also been investigated by Popovici. He found the maximal reducing subspace where the operators doubly commute and the Słociński theorem can be applied. Moreover, he decomposed the orthogonal complement of that subspace into a *modified bi-shift subspace* and an *evanescent subspace*.

In the present paper we continue the investigation of decompositions for commuting pairs of isometries. The evanescent subspace is decomposed according to the existence of "wandering vectors". We also give some properties of the parts of the resulting decomposition.

2. The known decompositions. Let G be a subsemigroup of an abelian group such that $G \cap -G = \{0\}$. Recall ([7]) that $\{T_g\}_{g\in G} \subset L(H)$ is a *semigroup of isometries* if $T_0 = I$, $T_{g_1+g_2} = T_{g_1}T_{g_2}$ for $g_1, g_2 \in G$ and T_g is an isometry for $g \in G$. Since G is an abelian semigroup, the isometries commute. A semigroup $\{T_g\}_{g\in G}$ of isometries is called *quasi-unitary* if the set

$$\bigcup_{f-g\notin G} T_f^* T_g H$$

is linearly dense in H. A quasi-unitary semigroup is called *strange* if there is no non-zero subspace reducing each isometry to a unitary operator. A semigroup of isometries is called *totally non-unitary* if there is no non-trivial subspace which reduces the semigroup to a quasi-unitary semigroup.

THEOREM 2.1 (Suciu [7]). *Let $\{T_g\}_{g\in G}$ be a semigroup of isometries on H. There is a unique decomposition*

$$H = H_{\mathrm{u}} \oplus H_{\mathrm{s}} \oplus H_{\mathrm{t}}$$

such that $H_{\mathrm{u}}, H_{\mathrm{s}}, H_{\mathrm{t}}$ reduce T_g for $g \in G$ and

- $\{T_g|_{H_{\mathrm{u}}}\}_{g\in G}$ *is a semigroup of unitary operators*,
- $\{T_g|_{H_{\mathrm{s}}}\}_{g\in G}$ *is a strange semigroup*,
- $\{T_g|_{H_{\mathrm{t}}}\}_{g\in G}$ *is a totally non-unitary semigroup.* ∎

Having a pair S_1, S_2 of commuting isometries we obtain a semigroup of isometries by setting $T_{(n,m)} := S_1^n S_2^m$ for $(n,m) \in (\mathbb{Z}_+ \cup \{0\})^2$. As a corollary of the above theorem we obtain the existence of a maximal subspace H_{u} of H such that $S_1|_{H_{\mathrm{u}}}, S_2|_{H_{\mathrm{u}}}$ are unitary operators. In [1] it was proved that this subspace is precisely the unitary subspace of the Wold decomposition for the product isometry S_1S_2 (i.e. $H_{\mathrm{u}} = \bigcap_{i\ge 0}(S_1S_2)^i H$). The decomposition theorem below proved by Słociński [6] additionally decomposes the strange subspace into two reducing subspaces. Note that the theorem is true only for pairs of doubly commuting isometries. Recall that isometries S_1, S_2 *doubly commute* if $S_1S_2 = S_2S_1$ and $S_1^*S_2 = S_2S_1^*$.

THEOREM 2.2 (Słociński [6]). *Suppose S_1, S_2 is a pair of doubly commuting isometries on H. There is a unique decomposition*

$$H = H_{\mathrm{uu}} \oplus H_{\mathrm{us}} \oplus H_{\mathrm{su}} \oplus H_{\mathrm{ss}},$$

where H_{uu}, H_{us}, H_{su}, H_{ss} are reducing subspaces for S_1 and S_2 such that

- $S_1|_{H_{\mathrm{uu}}}$, $S_2|_{H_{\mathrm{uu}}}$ *are unitary operators*,
- $S_1|_{H_{\mathrm{us}}}$ *is a unitary operator*, $S_2|_{H_{\mathrm{su}}}$ *is a unilateral shift*,
- $S_1|_{H_{\mathrm{su}}}$ *is a unilateral shift*, $S_2|_{H_{\mathrm{su}}}$ *is a unitary operator*,
- $S_1|_{H_{\mathrm{ss}}}$, $S_2|_{H_{\mathrm{ss}}}$ *are unilateral shifts.* ∎

The following relations hold between the subspaces considered by Słociński and Suciu:

$$H_{\mathrm{u}} = H_{\mathrm{uu}}, \quad H_{\mathrm{s}} = H_{\mathrm{us}} \oplus H_{\mathrm{su}}, \quad H_{\mathrm{t}} = H_{\mathrm{ss}}.$$

The general case has been investigated by Popovici. Set

$$K_1 := \bigcap_{i\ge 0} \ker S_1^*S_2^i, \quad K_2 := \bigcap_{i\ge 0} \ker S_2^*S_1^i, \tag{1}$$

and recall the following definition from [4].

DEFINITION 2.3. A pair of isometries S_1, S_2 is called a *weak bi-shift* if $S_1|_{K_2}, S_2|_{K_1}$, and the product isometry S_1S_2 are shifts.

Observe that if a pair of isometries S_1, S_2 doubly commute then $K_i = \ker S_i^*$ for $i = 1, 2$, and a weak bi-shift is precisely a pair of doubly commuting shifts. By [2] for any pair of isometries there is a space $\mathcal{H}$ including H and unitary operators U_1, U_2 on $\mathcal{H}$ such that $S_i = U_i|_{\mathcal{H}}$ for $i = 1, 2$. The space $\mathcal{H}$ can be chosen minimal among all having this property. Then $U_1^*|_{\mathcal{H}\ominus H}, U_2^*|_{\mathcal{H}\ominus H}$ is called a *dual pair of isometries* to S_1, S_2, and the space $\mathcal{H} \ominus H$ a *dual space* to H.

THEOREM 2.4 (Popovici). *For any pair of commuting isometries S_1, S_2 on H there is a unique decomposition*

$$H = H_{\mathrm{uu}} \oplus H_{\mathrm{us}} \oplus H_{\mathrm{su}} \oplus H_{\mathrm{ws}}$$

such that H_{uu}, H_{us}, H_{su}, H_{ws} reduce S_1 and S_2 and

- $S_1|_{H_{\mathrm{uu}}}, S_2|_{H_{\mathrm{uu}}}$ *are unitary operators*,
- $S_1|_{H_{\mathrm{us}}}$ *is a unitary operator*, $S_2|_{H_{\mathrm{us}}}$ *is a unilateral shift*,
- $S_1|_{H_{\mathrm{su}}}$ *is a unilateral shift*, $S_2|_{H_{\mathrm{su}}}$ *is a unitary operator*,
- $S_1|_{H_{\mathrm{ws}}}, S_2|_{H_{\mathrm{ws}}}$ *is a weak bi-shift.*

