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Proper holomorphic self-mappings of the symmetrized bidisc

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Abstract. We characterize proper holomorphic self-mappings $\mathbb{G}_2 \to \mathbb{G}_2$ for the symmetrized bidisc $\mathbb{G}_2 = \{(\lambda_1 + \lambda_2, \lambda_1\lambda_2) : |\lambda_1|, |\lambda_2| < 1\} \subset \mathbb{C}^2$.

Let $\mathbb{D}$ be the unit disc. Set

$$\pi : \mathbb{C}^2 \ni (\lambda_1, \lambda_2) \mapsto (\lambda_1 + \lambda_2, \lambda_1\lambda_2) \in \mathbb{C}^2$$

and $\mathbb{G}_2 = \pi(\mathbb{D}^2)$. The domain $\mathbb{G}_2$ is called the *symmetrized bidisc*. It has recently been studied by many authors, e.g. [1], [7], [10], [3], [4]. The original motivation for the study of the complex geometry of the symmetrized bidisc comes from control engineering [2].

From the complex analysis point of view the symmetrized bidisc is important since it is the first known example of a bounded pseudoconvex domain for which the Carathéodory and Lempert functions coincide, but which cannot be exhausted by domains biholomorphic to convex ones (see [6], [3], [4]).

Recall that a mapping $f : X \to Y$ between topological spaces X, Y is called *proper* if $f^{-1}(K)$ is a compact subset of X for any compact set $K \subset Y$. The main purpose of the paper is to give the following characterization of proper holomorphic self-mappings of the symmetrized bidisc.

THEOREM 1. *Let $f : \mathbb{G}_2 \to \mathbb{G}_2$ be a proper holomorphic mapping. Then there exists a finite Blaschke product B such that*

$$f\big(\pi(\lambda_1, \lambda_2)\big) = \pi(B(\lambda_1), B(\lambda_2)) \tag{1}$$

for any $\lambda_1, \lambda_2 \in \mathbb{D}$.

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By a (finite) Blaschke product we mean a function of the form

$$B(\lambda) = e^{i\tau} \prod_{j=1}^{m} \frac{\lambda - a_j}{1 - \overline{a}_j \lambda} \tag{2}$$

where $\tau \in \mathbb{R}$, $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in \mathbb{D}$. Recall that a holomorphic function $g : \mathbb{D} \to \mathbb{D}$ is proper if and only if it is a finite Blaschke product.

Theorem 1 implies that if f is an automorphism then $f\big(\pi(\lambda_1, \lambda_2)\big) = \pi(h(\lambda_1), h(\lambda_2)\big)$, where h is an automorphism of the unit disc $\mathbb{D}$ (see [7]).

The above result is a corollary of the following:

THEOREM 2. *Let $f : \mathbb{D}^2 \to \mathbb{G}_2$ be a proper holomorphic mapping. Then there exist finite Blaschke products B_1, B_2 such that*

$$f(\lambda_1, \lambda_2) = \big(B_1(\lambda_1) + B_2(\lambda_2), B_1(\lambda_1)B_2(\lambda_2)\big), \tag{3}$$

for any $\lambda_1, \lambda_2 \in \mathbb{D}$.

Note that $\pi : \mathbb{C}^2 \to \mathbb{C}^2$ is a proper holomorphic mapping and the singular set is equal to $\Sigma_2 = \pi(\Delta)$, where $\Delta = \{(\lambda, \lambda) : \lambda \in \mathbb{C}\}$, i.e. $\pi : \mathbb{C}^2 \setminus \Delta \to \mathbb{C}^2 \setminus \Sigma_2$ is a holomorphic covering.

Let us first gather some elementary properties of the symmetrized bidisc (see e.g. [1]).

PROPOSITION 3. (1) $(s, p) \in \mathbb{G}_2$ *if and only if* $|s - \overline{s}p| + |p|^2 < 1$;
(2) *if* $(s, p) \in \partial\mathbb{G}_2$ *then* $|s - \overline{s}p| + |p|^2 = 1$;
(3) $\pi^{-1}(\mathbb{G}_2) = \mathbb{D}^2$;
(4) $\pi^{-1}(\partial\mathbb{G}_2) = \partial(\mathbb{D}^2)$;
(5) $\Sigma_2 \cap \partial\mathbb{G}_2 = \{(2\lambda, \lambda^2) : |\lambda| = 1\}$.

LEMMA 4. *Assume that $\varphi : \mathbb{D} \to \partial\mathbb{G}_2$ is a holomorphic mapping. Then there exist a $\theta \in \mathbb{R}$ and a holomorphic function $\psi : \mathbb{D} \to \overline{\mathbb{D}}$ such that $\varphi(\lambda) = (e^{i\theta} + \psi(\lambda), e^{i\theta}\psi(\lambda))$ for any $\lambda \in \mathbb{D}$.*

Proof. We know that $\partial\mathbb{G}_2 \subset \{(s, p) : |s - \bar{s}p| + |p|^2 = 1\}$. If $\varphi(\lambda_0) \in \Sigma_2 \cap \partial\mathbb{G}_2$ for some $\lambda_0 \in \mathbb{D}$ then $\varphi_2 = e^{i\tau}$ for some $\tau \in \mathbb{R}$ and therefore $\varphi_1 = \text{const}$.

