ANNALES
POLONICI MATHEMATICI
105.3 (2012)

Real hypersurfaces with a special transversal vector field

by Zuzanna Szancer (Kraków)

Abstract. Real affine hypersurfaces of the complex space $\mathbb{C}^{n+1}$ are studied. Some properties of the structure determined by a J-tangent transversal vector field are proved. Moreover, some generalizations of the results obtained by V. Cruceanu are given.

1. Introduction. In [C], V. Cruceanu studied centro-affine real hypersurfaces in complex affine spaces. He called hypersurfaces for which a centro-affine transversal vector field is J-tangent "special hypersurfaces". In particular, he gave the local characterization of such hypersurfaces and proved that an almost contact structure (φ, ξ, η) induced by a centro-affine J-tangent transversal vector field is always normal.

The main purpose of this paper is to generalize the results of [C] to affine hypersurfaces with a J-tangent transversal vector field of a special form. More precisely, we study real hypersurfaces $f\colon M \to \mathbb{R}^{2n+2}$ with a J-tangent transversal vector field

$$C = -\alpha_1 f - \alpha_2 J f,$$

where α_1, α_2 are smooth functions on M and J is the standard complex structure on $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$. It is easy to see that the case studied by Cruceanu is obtained by setting $\alpha_1 \equiv 1$ and $\alpha_2 \equiv 0$.

In Section 2 we briefly recall basic formulas of affine differential geometry, we introduce the notion of a J-tangent transversal vector field and prove a lemma required in the next section.

In Section 3 we recall some results obtained in [SS] and prove necessary and sufficient conditions for an affine normal to be J-tangent.

Section 4 contains the main results of this paper. In particular, we prove some properties of special vector fields and give a local characterization of hypersurfaces with a locally equiaffine special vector field. We also prove that the affine normal is always J-tangent for hypersurfaces with a centro-affine

2010 *Mathematics Subject Classification*: 53A15, 53D15.
Key words and phrases: affine hypersurface, almost contact structure, special hypersurface.

DOI: 10.4064/ap105-3-3

J-tangent transversal vector field and give an example of a J-tangent affine hypersphere.

Throughout the paper we write $\alpha \equiv 0$ if $\alpha(x) = 0$ for all $x \in M$, and $\alpha \neq 0$ if $\alpha(x) \neq 0$ for every $x \in M$ (i.e. α is a nowhere vanishing function on M).

2. Preliminaries. We briefly recall the basic formulas of affine differential geometry. For more details, we refer to [NS]. Let $f\colon M \to \mathbb{R}^{n+1}$ be an orientable connected differentiable n-dimensional hypersurface immersed in the affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D. Then for any transversal vector field C we have

$$\mathrm{D}_X f_*Y = f_*(\nabla_X Y) + h(X,Y)C, \tag{2.1}$$

$$\mathrm{D}_X C = -f_*(SX) + \tau(X)C, \tag{2.2}$$

where X, Y are vector fields tangent to M. It is known that ∇ is a torsion-free connection, h is a symmetric bilinear form on M, called the *second fundamental form*, S is a tensor of type $(1,1)$, called the *shape operator*, and τ is a 1-form, called the *transversal connection form*.

We shall now consider the change of a transversal vector field for a given immersion f.

THEOREM 2.1 ([NS]). *Suppose we change a transversal vector field C to*

$$\bar{C} = \Phi C + f_*(Z),$$

where Z is a tangent vector field on M and Φ is a nowhere vanishing function. Then the affine fundamental form, the induced connection, the transversal connection form, and the affine shape operator change as follows:

$$\bar{h} = \frac{1}{\Phi}h; \tag{2.3}$$

$$\bar{\nabla}_X Y = \nabla_X Y - \frac{1}{\Phi}h(X,Y)Z; \tag{2.4}$$

$$\bar{\tau} = \tau + \frac{1}{\Phi}h(Z,\cdot) + d\ln|\Phi|; \tag{2.5}$$

$$\bar{S} = \Phi S - \nabla.Z + \bar{\tau}(\cdot)Z. \tag{2.6}$$

We assume that h is nondegenerate so that h defines a semi-Riemannian metric on M. If h is nondegenerate, then we say that the hypersurface or the hypersurface immersion is *nondegenerate*. We have the following

THEOREM 2.2 ([NS, Fundamental equations]). *For an arbitrary transversal vector field C the induced connection ∇, the second fundamental* *form h, the shape operator S, and the 1-form τ satisfy the following equations:*

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY, \tag{2.7}$$

$$(\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z), \tag{2.8}$$

$$(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX, \tag{2.9}$$

$$h(X,SY) - h(SX,Y) = 2d\tau(X,Y). \tag{2.10}$$

The equations (2.7), (2.8), (2.9), and (2.10) are called the equations of Gauss, Codazzi for h, Codazzi for S, and Ricci, respectively.