Moreover, the subspace H_{ws} *can be uniquely decomposed as*

$$H_{\mathrm{ws}} = H_{\mathrm{ss}} \oplus H_{\mathrm{m}} \oplus H_{\mathrm{e}},$$

where H_{ss}, H_{m}, H_{e} *reduce* S_1 *and* S_2 *and*

- H_{ss} *is a maximal subspace such that the restrictions* $S_1|_{H_{\mathrm{ss}}}$ *and* $S_2|_{H_{\mathrm{ss}}}$ *are doubly commuting unilateral shifts*,
- H_{m} *is a maximal subspace such that a pair of isometries dual to* $S_1|_{H_{\mathrm{m}}}$, $S_2|_{H_{\mathrm{m}}}$ *is a doubly commuting pair of unilateral shifts*,
- $H_{\mathrm{e}} := H_{\mathrm{ws}} \ominus (H_{\mathrm{ss}} \oplus H_{\mathrm{m}})$. ∎

The above subspaces can be described in the following way:

$$H_{\mathrm{uu}} = \bigcap_{n\geq 0} (S_1S_2)^n H, \tag{2}$$

$$H_{\mathrm{us}} = \bigoplus_{n\geq 0} S_2^n \Big(\bigcap_{m\geq 0} S_1^m(K_2)\Big), \tag{3}$$

$$H_{\mathrm{su}} = \bigoplus_{n\geq 0} S_1^n \Big(\bigcap_{m\geq 0} S_2^m(K_1)\Big), \tag{4}$$

$$H_{\mathrm{ss}} = \bigoplus_{n,m\geq 0} S_1^n S_2^m (K_1 \cap K_2), \tag{5}$$

$$H_{\mathrm{m}} = \bigoplus_{(m,n)\in\mathbb{Z}^2\setminus(\mathbb{Z}_-)^2} (S_1,S_2)^{(m,n)} (H_{\mathrm{u}}^1 \cap H_{\mathrm{u}}^2 \cap \ker (S_1S_2)^*), \tag{6}$$

where H_{u}^i is the subspace of the Wold decomposition for the single isometry S_i where S_i is a unitary operator, and

$$(S_1,S_2)^{(m,n)} = \begin{cases} S_1^m S_2^n & \text{for } m\geq 0,\ n\geq 0,\\ S_1^{*|m|} S_2^n & \text{for } m<0,\ n\geq 0,\\ S_2^{*|n|} S_1^m & \text{for } m\geq 0,\ n<0.\end{cases}$$

All the subspaces considered except H_{m} and H_{e} are denoted as in the Słociński theorem. They are in fact the same subspaces in the case of doubly commuting isometries. Moreover for any pair of commuting isometries the orthogonal sum $H_{\mathrm{uu}} \oplus H_{\mathrm{us}} \oplus H_{\mathrm{su}} \oplus H_{\mathrm{ss}}$ is a maximal reducing subspace where the isometries doubly commute. Comparing the Popovici and Suciu results, we easily get $H_{\mathrm{u}} = H_{\mathrm{uu}}$. We also have the following

Corollary 2.5. *Let* S_1, S_2 *be a pair of commuting isometries. The subspace* H_{t} *of the Suciu decomposition is equal to the subspace* H_{ss} *of the Popovici decomposition.*

Proof. Put $G := (\mathbb{Z}_+ \cup \{0\})^2$ and $T_{(n,m)} := S_1^n S_2^m$. According to the proof of existence of a maximal totally non-unitary subspace (see [7]),

$H_{\mathrm{t}} := \bigoplus_{f\in G} T_f N$, where $N = \big(\bigcup_{f-g\notin G} T_f^* T_g H\big)^{\perp}$. It can be proved that

$$N = \bigcap_{f-g\notin G} \ker (T_f^* T_g)^* = \bigcap_{f-g\notin G} \ker (T_g^* T_f).$$

Then

$$\begin{aligned} N &= \bigcap_{(k,l)-(n,m)\notin(\mathbb{Z}_+\cup\{0\})^2} \ker S_1^{*n} S_2^{*m} S_1^k S_2^l \\ &= \bigcap_{n>k,\, m\le l} \ker S_1^{*n-k} S_2^{l-m} \cap \bigcap_{m>l,\, n\le k} \ker S_2^{*m-l} S_1^{k-n} \cap \bigcap_{n>k,\, m>l} \ker S_1^{*n-k} S_2^{*m-l}. \end{aligned}$$

Since $\ker S_1^* S_2^i \subset \ker S_1^{*j} S_2^i$ for $i\ge 0$, $j\ge 1$, we have

$$\begin{aligned} N &= \bigcap_{n>k,\, m\le l} \ker S_1^* S_2^{l-m} \cap \bigcap_{m>l,\, n\le k} \ker S_2^* S_1^{k-n} \\ &\qquad \cap \bigcap_{n>k,\, m>l} \ker S_1^{*n-k} S_2^{*m-l} \\ &= \bigcap_{i\ge 0} \ker S_1^* S_2^i \cap \bigcap_{i\ge 0} \ker S_2^* S_1^i \cap \bigcap_{i,j\ge 1} \ker S_1^{*i} S_2^{*j} \\ &= K_1 \cap K_2 \cap \ker S_1^* S_2^* = K_1 \cap K_2. \end{aligned}$$

Since $G = (\mathbb{Z}_+ \cup \{0\})^2$ and $T_{(n,m)} = S_1^n S_2^m$, it follows that $N = K_1 \cap K_2$ and we obtain

$$H_{\mathrm{t}} = \bigoplus_{g\in G} T_g(N) = \bigoplus_{(n,m)\in(\mathbb{Z}_+\cup\{0\})^2} S_1^n S_2^m (K_1 \cap K_2),$$

which finishes the proof. ■

By this theorem and $H_{\mathrm{uu}} = H_{\mathrm{u}}$ the strange part in the case of a pair of commuting isometries decomposes into four orthogonal subspaces reducing both isometries S_1, S_2:

$$H_{\mathrm{s}} = H_{\mathrm{us}} \oplus H_{\mathrm{su}} \oplus H_{\mathrm{m}} \oplus H_{\mathrm{e}},$$

where H_{e} is called the *evanescent subspace*.

3. New results

3.1. *Examples.* The evanescent subspace H_{e} considered by Popovici has not been characterized. Let us consider a few examples of pairs of isometries on H such that $H = H_{\mathrm{e}}$. Recall that K_1, K_2 are the subspaces given by (1). An easy consequence of (3)–(5) may be helpful.

COROLLARY 3.1. *If $K_1 = K_2 = \{0\}$ then $H_{\mathrm{su}} = H_{\mathrm{us}} = H_{\mathrm{ss}} = \{0\}$.*

EXAMPLE 3.2. Fix $n, m \in \mathbb{Z}_+$ and take a pair S^n, S^m, where S is a completely non-unitary isometry (i.e. $\bigcap_{i\ge 0} S^i H = \{0\}$). Hence the isometries

S^n, S^m are completely non-unitary. Therefore $H_{\mathrm{uu}} = \{0\}$ and by (6) we have $H_{\mathrm{m}} = \{0\}$. By Corollary 3.1 we get $H_{\mathrm{su}} = H_{\mathrm{us}} = H_{\mathrm{ss}} = \{0\}$. Hence $H = H_{\mathrm{e}}$. Note that in this example there are k, l such that $S_1^k = S_2^l$.