So, assume that $\varphi(\mathbb{D}) \subset \mathbb{C}^2 \setminus \Sigma_2$. Hence, there exists a holomorphic mapping $\widetilde{\varphi} : \mathbb{D} \to \mathbb{C}^2$ such that $\varphi = \pi \circ \widetilde{\varphi}$. Now, it suffices to note that $\widetilde{\varphi} : \mathbb{D} \to \partial(\mathbb{D}^2)$. ■

Proof of Theorem 2. We use similar methods to those in the proof of the Remmert–Stein theorem (see e.g. [9, p. 71]).

Let $f = (f_1, f_2)$. Assume that $\mathbb{D} \ni w_\nu \to w_0 \in \partial\mathbb{D}$, $\nu \in \mathbb{N}$, is any sequence. The functions $\varphi_{j\nu}(z) = f_j(z, w_\nu)$, $j \in \{1, 2\}$, $\nu \geq 1$, are holomorphic in $\mathbb{D}$. By Montel's theorem there is a subsequence $\{\nu_k\}$ so that $\varphi_{j\nu_k} \to \varphi_j$ uniformly on compact subsets of $\mathbb{D}$. Moreover, $(\varphi_1(z), \varphi_2(z)) \in \partial\mathbb{G}_2$ for any $z \in \mathbb{D}$ (here we use the properness of f).

By Lemma 4, $\varphi_1 = e^{i\theta} + \psi$ and $\varphi_2 = e^{i\theta}\psi$. Now, by the Weierstrass theorem,

$$\frac{\partial f_1(z, w_{\nu_k})}{\partial z} \to \psi'(z), \qquad \frac{\partial^2 f_1(z, w_{\nu_k})}{\partial z^2} \to \psi''(z), \tag{4}$$

$$\frac{\partial f_2(z, w_{\nu_k})}{\partial z} \to e^{i\theta}\psi'(z), \qquad \frac{\partial^2 f_2(z, w_{\nu_k})}{\partial z^2} \to e^{i\theta}\psi''(z). \tag{5}$$

Set

$$H_1(z,w) = \frac{\partial f_1(z,w)}{\partial z}\,\frac{\partial^2 f_2(z,w)}{\partial z^2} - \frac{\partial^2 f_1(z,w)}{\partial z^2}\,\frac{\partial f_2(z,w)}{\partial z}, \tag{6}$$

$$H_2(z,w) = f_2(z,w)\Big(\frac{\partial f_1(z,w)}{\partial z}\Big)^2 + \Big(\frac{\partial f_2(z,w)}{\partial z}\Big)^2 - f_1(z,w)\,\frac{\partial f_1(z,w)}{\partial z}\,\frac{\partial f_2(z,w)}{\partial z}. \tag{7}$$

From (4) and (5) we get

$$H_1(z, w_{\nu_k}) \to 0, \qquad H_2(z, w_{\nu_k}) \to 0. \tag{8}$$

Hence, $H_1(z,w) \equiv 0$ and $H_2(z,w) \equiv 0$.

Set $A = \{(z,w) \in \mathbb{D}^2 : \frac{\partial f_1}{\partial z}(z,w) = 0\}$. Note that A is a proper analytic subset of $\mathbb{D}^2$. Indeed, if $A = \mathbb{D}^2$ then the function ψ in (4) is identically zero, so from (5) we have $\partial f_2/\partial z \equiv 0$. Hence, $f_1(z,w) = g_1(w)$ and $f_2(z,w) = g_2(w)$ for $(z,w) \in \mathbb{D}^2$, where g_1, g_2 are holomorphic functions on $\mathbb{D}$. This contradicts the properness of f (for a fixed $w \in \mathbb{D}$ take $z \to \partial\mathbb{D}$).

Note that $\mathbb{D}^2 \setminus A$ is a domain (i.e. an open connected set). By (6) there exists a holomorphic function g_1 such that

$$\frac{\partial f_2(z,w)}{\partial z} = g_1(w)\,\frac{\partial f_1(z,w)}{\partial z} \tag{9}$$

on $\mathbb{D}^2 \setminus A$. From (7) we get

$$f_2(z,w) = g_1(w) f_1(z,w) - g_1^2(w) \tag{10}$$

for $(z,w) \in \mathbb{D}^2 \setminus A$. Note that $f(z,w) = \pi(g_1(w), f_1(z,w) - g_1(w))$. So, $\widetilde{f}(z,w) = (g_1(w), f_1(z,w) - g_1(w))$ is a holomorphic mapping $\mathbb{D}^2 \setminus A \to \mathbb{D}^2$. Since g_1 is bounded on $\mathbb{D}^2 \setminus A$, it extends holomorphically to $\mathbb{D}^2$. So, $f = \pi \circ \widetilde{f}$ where $\widetilde{f} = (g_1, f_1 - g_1)$.

Repeating similar arguments for w we show that there exists a holomorphic mapping g_2 such that

$$f_2(z,w) = g_2(z) f_1(z,w) - g_2^2(z). \tag{11}$$

From (10) and (11) we get $f_1(z,w) = g_1(w) + g_2(z)$ and $f_2(z,w) = g_1(w)g_2(z)$. Now, it suffices to note that $g_1, g_2 : \mathbb{D} \to \mathbb{D}$ are proper holomorphic functions and, therefore, they are finite Blaschke products. ∎

Remark 5. Note that one may consider proper holomorphic self-mappings of the symmetrized polydisc (see e.g. [8]). In [5], we will show that they have a similar description.

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