For a hypersurface immersion $f\colon M \to \mathbb{R}^{n+1}$ a transversal vector field C is said to be *equiaffine* (resp. *locally equiaffine*) if $\tau = 0$ (resp. $d\tau = 0$).

When f is nondegenerate, there exists a canonical transversal vector field C, called the *affine normal*. The affine normal is uniquely determined up to sign by the following conditions: the metric volume form ω_h of h is ∇-parallel and coincides with the induced volume form Θ, where ω_h is defined by $\omega_h(X_1,\dots,X_n) = |\det[h(X_i,X_j)]|^{1/2}$ and Θ is defined by $\Theta(X_1,\dots,X_n) = \det[f_*X_1,\dots,f_*X_n,C]$ for tangent vectors X_i $(i=1,\dots,n)$.

Let $\dim M = 2n+1$ and $f\colon (M,g) \to (\mathbb{R}^{2n+2},\tilde{g})$ be a nondegenerate isometric immersion, where $\tilde{g}$ is the standard inner product on $\mathbb{R}^{2n+2}$. We always assume that $\mathbb{R}^{2m} \simeq \mathbb{C}^m$ is endowed with the standard complex structure J. In particular, if $m = n+1$ we have

$$J(x_1,\dots,x_{n+1},y_1,\dots,y_{n+1}) = (-y_1,\dots,-y_{n+1},x_1,\dots,x_{n+1}).$$

Let C be a transversal vector field on M. We say that C is *J-tangent* if $JC_x \in f_*(T_xM)$ for every $x \in M$. We also define a distribution $\mathcal{D}$ on M as the biggest J invariant distribution on M, that is,

$$\mathcal{D}_x = f_*^{-1}(f_*(T_xM) \cap J(f_*(T_xM)))$$

for every $x \in M$. It is clear that $\dim \mathcal{D} = 2n$. A vector field X is called a *$\mathcal{D}$-field* if $X_x \in \mathcal{D}_x$ for every $x \in M$. We use the notation $X \in \mathcal{D}$ for vectors as well as for $\mathcal{D}$-fields. We define two 1-dimensional distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ by

$$\mathcal{D}_{1x} := \{X \in T_xM\colon g(X,Y) = 0 \ \forall Y \in \mathcal{D}_x\},$$
$$\mathcal{D}_{2x} := \{X \in T_xM\colon h(X,Y) = 0 \ \forall Y \in \mathcal{D}_x\},$$

where h is the second fundamental form on M relative to any transversal vector field. It follows from Theorem 2.1 that the definition of $\mathcal{D}_2$ is independent of the choice of the transversal vector field. We say that the distribution $\mathcal{D}$ is nondegenerate if h is nondegenerate on $\mathcal{D}$. It is easy to see that $\mathcal{D}$ is nondegenerate if and only if $\mathcal{D} \oplus \mathcal{D}_2 = TM$. In this paper we assume that f

as well as D are always nondegenerate. To simplify the writing, we will be omitting f_* in front of vector fields in most cases.

We conclude this section with the following useful lemma relating to differential equations:

LEMMA 2.3. *Let $F\colon I \to \mathbb{R}^{2n}$ be a smooth function on an interval I and let $\alpha, \beta \in C^\infty(I, \mathbb{R})$ be such that $\alpha^2 + \beta^2 \neq 0$ on I. If F satisfies the differential equation*

$$F'(y) = -\alpha(y)JF(y) + \beta(y)F(y), \tag{2.11}$$

then

$$F(y) = Jve^{\hat{\beta}(y)}\cos(\hat{\alpha}(y)) + ve^{\hat{\beta}(y)}\sin(\hat{\alpha}(y)), \tag{2.12}$$

where $v \in \mathbb{R}^{2n}$ and $\hat{\alpha}$, $\hat{\beta}$ are any primitives of α and β on I respectively.