Recall from [5] that $J \subset \mathbb{Z}^2$ is called a *diagram* (in $\mathbb{Z}^2$) if for any $g \in (\mathbb{Z}_+ \cup \{0\})^2$ and any $j \in J$ the element $g + j$ belongs to J.

EXAMPLE 3.3. Let us fix a diagram J in $\mathbb{Z}^2$ and orthonormal vectors $\{e_{i,j}\}_{(i,j)\in J}$ in a complex Hilbert space. We can define a new Hilbert space

$$H = \bigoplus_{(i,j)\in J} \mathbb{C}e_{i,j}$$

and isometries

$$S_1(e_{i,j}) = e_{i+1,j}, \quad S_2(e_{i,j}) = e_{i,j+1}.$$

The properties of isometries given in this example depend on the diagram J. Some examples in this paper are obtained by specifying J. The following one shows that the converse of Corollary 3.1 is not true.

EXAMPLE 3.4 ([3]). Let $J = (\mathbb{Z}_+ \cup \{0\})^2 \backslash (0,0)$ in Example 3.3. We have $K_1 = \bigoplus_{i\in\mathbb{Z}_+} \mathbb{C}e_{0,i}$ and $K_2 = \bigoplus_{j\in\mathbb{Z}_+} \mathbb{C}e_{j,0}$. But $K_1 \cap K_2 = \{0\}$. By (5) we have $H_{\mathrm{ss}} = \{0\}$. The operators S_1, S_2 are unilateral shifts so $H_{\mathrm{us}} = H_{\mathrm{su}} = H_{\mathrm{uu}} = \{0\}$. Moreover $H_{\mathrm{u}}^1 = H_{\mathrm{u}}^2 = \{0\}$ implies $H_{\mathrm{u}}^1 \cap H_{\mathrm{u}}^2 \cap \ker (S_1S_2)^* = \{0\}$ and using (6) we get $H_{\mathrm{m}} = \{0\}$. So $H = H_{\mathrm{e}}$.

Let G be a semigroup and $\{T_g\}_{g\in G}$ be a semigroup of isometries on H. A vector $x \in H$ is called a *wandering vector* (for a given semigroup of isometries) if $(T_{g_1}x, T_{g_2}x) = 0$ for any distinct $g_1, g_2 \in G$. In Example 3.4 each $e_{i,j}$ is a wandering vector for the semigroup $T_{(n,m)} = S_1^n S_2^m$, where $(n,m) \in \mathbb{Z}_+$. Hence there is no relation like that in Example 3.2. These two examples show two quite different types of pairs of isometries. Now consider the next example, similar to Example 3.4, but having $K_1 = K_2 = \{0\}$.

EXAMPLE 3.5. Let $J = \{(i,j) \in (\mathbb{Z})^2 : j \geq -i\}$ in Example 3.3. Then $K_1 = K_2 = \{0\}$. By similar arguments to those in Example 3.4 we obtain $H = H_{\mathrm{e}}$.

3.2. *New spaces.* Let $\mathcal{S}$ be a subset of H. Denote by $\langle\mathcal{S}\rangle$ the smallest closed linear subspace containing $\mathcal{S}$. Then for any $z \in H$ we have $\langle z\rangle = \mathbb{C}z$. For every $x \in H$ we define

$$H(x) := \langle\{S_1^i S_2^j x : i,j \geq 0\}\rangle, \tag{7}$$

$$K_1(x) := \bigcap_{n\geq 0} \ker (S_1|_{H(x)})^* S_2^n, \qquad K_2(x) := \bigcap_{n\geq 0} \ker (S_2|_{H(x)})^* S_1^n. \tag{8}$$

LEMMA 3.6. *For any $x \in H_{\mathrm{e}}$ there are $y, z \in H(x)$ such that $H(x) = H(z) \oplus H(y)$ and $K_1(z) \cap K_2(z) = \langle z\rangle$, $K_1(y) \cap K_2(y) = \{0\}$. Moreover the* *vector z is either zero or the orthogonal projection of x on* $\ker(S_1|_{H(x)})^* \cap \ker(S_2|_{H(x)})^*$.

Proof. Let $x \in H_{\mathrm{e}}$. Define $w := P_F x$, where $F = \ker(S_1|_{H(x)})^* \cap \ker(S_2|_{H(x)})^*$. Note that

$$F \perp S_1^i S_2^j(H(x)) \quad \text{for } i, j \geq 0,\ i+j \geq 1. \tag{9}$$

Therefore if $w = 0$ then $F = \{0\}$. Since $K_1(x) \cap K_2(x) \subset F$, taking $y = x$, $z = 0$ we finish the proof for $w = 0$.

Now consider the case $w \neq 0$. We can assume that $\|w\| = 1$. Let $u \in F$. The projection of u on the space orthogonal to $\langle w\rangle$ is $(I - P_{\langle w\rangle})u = u - (u,w)w$. Since $u, w \in F$, we have $(I - P_{\langle w\rangle})u \in F$ as well. Hence by (9), $((I - P_{\langle w\rangle})u, S_1^k S_2^l x) = 0$ for $(k,l) \neq (0,0)$, while for $(k,l) = (0,0)$ we have $0 = ((I - P_{\langle w\rangle})u, w) = ((I - P_{\langle w\rangle})u, P_F x) = (P_F(I - P_{\langle w\rangle})u, x) = ((I - P_{\langle w\rangle})u, x)$. Thus $(I - P_{\langle w\rangle})u \perp H(x)$ so $0 = (I - P_{\langle w\rangle})u = u - (u,w)w$ and $u = (u,w)w$. Since u was an arbitrary vector in F this proves the inclusion $F \subset \langle w\rangle$. Hence also $K_1(x) \cap K_2(x) \subset \langle w\rangle$. If $K_1(x) \cap K_2(x) = \{0\}$ then taking $z = 0$, $y = x$ we finish the proof.

Assume $K_1(x) \cap K_2(x) = \langle w\rangle$. Decompose $x = x_w + v$, where $x_w := P_{H(w)}x$. Since $w \in H(x)$ the vectors x_w, v belong to $H(x)$ as well. Moreover, $w \perp v$ and by (9) we also have $w \perp R(S_i|_{H(x)})$ for $i = 1, 2$ and thus $w \perp H(v)$. Note also that $v \perp H(w)$ by definition. Since $w \in K_1(x) \cap K_2(x)$, by the following calculation we obtain $H(w) \perp H(v)$:

$$(S_1^k S_2^l w, S_1^n S_2^m v) = \begin{cases} (S_1^{k-n} S_2^{l-m} w, v) = 0 & \text{for } k \geq n,\ l \geq m, \\ (S_2^* S_1^{k-n} w, S_2^{m-l-1} v) = (0, S_2^{m-l-1} v) = 0 & \text{for } k > n,\ l < m, \\ (w, S_1^{n-k} S_2^{m-l} v) = 0 & \text{for } k \leq n,\ l \leq m, \\ (S_1^* S_2^{l-m} w, S_1^{n-k-1} v) = (0, S_1^{n-k-1} v) = 0 & \text{for } k < n,\ l > m. \end{cases}$$

Since $x \in H(w) \oplus H(v) \subset H(x)$, we have $H(x) = H(w) \oplus H(v)$. On the other hand, by the definition $H(w), H(v)$ are S_1, S_2 invariant. Therefore, both are reducing for $S_1|_{H(x)}, S_2|_{H(x)}$. Thus $(S_1|_{H(v)})^* = (S_1|_{H(x)})^*|_{H(v)}$ and $(S_2|_{H(v)})^* = (S_2|_{H(x)})^*|_{H(v)}$. Hence $K_1(v) \cap K_2(v) = K_1(x) \cap K_2(x) \cap H(v) = \{0\}$. The same arguments show that $K_1(w) \cap K_2(w) = \langle w\rangle$. Therefore taking $y = v$ and $z = w$ we finish the proof. ∎

Although the evanescent subspace H_{e} does not contain any non-zero subspace reducing the isometries to a doubly commuting pair, we can look for invariant subspaces where the restrictions of the isometries doubly commute. If both restrictions were unitary, the subspace would be not only invariant but also reducing. This is impossible. The following proposition helps us find an invariant subspace where the restrictions are doubly commuting unilateral shifts.

PROPOSITION 3.7. *Let S_1, S_2 be a pair of commuting isometries on H. Let $x \in H$. There is a vector z such that $H(z)$ is a maximal subspace of $H(x)$ which reduces $S_1|_{H(x)}, S_2|_{H(x)}$ to doubly commuting unilateral shifts. Moreover, $S_1|_{H(x)}, S_2|_{H(x)}$ are doubly commuting unilateral shifts on $H(x)$ if and only if $(S_1^n S_2^m x, S_1^k S_2^l x) = 0$ for any $(n, m) \neq (k, l)$.*

Proof. By the proof of Lemma 3.6, $K_1(x) \cap K_2(x) = \langle z\rangle$, where $z = 0$ or z is the orthogonal projection of x onto $\ker (S_1|_{H_x})^* \cap \ker (S_2|_{H_x})^*$. Then $H(z) = \bigoplus_{n,m\geq 0} S_1^n S_2^m (K_1(x)\cap K_2(x))$, which, by (5), is a maximal subspace of $H(x)$ reducing $S_1|_{H(x)}$, $S_2|_{H(x)}$ to doubly commuting unilateral shifts.

By similar arguments, for the second part of the theorem, it is enough to show that the condition $(S_1^n S_2^m x, S_1^k S_2^l x) = 0$ for any $(n, m) \neq (k, l)$ is equivalent to $K_1(x) \cap K_2(x) = \langle x\rangle$. Assume $K_1(x) \cap K_2(x) = \langle x\rangle$. We can rewrite $(S_1^k S_2^l x, S_1^m S_2^n x) = 0$ as either $(S_1^i S_2^j x, x) = 0$ or $(S_1^i x, S_2^j x) = 0$, where $i := |k - m|$ and $j := |l - n|$. Note that $(n, m) \neq (k, l)$ implies $(i, j) \neq (0, 0)$. Since $x \in K_1(x)$, for $i > 0$ we have either $(S_1^{i-1} S_2^j x, S_1^* x) = (S_1^{i-1} S_2^j x, 0) = 0$ or $(S_1^{i-1} x, S_1^* S_2^j x) = (S_1^{i-1} x, 0) = 0$, and similarly for $i = 0$, $j > 0$. Conversely, if $(S_1^k S_2^l x, S_1^m S_2^n x) = 0$ for any non-negative $(k, l) \neq (m, n)$, then $(S_1^* S_2^j x, S_1^n S_2^m x) = (S_2^j x, S_1^{n+1} S_2^m x) = 0$. Since n, m are arbitrary, $S_1^* S_2^j x \perp H(x)$, so $(S_1|_{H(x)})^* S_2^j x = 0$. Similarly we can prove that $(S_2|_{H(x)})^* S_1^i x = 0$. ∎

Let us make a few observations.

REMARK 3.8. Let $x \in H$ and $z \in H(x)$. Consider three conditions:

(1) $K_1(x) \cap K_2(x) = \langle z\rangle$,
(2) $(S_1^k S_2^l z, S_1^m S_2^n z) = 0$ for any $(n, m) \neq (k, l)$, $n, m, k, l \geq 0$,
(3) $K_1(z) \cap K_2(z) = \langle z\rangle$.

By the proof of Lemma 3.6 if $K_1(x) \cap K_2(x) \neq \{0\}$ then $K_1(x) \cap K_2(x) = \langle z\rangle$, where z is such that $K_1(z) \cap K_2(z) = \langle z\rangle$. Therefore (1) implies (3) if $K_1(x) \cap K_2(x) \neq \{0\}$. However, $K_1(0) \cap K_2(0) = \{0\}$, which shows the implication in case $K_1(x) \cap K_2(x) = \{0\}$. The equivalence of (2) and (3) has been shown in the proof of Proposition 3.7. To show that (2) and (3) do not always imply (1) take $x \in H$ such that $K_1(x) \cap K_2(x) = \langle v_0\rangle$, where $v_0 \neq x$. Then $z = S_i v_0 \perp \langle v_0\rangle$ for $i = 1, 2$. However, $(S_1^n S_2^m z, S_1^k S_2^l z) = (S_1^n S_2^m S_i v_0, S_1^k S_2^l S_i v_0) = (S_1^n S_2^m v_0, S_1^k S_2^l v_0) = 0$ for any $(n, m) \neq (k, l)$.

REMARK 3.9. The sets $\{z : K_1(z) \cap K_2(z) = \langle z\rangle\}$ and $\{z : K_1(z) \cap K_2(z) \neq \langle z\rangle\} \cup \{0\}$ are linear manifolds if and only if $\{z : K_1(z) \cap K_2(z) = \langle z\rangle\} = \{0\}$.

The set of vectors such that condition (2) of Remark 3.8 holds is S_1, S_2 invariant. Since conditions (2) and (3) of Remark 3.8 are equivalent the set $\{z : K_1(z) \cap K_2(z) = \langle z \rangle\}$ is S_1, S_2 invariant. Assume there is a non-zero vector $w \in \{z : K_1(z) \cap K_2(z) = \langle z \rangle\}$. Then $S_1 w \in \{z : K_1(z) \cap K_2(z) = \langle z \rangle\}$. Since $(w + S_1 w, S_1(w + S_1 w)) = (S_1 w, S_1 w) = \|w\|^2 \neq 0$ the vector $w + S_1 w$ does not satisfy condition (2) of Remark 3.8 for the pairs of integers $(0,0)$, $(1,0)$. Consequently, $\{z : K_1(z) \cap K_2(z) = \langle z \rangle\}$ is not a linear manifold, unless it is $\{0\}$.