Proof. It is easily seen that functions of the form (2.12) satisfy the differential equation (2.11). On the other hand, since (2.11) is a first order ordinary differential equation, the Picard–Lindelöf theorem implies that any solution of (2.11) must be of the form (2.12). ∎

3. Induced almost contact structures. First, we recall some definitions from [B]. A $(2n+1)$-dimensional manifold M is said to have an *almost contact structure* if there exist on M a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η which satisfy

$$\varphi^2(X) = -X + \eta(X)\xi, \tag{3.1}$$

$$\eta(\xi) = 1 \tag{3.2}$$

for every $X \in TM$. We say that an almost contact structure (φ, ξ, η) is *normal* if

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis tensor for φ.

Let $f\colon M \to \mathbb{R}^{2n+2}$ be a nondegenerate hypersurface with a J-tangent transversal vector field C. Then we can define a vector field ξ, a 1-form η and a tensor field φ of type (1,1) as follows:

$$\xi := JC, \tag{3.3}$$

$$\eta|_{\mathcal{D}} = 0 \quad \text{and} \quad \eta(\xi) = 1, \tag{3.4}$$

$$\varphi|_{\mathcal{D}} = J|_{\mathcal{D}} \quad \text{and} \quad \varphi(\xi) = 0. \tag{3.5}$$

It is easy to see that (φ, ξ, η) is an almost contact structure on M. This structure is called the *almost contact structure on M induced by C*.

For an induced almost contact structure we have the following theorem:

THEOREM 3.1 ([SS]). *If (φ, ξ, η) is an induced almost contact structure on M then the following equations hold:*

$$\eta(\nabla_X Y) = -h(X, \varphi Y) + X(\eta(Y)) + \eta(Y)\tau(X), \tag{3.6}$$

$$\varphi(\nabla_X Y) = \nabla_X \varphi Y + \eta(Y) SX - h(X, Y)\xi, \tag{3.7}$$

$$\eta([X, Y]) = -h(X, \varphi Y) + h(Y, \varphi X) + X(\eta(Y)) - Y(\eta(X)) + \eta(Y)\tau(X) - \eta(X)\tau(Y), \tag{3.8}$$

$$\varphi([X, Y]) = \nabla_X \varphi Y - \nabla_Y \varphi X - \eta(X) SY + \eta(Y) SX, \tag{3.9}$$

$$\eta(\nabla_X \xi) = \tau(X), \tag{3.10}$$

$$\eta(SX) = h(X, \xi), \tag{3.11}$$

for every $X, Y \in \mathcal{X}(M)$.

Normal almost contact structures can be characterized as follows

THEOREM 3.2 ([YI, Th. 3.3]). *The induced almost contact structure (φ, ξ, η) is normal if and only if*

$$S\varphi Z - \varphi S Z + \tau(Z)\xi = 0 \quad \textit{for every } Z \in \mathcal{D}.$$

It is interesting to ask about a necessary and sufficient condition for the affine normal to be J-tangent. An answer is the following:

THEOREM 3.3. *Let $f \colon M \to \mathbb{R}^{2n+2}$ be the Blaschke hypersurface with an affine normal field C. Then C is J-tangent if and only if the Gauss–Kronecker curvature is constant in the direction of the distribution $\mathcal{D}_2$.*

Proof. Let N^0 be the metric normal vector field on M. Then there exist a nonvanishing function Φ on M and a vector field Z such that

$$C = \Phi N^0 + f_* Z.$$

From (2.5) and the fact that C and N^0 are both equiaffine we have

$$h(Z, X) = -X(\Phi) \tag{3.12}$$

for all $X \in \mathcal{X}(M)$. The affine normal C is J-tangent if and only if $Z \in \mathcal{D}$, or equivalently $h(Z, X) = 0$ for every $X \in \mathcal{D}_2$. By (3.12) is equivalent with constancy of Φ in the direction of $\mathcal{D}_2$.