The set $\{z : K_1(z) \cap K_2(z) \neq \langle z \rangle\} \cup \{0\}$ is a linear manifold if and only if $\{z : K_1(z) \cap K_2(z) = \langle z \rangle\} = \{0\}$. If $\{z : K_1(z) \cap K_2(z) = \langle z \rangle\} = \{0\}$ then $\{z : K_1(z) \cap K_2(z) \neq \langle z \rangle\} \cup \{0\}$ is the whole space. Conversely, assume there is $w \neq 0$ such that $K_1(w) \cap K_2(w) = \langle w \rangle$. Set $y_1 := w + S_i w$ and $y_2 := w - S_i w$. We have already shown that y_1 does not satisfy condition (2) of Remark 3.8 for the pairs of integers $(0,0)$, $(1,0)$. Hence $K_1(y_1) \cap K_2(y_1) \neq \langle y_1 \rangle$. By similar arguments $K_1(y_2) \cap K_2(y_2) \neq \langle y_2 \rangle$. The equality $y_1 + y_2 = 2w$ implies $H(y_1 + y_2) = H(w)$ and $K_1(y_1 + y_2) \cap K_2(y_1 + y_2) = \langle y_1 + y_2 \rangle$. The set $\{x : K_1(x) \cap K_2(x) \neq \langle z \rangle\}$ is not a linear manifold either.

The smallest subspace containing $\{x \in H_{\mathrm{e}} : K_1(x) \cap K_2(x) = \langle x \rangle\}$ and reducing for S_1, S_2 is denoted by H_{sbs} and called a *sub-bi-shift* subspace. The orthogonal complement $H_{\mathrm{tno}} := H_{\mathrm{e}} \ominus H_{\mathrm{sbs}}$ is called *totally non-orthogonal*. Note that H_{tno} does not contain any non-zero wandering vector.

3.3. *The decomposition theorem.* Let us first decompose the operators given in examples stated in Section 3.1 according to the subspaces introduced in the previous section.

Example 3.10. Let S be a totally non-unitary isometry. Fix $n, m \in \mathbb{Z}_+$ and set $S_1 = S^n$, $S_2 = S^m$. It was proved in Example 3.2 that $H = H_{\mathrm{e}}$. There are k, l such that $S_1^k x = S_2^l x$ for any vector x (e.g. $k = m$, $l = n$). Therefore for any non-zero vector x we have $(S_1^k x, S_2^l x) = \|S^{kn} x\| = \|x\| \neq 0$. By Proposition 3.7 there are no non-zero generators of H_{sbs}. It follows that $H_{\mathrm{sbs}} = \{0\}$ and $H = H_{\mathrm{tno}}$.

Example 3.11. Take a pair S_1, S_2 of commuting isometries and a Hilbert space H as in Example 3.4. Each vector $e_{i,j}$ satisfies $(S_1^k S_2^l x, S_1^n S_2^m x) = 0$ for every $(n,m) \neq (k,l)$ so by Proposition 3.7, $e_{i,j} \in H_{\mathrm{sbs}}$ for every $(i,j) \in J$. Since these vectors generate the whole space, we have $H_{\mathrm{e}} = H_{\mathrm{sbs}}$ and consequently $H_{\mathrm{tno}} = \{0\}$. In Example 3.4 it was shown that $H = H_{\mathrm{e}}$. Hence $H = H_{\mathrm{sbs}}$.

Example 3.12. Take a pair S_1, S_2 of commuting isometries and a Hilbert space H as in Example 3.5. Then $H = H_{\mathrm{sbs}}$ and $H_{\mathrm{tno}} = \{0\}$ by the same argument as above.

We now state the decomposition theorem.

Theorem 3.13. *Let S_1, S_2 be a pair of commuting isometries. Then there is a unique decomposition*

$$H = H_{\mathrm{uu}} \oplus H_{\mathrm{us}} \oplus H_{\mathrm{su}} \oplus H_{\mathrm{ss}} \oplus H_{\mathrm{m}} \oplus H_{\mathrm{sbs}} \oplus H_{\mathrm{tno}},$$

where $H_{\mathrm{uu}}, H_{\mathrm{us}}, H_{\mathrm{su}}, H_{\mathrm{ss}}, H_{\mathrm{m}}, H_{\mathrm{sbs}}, H_{\mathrm{tno}}$ are reducing subspaces for S_1 and S_2 and:

- *$S_1|_{H_{\mathrm{uu}}}, S_2|_{H_{\mathrm{uu}}}$ are unitary operators,*
- *$S_1|_{H_{\mathrm{us}}}$ is a unitary operator, $S_2|_{H_{\mathrm{us}}}$ is a unilateral shift,*
- *$S_1|_{H_{\mathrm{su}}}$ is a unilateral shift, $S_2|_{H_{\mathrm{su}}}$ is a unitary operator,*
- *H_{ss} is a maximal subspace such that the restrictions $S_1|_{H_{\mathrm{ss}}}, S_2|_{H_{\mathrm{ss}}}$ are doubly commuting unilateral shifts,*
- *H_{m} is a maximal subspace such that a pair of isometries dual to $S_1|_{H_{\mathrm{m}}}$, $S_2|_{H_{\mathrm{m}}}$ is a doubly commuting pair of unilateral shifts,*
- *H_{sbs} is a sub-bi-shift subspace,*
- *H_{tno} is a totally non-orthogonal subspace.*

The space H_{sbs} is a maximal sub-bi-shift subspace. However, the subspace H_{ss} is also generated by wandering vectors. The following example shows that H_{m} can also be generated by wandering vectors.

Example 3.14. Let $J = \{(i,j) \in \mathbb{Z} : i \geq 0 \text{ or } j \geq 0\}$ in Example 3.3. Since the operators are unilateral shifts, $H_{\mathrm{uu}} = H_{\mathrm{us}} = H_{\mathrm{su}} = \{0\}$. Since $K_1 = \bigoplus_{n<0} \mathbb{C}e_{0,n}$ and $K_2 = \bigoplus_{n<0} \mathbb{C}e_{n,0}$, we have $K_1 \cap K_2 = \{0\}$. Consequently, $H_{\mathrm{ss}} = \{0\}$. The dual space is $\widetilde{H} = \bigoplus_{(i,j)\in\mathbb{Z}^2_-} \mathbb{C}e_{i,j}$ and a dual pair of isometries is

$$\widetilde{S}_1(e_{i,j}) = e_{i-1,j}, \quad \widetilde{S}_2(e_{i,j}) = e_{i,j-1}.$$

Therefore, as the dual isometries are doubly commuting unilateral shifts, the pair S_1, S_2 is a modified bi-shift. Moreover, each vector $e_{i,j}$ is a wandering vector.

The space H_{sbs} is a maximal subspace, among subspaces of H_{e}, generated by wandering vectors. The maximal totally non-orthogonal subspace H_{tno} is a maximal reducing subspace, among subspaces of H_{e}, not containing any wandering vector. If S_1 is the identity then there is no wandering vector, while the whole space is decomposed as $H = H_{\mathrm{uu}} \oplus H_{\mathrm{us}}$. Therefore the space H_{tno} is not always a maximal reducing subspace in H not containing any wandering vector.

3.4. *Properties of the decomposing subspaces.* Examples 3.11 and 3.12 show that H_{sbs} can be further non-trivially decomposed into subspaces reducing for S_1, S_2: the subspace containing $(K_1 \cup K_2) \cap H_{\mathrm{e}}$, and the subspace orthogonal to both K_i for $i = 1, 2$. The subspace containing $(K_1 \cup K_2) \cap H_{\mathrm{e}}$

in Example 3.11 is the whole space while in Example 3.12 the whole space is orthogonal to $(K_1 \cup K_2) \cap H_{\mathrm{e}}$.