Recall ([NS]) that $\Phi = |K|^{1/(2n+3)}$, where K is the Gauss–Kronecker curvature function on M. So Φ is constant in the direction $\mathcal{D}_2$ if and only if the Gauss–Kronecker curvature is constant along $\mathcal{D}_2$. ∎

4. Special hypersurfaces. Let $f \colon M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with a J-tangent transversal vector field C. The transversal vector field C will be called *special* if there exist smooth functions $\alpha_1, \alpha_2 \in C^\infty(M)$

such that $\alpha_1^2 + \alpha_2^2 \neq 0$ and

$$C_x = -\alpha_1(x)\overrightarrow{of(x)} - \alpha_2(x)J\overrightarrow{of(x)}$$

is J-tangent for every $x \in M$. A hypersurface equipped with a special J-tangent transversal vector field will be called a *special hypersurface.*

It is easy to see that if C is a special transversal vector field for f then for every nonvanishing function Φ on M, a vector field ΦC is also special. It turns out that the converse is also true.

LEMMA 4.1. *Let $f\colon M \to \mathbb{R}^{2n+2}$ be a hypersurface. Assume that C and C' are both special vector fields for f. Then there exists a nonvanishing function Φ on M such that $C' = \Phi C$.*

Proof. Directly by the definition of a special vector field we have

$$C_x, C'_x \in \operatorname{span}\{\overrightarrow{of(x)}, J\overrightarrow{of(x)}\}$$

for every $x \in M$. Since C_x and C'_x are both transversal to $f_*(T_xM)$ and JC_x, JC'_x belong to $f_*(T_xM)$ and moreover

$$JC_x, JC'_x \in \operatorname{span}\{\overrightarrow{of(x)}, J\overrightarrow{of(x)}\},$$

thus there must exist a nonzero constant $\Phi(x)$ such that $C'_x = \Phi(x)C_x$. It is easy to verify that Φ is a smooth function on M. ∎

To avoid nonuniqueness we introduce a notion of a *normalized special vector field*, i.e. a special vector field with the property that $\alpha_1^2 + \alpha_2^2 = 1$. It can be shown that such a vector field is unique up to sign. From now on we shall be considering only hypersurfaces with a special vector field C satisfying $\alpha_1 \neq 0$. Since α_1 is a smooth function, without loss of generality we can assume that $\alpha_1 > 0$. Moreover, if C is a normalized special vector field there exists $\alpha \in C^\infty(M, (-\pi/2, \pi/2))$ such that $\alpha_1 = \cos\alpha$ and $\alpha_2 = \sin\alpha$, so the normalized special vector field can be expressed in the form

$$C_x = -\cos\alpha(x)\,\overrightarrow{of(x)} - \sin\alpha(x)\,J\overrightarrow{of(x)}$$

for every $x \in M$.

The following lemma characterizes the shape operator S and the 1-form τ for a hypersurface with a normalized special vector field.

LEMMA 4.2. *Let $f\colon M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with a normalized special vector field*

$$C = -\cos\alpha\, f - \sin\alpha\, Jf$$

and let (φ, ξ, η) be an almost contact structure induced by C. Then

$$SX = \cos\alpha\, X + \sin\alpha\,\varphi X - X(\alpha)\xi, \tag{4.1}$$

$$\tau(X) = \sin\alpha\,\eta(X), \tag{4.2}$$

for all $X \in \mathcal{X}(M)$.

Proof. The form of C and the fact that $\mathrm{D}\,J=0$ imply that

$$\begin{aligned}\mathrm{D}_X\,C &= -X(\cos\alpha)f-\cos\alpha\,f_*(X)-X(\sin\alpha)Jf-\sin\alpha\,Jf_*(X)\\ &= -\cos\alpha\,f_*(X)-\sin\alpha\,Jf_*(X))+X(\alpha)(\sin\alpha\,f-\cos\alpha\,Jf)\\ &= -\cos\alpha\,f_*(X)-\sin\alpha\,(f_*(\varphi X)-\eta(X)C)+X(\alpha)f_*(\xi)\\ &= -f_*(\cos\alpha\,X+\sin\alpha\,\varphi X-X(\alpha)\xi)+\sin\alpha\,\eta(X)C\end{aligned}$$

for every $X\in\mathcal{X}(M)$. Now using the Weingarten formula (2.2) and comparing the tangent and transversal parts we easily get (4.1) and (4.2). ∎

It is interesting to ask when an induced almost contact structure is normal. The following theorem gives some conditions equivalent to the normality.

THEOREM 4.3. *Let $f\colon M\to\mathbb{R}^{2n+2}$ be an affine hypersurface with a normalized special vector field*

$$C=-\cos\alpha\,f-\sin\alpha\,Jf$$

and let (φ,ξ,η) be an almost contact structure induced by C. Then the following conditions are equivalent:

(i) (φ,ξ,η) *is normal*,

(ii) $\xi\in\mathcal{D}_2$,

(iii) α *is constant in the direction of* $\mathcal{D}$.