The following lemma helps us find out whether a given reducing subspace is orthogonal to the space K_i given by (1) for $i = 1, 2$.

LEMMA 3.15. *Let S_1, S_2 be a pair of isometries on $H = H_1 \oplus H_2$, where H_1, H_2 are reducing for S_1, S_2. The subspaces K_i given by* (1) *are orthogonal to H_1 if and only if $K_i \cap H_1 = \{0\}$ for $i = 1, 2$.*

Proof. One implication is trivial. For the converse, assume $K_i \cap H_1 = \{0\}$ for $i = 1, 2$. It is enough to show that $P_{H_1} K_i \subset K_i$ for $i = 1, 2$. We show this for K_1; the proof for K_2 is analogous. Consider a vector $v \in K_1$ and its decomposition $v = v_1 \oplus v_2 \in H_1 \oplus H_2$. For any n we have $0 = S_1^* S_2^n v = S_1^* S_2^n (v_1 \oplus v_2) = S_1^* (S_2^n v_1 \oplus S_2^n v_2) = S_1^* S_2^n v_1 + S_1^* S_2^n v_2$. Since H_1, H_2 are orthogonal and S_1, S_2 reducing, we have $S_1^* S_2^n v_1 \perp S_1^* S_2^n v_2$. Hence $0 = S_1^* S_2^n v_1 \oplus S_1^* S_2^n v_2$ and consequently $0 = S_1^* S_2^n v_1 = S_1^* S_2^n v_2$. Therefore $P_{H_1} K_1 \subset K_1$. ∎

THEOREM 3.16. *Let S_1, S_2 be a pair of commuting isometries on H such that $H = H_{\mathrm{sbs}} \oplus H_{\mathrm{tno}}$. Then $H_{\mathrm{tno}} \subset \{x : K_1(x) \cap K_2(x) = \{0\}\}$. Moreover $K_i \subset H_{\mathrm{sbs}}$ for $i = 1, 2$.*

Proof. The inclusion $H_{\mathrm{tno}} \subset \{x : K_1(x) \cap K_2(x) = \{0\}\}$ is a consequence of Lemma 3.6 and orthogonality of H_{tno} to $\{x : K_1(x) \cap K_2(x) = \langle x \rangle\}$. For the second part of the theorem, note that by Lemma 3.15, it is enough to show $K_i \cap H_{\mathrm{tno}} = \{0\}$ for $i = 1, 2$. Let $y \in H_{\mathrm{tno}} \cap K_1$. By Proposition 3.7 since $y \in H_{\mathrm{tno}}$ there are $(n, m) \neq (k, l)$ such that $(S_1^k S_2^l y, S_1^n S_2^m y) \neq 0$. We may assume that $k = 0$. Since $y \in K_1$ we have $0 = (S_1^* S_2^l y, S_1^i S_2^m v) = (S_2^l y, S_1^{i+1} S_2^m v)$ for any $v \in H_{\mathrm{tno}}$ and $i \geq 0$. Since $y \in H_{\mathrm{tno}}$ we can take $v = y$ to obtain $(S_2^l y, S_1^{i+1} S_2^m y) = 0$ for any $i \geq 0$. On the other hand, $(S_2^l y, S_1^n S_2^m y) \neq 0$. Therefore $n \neq i + 1$ for any $i \geq 0$, so n cannot be positive. Hence $n = 0$ and so $(S_2^l y, S_2^m y) \neq 0$. We conclude that for any $y \in K_1 \cap H_{\mathrm{tno}}$ either there is $j \geq 1$ such that $(y, S_2^j y) \neq 0$, or $y = 0$. Consider $\langle \{S_2^j y : j \geq 0\} \rangle$. Since $\langle \{S_2^j y : j \geq 1\} \rangle \subset \langle \{S_2^j y : j \geq 0\} \rangle \subset K_1 \cap H_{\mathrm{tno}}$, the vector $y_0 := y - P_{\langle \{S_2^j y : j \geq 1\} \rangle} y$ belongs to $K_1 \cap H_{\mathrm{tno}}$. Expanding

$$P_{\langle \{S_2^j y : j \geq 1\} \rangle} y = \sum_{j \geq 1} \alpha_j S_2^j y,$$

for any $k \geq 1$ we have

$$\begin{aligned} 0 = (y_0, S_2^k y) &= (y_0, S_2^k y_0) + \Big(y_0, S_2^k \Big(\sum_{j \geq 1} \alpha_j S_2^j y \Big) \Big) \\ &= (y_0, S_2^k y_0) + \sum_{j \geq 1} \overline{\alpha}_j (y_0, S_2^{k+j} y) = (y_0, S_2^k y_0). \end{aligned}$$

Since $y_0 \in K_1 \cap H_{\mathrm{tno}}$ the previous conclusion implies $y_0 = 0$. Therefore $y \in \langle\{S_2^j y : j \geq 1\}\rangle$, and consequently $\langle\{S_2^j y : j \geq 0\}\rangle = \langle\{S_2^j y : j \geq 1\}\rangle$. Using the same arguments for $S_2^l y$ which belongs to $K_1 \cap H_{\mathrm{tno}}$ for $l = 1, 2, \ldots$ we have $\langle\{S_2^j y : j \geq 0\}\rangle = \langle\{S_2^j y : j \geq k\}\rangle$ for any $k \geq 0$. It follows that for any $k \geq 0$ we can represent $y = S_2^k(y_k)$ for some $y_k = \sum_{j\geq 0} \alpha_j S_2^j y$ in $K_1 \cap H_{\mathrm{tno}}$. Therefore $y \in \bigcap_{m\geq 0} S_2^m(K_1)$. But $\bigcap_{m\geq 0} S_2^m(K_1) \subset H_{\mathrm{su}}$, which is orthogonal to $H_{\mathrm{e}} = H_{\mathrm{tno}} \oplus H_{\mathrm{sbs}}$. Hence $y = 0$. ∎

For completeness recall from [4] the following property of H_{m}.

Lemma 3.17. *Let S_1, S_2 be a pair of commuting isometries on a Hilbert space H such that $H = H_{\mathrm{m}}$. Then* $\ker S_1^*$ *is orthogonal to* $\ker S_2^*$.

Proof. Let $U_1, U_2 \in L(\mathcal{H})$ be the minimal unitary extension of S_1, S_2. Set

$$\widetilde{H} := \mathcal{H} \ominus H, \qquad \widetilde{S}_i := U_i^*|_{\widetilde{H}} \quad \text{for } i = 1, 2.$$

Note that $\widetilde{S}_i \in L(\widetilde{H})$. Moreover $(\widetilde{S}_i)^* x = P_{\widetilde{H}} U_i x$ and $S_i^* y = P_H U_i^* y$ for $x \in \widetilde{H}$, $y \in H$ and $i = 1, 2$. Let $x \in \ker(\widetilde{S}_1)^*$. Then $0 = (x, \widetilde{S}_1 v) = (x, U_1^* v) = (U_1 x, v)$ for any $v \in \widetilde{H}$. Therefore $P_{\widetilde{H}} U_1 x = 0$, and consequently $U_1 x \in H$. Moreover $S_1^* U_1 x = P_H U_1^* U_1 x = P_H x = 0$. We have shown $U_1(\ker(\widetilde{S}_1)^*) \subset \ker S_1^*$, so that $\ker(\widetilde{S}_1)^* \subset U_1^*(\ker S_1^*)$. Conversely, let $y \in \ker S_1^*$. Then $0 = S_1^* y = P_H U_1^* y$. Therefore $U_1^* y \in \widetilde{H}$ and $(\widetilde{S}_1)^* U_1^* y = P_{\widetilde{H}} U_1 U_1^* y = P_{\widetilde{H}} y = 0$. Thus $\ker(\widetilde{S}_1)^* \supset U_1^*(\ker S_1^*)$. We have obtained

$$\ker(\widetilde{S}_1)^* = U_1^*(\ker S_1^*).$$

Since $\widetilde{S}_1$ and $\widetilde{S}_2$ doubly commute, $\widetilde{S}_2(\ker(\widetilde{S}_1)^*) \subset \ker(\widetilde{S}_1)^*$. By the above equality,

$$U_2^* U_1^*(\ker S_1^*) = U_2^*(\ker(\widetilde{S}_1)^*) = \widetilde{S}_2(\ker(\widetilde{S}_1)^*) \subset \ker(\widetilde{S}_1)^* = U_1^*(\ker S_1^*).$$

Hence applying $U_2 U_1$ to the inclusion $U_2^* U_1^*(\ker S_1^*) \subset U_1^*(\ker S_1^*)$ we obtain $\ker S_1^* \subset U_2(\ker S_1^*) = S_2(\ker S_1^*)$. Consequently, $\ker S_1^*$ is orthogonal to $\ker S_2^*$. ∎

Finally, we obtain the following theorem.

Theorem 3.18. *For any pair S_1, S_2 of commuting isometries on H the following inclusions hold*:

$$K_1 \subset H_{\mathrm{su}} \oplus H_{\mathrm{ss}} \oplus H_{\mathrm{sbs}}, \quad K_2 \subset H_{\mathrm{us}} \oplus H_{\mathrm{ss}} \oplus H_{\mathrm{sbs}}.$$

Proof. Since $K_1 \subset \ker S_1^*$, it is orthogonal to $H_{\mathrm{uu}} \oplus H_{\mathrm{us}}$. Similarly K_2 is orthogonal to $H_{\mathrm{uu}} \oplus H_{\mathrm{su}}$. By Theorem 3.16, $K_i \perp H_{\mathrm{tno}}$ for $i = 1, 2$. We need to show $K_i \perp H_{\mathrm{m}}$ for $i = 1, 2$. By the previous lemma $(\ker S_1^* \cap H_{\mathrm{m}}) \perp (\ker S_2^* \cap H_{\mathrm{m}})$. Hence $K_1 \cap H_{\mathrm{m}} \subset \ker S_1^* \cap H_{\mathrm{m}} \subset S_2(H_{\mathrm{m}})$. Therefore for any $x \in K_1 \cap H_{\mathrm{m}}$ and any $n \geq 1$ we have $S_1^* S_2^n S_2^* x = S_1^* S_2^{n-1} x = 0$. Moreover $S_1^* S_2^* x = S_2^* S_1^* x = 0$. Therefore $K_1 \cap H_{\mathrm{m}}$ is S_2^* invariant. To show that K_1 is S_2 invariant, take any $x \in K_1$ and integer $n \geq 0$. Then $0 = S_1^* S_2^{n+1} x = S_1^* S_2^n S_2 x$. Since n is arbitrary, $S_2 x \in K_1$, and consequently $K_1 \cap H_{\mathrm{m}}$ is S_2 invariant as intersection of S_2 invariant subspaces. We have shown that $K_1 \cap H_{\mathrm{m}}$ reduces S_2.

Consider the space $\bigoplus_{n\geq 0} S_1^n(K_1 \cap H_{\mathrm{m}})$. Since $K_1 \cap H_{\mathrm{m}} \subset \ker S_1^*$, the space $\bigoplus_{n\geq 0} S_1^n(K_1 \cap H_{\mathrm{m}})$ reduces S_1 to a unilateral shift. For any $x \in K_1 \cap H_{\mathrm{m}}$ and $n \geq 0$ we have $S_2^* S_1^n x = S_2^* S_1^n S_2 S_2^* x = S_1^n S_2^* x \in S_1^n(K_1 \cap H_{\mathrm{m}})$ and $S_2 S_1^n x = S_1^n S_2 x \in S_1^n(K_1 \cap H_{\mathrm{m}})$. Therefore for any $n \geq 0$ the subspace $S_1^n(K_1 \cap H_{\mathrm{m}})$ is reducing for S_2. Hence $\bigoplus_{n\geq 0} S_1^n(K_1 \cap H_{\mathrm{m}})$ is reducing for S_1 and S_2. Moreover,

$$S_1^n(K_1 \cap H_{\mathrm{m}}) = S_1^n S_2^i S_2^{*i}(K_1 \cap H_{\mathrm{m}}) = S_2^i S_1^n S_2^{*i}(K_1 \cap H_{\mathrm{m}}) \subset S_2^i S_1^n (K_1 \cap H_{\mathrm{m}})$$

for any $i, n \geq 0$. Consequently,

$$\bigoplus_{n\geq 0} S_1^n(K_1 \cap H_{\mathrm{m}}) = \bigcap_{i\geq 0} S_2^i\Big(\bigoplus_{n\geq 0} S_1^n(K_1 \cap H_{\mathrm{m}})\Big).$$

Hence S_2 is unitary on $\bigoplus_{n\geq 0} S_1^n(K_1 \cap H_{\mathrm{m}})$. Since S_1 is a unilateral shift on $\bigoplus_{n\geq 0} S_1^n(K_1 \cap H_{\mathrm{m}})$, we have $\bigoplus_{n\geq 0} S_1^n(K_1 \cap H_{\mathrm{m}}) \subset H_{\mathrm{su}} \cap H_{\mathrm{m}} = \{0\}$. Hence $K_1 \cap H_{\mathrm{m}} = \{0\}$ and so $K_1 \perp H_{\mathrm{m}}$ by Lemma 3.15. Similarly we can prove that $K_2 \perp H_{\mathrm{m}}$. ■

The following theorem helps us find a subspace reducing for isometries S_1, S_2 and orthogonal to the space where S_1, S_2 are unitary operators.

THEOREM 3.19. *Let H_0 be a subspace reducing for S_1 and S_2 and orthogonal to H_{uu}. Then*

$$H_0 = \bigoplus_{n\geq 0} S_1^n S_2^n (H_0 \cap \ker S_1^* S_2^*).$$

Assume that W is a subspace of $\ker S_1^* S_2^*$. Then $\bigoplus_{n\geq 0} S_1^n S_2^n(W)$ reduces S_1, S_2 if and only if W is $S_1^*, S_2^*, S_1(I - S_2 S_2^*), S_2(I - S_1 S_1^*)$ invariant.

Proof. The inclusion $\bigoplus_{n\geq 0} S_1^n S_2^n (H_0 \cap \ker S_1^* S_2^*) \subset H_0$ is obvious. For the reverse inclusion note that H_{uu} is precisely the subspace of Wold's decomposition for the isometry $S_1 S_2$ where $S_1 S_2$ is a unitary operator. Therefore its orthogonal complement is $\bigoplus_{n\geq 0} S_1^n S_2^n (\ker S_1^* S_2^*)$. Let $x \in H_0$. Then $x = \sum_{n\geq 0} S_1^n S_2^n x_n$, where $x_n \in \ker S_1^* S_2^*$ for all $n \geq 0$. To prove that $H_0 \subset \bigoplus_{n\geq 0} S_1^n S_2^n (H_0 \cap \ker S_1^* S_2^*)$ it is enough to show that $x_n \in H_0$ for all $n \geq 0$. For $n = 0$ we have $x_0 = x - S_1 S_2 S_1^* S_2^* x \in H_0$. Assume that $x_0, \ldots, x_{n-1} \in H_0$. Then $\sum_{i=0}^{n-1} S_1^i S_2^i x_i \in H_0$. On the other hand,

$\sum_{i=0}^{n} S_1^i S_2^i x_i = x - S_1^n S_2^n S_1^{*n} S_2^{*n} x \in H_0$. Finally,

$$x_n = S_1^{*n} S_2^{*n} (S_1^n S_2^n x_n) = S_1^{*n} S_2^{*n} \Big(\sum_{i=0}^{n} S_1^i S_2^i x_i - \sum_{i=0}^{n-1} S_1^i S_2^i x_i \Big) \in H_0.$$

The first part of the theorem has been proved.

For the second part, set $L = \bigoplus_{n\geq 0} S_1^n S_2^n W$. Let $x \in \ker S_1^* S_2^*$. Since $S_2^* S_1^* x = 0$ implies $S_1^* x \in \ker S_2^* \subset \ker S_1^* S_2^*$, the space $\ker S_1^* S_2^*$ is S_1^* invariant. Since $S_1(I - S_2 S_2^*)H \subset \ker S_1^* S_2^*$, it is also invariant for $S_1(I - S_2 S_2^*)$. Similarly we can prove that it is invariant for $S_2^*, S_2(I - S_1 S_1^*)$. Assume L is S_1, S_2 reducing. Then it is $S_1^*, S_2^*, S_2(I - S_1 S_1^*), S_1(I - S_2 S_2^*)$ invariant. By the first part of the theorem, $W = L \cap \ker S_1^* S_2^*$. Hence W is also S_1^*, S_2^*, $S_2(I - S_1 S_1^*), S_1(I - S_2 S_2^*)$ invariant.

Conversely, let $x \in \bigoplus_{n\geq 0} S_1^n S_2^n W$. Then $x = \sum_{n\geq 0} S_1^n S_2^m x_n$, where $x_n \in W$ for any $n \geq 0$. Note that $\bigoplus_{n\geq 0} S_1^n S_2^n W$ is invariant for the product isometry $S_1 S_2$ and $S_1 x = S_2^* S_1 S_2 x$. Therefore to show that $\bigoplus_{n\geq 0} S_1^n S_2^n W$ is S_1, S_2 reducing it is enough to show that it is S_1^*, S_2^* invariant. This is a consequence of the following equalities:

$$\begin{aligned}
S_1^* x &= S_1^* \Big(\sum_{n\geq 0} S_1^n S_2^n x_n \Big) \\
&= S_1^* x_0 + \sum_{n\geq 1} S_1^{n-1} S_2^n (S_1 S_1^* x_n + (I - S_1 S_1^*) x_n) \\
&= S_1^* x_0 + \sum_{n\geq 1} S_1^n S_2^n (S_1^* x_n) + \sum_{n\geq 1} S_1^{n-1} S_2^{n-1} (S_2 (I - S_1 S_1^*) x_n) \\
&= \sum_{n\geq 0} S_1^n S_2^n (S_1^* x_n) + \sum_{n\geq 1} S_1^{n-1} S_2^{n-1} (S_2 (I - S_1 S_1^*) x_n).
\end{aligned}$$

Since the vectors $S_1^* x_n, S_2(I - S_1 S_1^*) x_n$ are in W for all $n \geq 0$, it follows that $S_1^* x \in \bigoplus_{n\geq 0} S_1^n S_2^n W$. ■

Recall that if $x \in H_{\mathrm{tno}}$ then $(S_1^n S_2^m x, S_1^k S_2^l x) \neq 0$ for some $(n, m) \neq (k, l)$. This can be restated as $(S_1^i x, S_2^j x) \neq 0$ or $(S_1^i S_2^j x, x) \neq 0$, where $i = |n - k|$, $j = |m - l|$.

Remark 3.20. In Example 3.2 there are $i, j \geq 0$ such that $i + j \geq 1$ and $(S_1^i x, S_2^j x) \neq 0$ for any $x \in H$, $x \neq 0$. The set $\{x : \text{there are } i, j \geq 0 \text{ with } i + j \geq 1 \text{ such that } (S_1^i x, S_2^j x) \neq 0\}$ equals $H_{\mathrm{tno}} \setminus \{0\}$.

In general the following theorem holds.

Theorem 3.21. *Let S_1, S_2 be a pair of commuting isometries on H such that $H = H_{\mathrm{tno}}$. Then*

$$H_{\mathrm{tno}} = \langle \{x : \textit{there are } i, j \geq 0 \textit{ with } i + j \geq 1 \textit{ such that } (S_1^i x, S_2^j x) \neq 0\} \rangle.$$

Proof. Let $x \in H_{\mathrm{tno}} \cap \ker S_1^* S_2^*$. Then $(x, S_1^n S_2^m x) = (S_1^* S_2^* x, S_1^{n-1} S_2^{m-1} x) = (0, S_1^{n-1} S_2^{m-1} x) = 0$ for $n, m \geq 1$. Since $x \in H_{\mathrm{tno}}$ there are $i, j \geq 0$ such that $(S_1^i x, S_2^j x) \neq 0$. Note that $(S_1^i S_1^n S_2^n x, S_2^j S_1^n S_2^n x) = (S_1^i x, S_2^j x) \neq 0$ for any positive n. Therefore, for any $n \geq 0$ and any $y \in S_1^n S_2^n (H_{\mathrm{tno}} \cap \ker S_1^* S_2^*)$, there are $i, j \geq 0$ such that $(S_1^i y, S_2^j y) \neq 0$. By Theorem 3.19 every vector in H_{tno} can be represented as an orthogonal sum of vectors from $\{x : \text{there are } i, j \geq 0, \text{ with } i + j \geq 1 \text{ such that } (S_1^i x, S_2^j x) \neq 0\}$. ∎

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Reçu par la Rédaction le 4.3.2004 (1507)