Proof. From (4.1) we have

$$S\varphi Z-\varphi SZ=-\varphi Z(\alpha)\xi$$

for all $Z\in\mathcal{D}$. Applying (3.11) to (4.1) we get

$$-\varphi Z(\alpha)=h(\varphi Z,\xi) \tag{4.3}$$

for all $Z\in\mathcal{D}$. The above equalities imply that

$$S\varphi Z-\varphi SZ-h(\varphi Z,\xi)\xi=0 \tag{4.4}$$

for all $Z\in\mathcal{D}$. If (φ,ξ,η) is normal, then (4.2), (4.4) and Theorem 3.2 yield $h(\varphi Z,\xi)=0$ for $Z\in\mathcal{D}$. Thus $\xi\in\mathcal{D}_2$. This completes the proof of (i)⇒(ii). The implication (ii)⇒(iii) easily follows from (4.3). The proof of (iii)⇒(i) can be deduced from Lemma 4.2 and Theorem 3.2. ∎

Formula (4.2) implies that a normalized special vector field is equiaffine if and only if $\alpha\equiv 0$, i.e. is centro-affine. Unfortunately, a change of a special vector field usually does not preserve the equiaffinity. More precisely, if C and C' are both equiaffine and special then there exists constant $a\neq 0$ such that $C'=aC$. The above considerations motivate studying hypersurfaces with a locally equiaffine special vector field, since this property is invariant relative to any change among special vector fields.

Theorem 4.4. *Let* $f\colon M \to \mathbb{R}^{2n+2}$ *be an affine hypersurface with a locally equiaffine normalized special vector field*

$$C = -\cos\alpha\, f - \sin\alpha\, Jf.$$

Then the function α *is constant in the direction of* $\mathcal{D}$. *Moreover, if* $\alpha \equiv 0$ *then* C *is centro-affine, whereas if* $\alpha \neq 0$ *then the distribution* $\mathcal{D}$ *is involutive.*

Proof. Let $X \in \mathcal{D}$. Using (3.11) and (4.1) we get

$$\begin{aligned} h(SX,\xi) &= h(\cos\alpha\, X + \sin\alpha\,\varphi X - X(\alpha)\xi, \xi) \\ &= \cos\alpha\, h(X,\xi) + \sin\alpha\, h(\varphi X,\xi) - X(\alpha)h(\xi,\xi) \\ &= -\cos\alpha\, X(\alpha) - \sin\alpha\,\varphi X(\alpha) - X(\alpha)(\cos\alpha - \xi(\alpha)) \end{aligned}$$

and

$$\begin{aligned} h(X,S\xi) = h(X,\cos\alpha\,\xi - \xi(\alpha)\xi) &= \cos\alpha\, h(X,\xi) - \xi(\alpha)h(X,\xi) \\ &= -\cos\alpha\, X(\alpha) + \xi(\alpha)X(\alpha). \end{aligned}$$

Since C is locally equiaffine the Ricci equation (2.10) and the above equalities imply

$$\cos\alpha\, X(\alpha) + \sin\alpha\,\varphi X(\alpha) = 0 \tag{4.5}$$

for all $X \in \mathcal{D}$. Let Y be any $\mathcal{D}$-field on M. Putting

$$X = \cos\alpha\, Y - \sin\alpha\,\varphi Y$$

in (4.5) we get

$$\begin{aligned} 0 = \cos\alpha\, X(\alpha) + \sin\alpha\,\varphi X(\alpha) = {} & \cos^2\alpha Y(\alpha) - \cos\alpha\sin\alpha\,\varphi Y(\alpha) \\ & + \sin\alpha\cos\alpha\,\varphi Y(\alpha) + \sin^2\alpha Y(\alpha) \\ = {} & Y(\alpha) \end{aligned}$$

for all $Y \in \mathcal{D}$. That is, α is constant in the direction of $\mathcal{D}$. Since $d\tau = 0$, formula (4.2) implies

$$0 = 2d\tau(X,Y) = X(\tau(Y)) - Y(\tau(X)) - \tau([X,Y]) = -\sin\alpha\,\eta([X,Y])$$

for all $X, Y \in \mathcal{D}$. Hence $\alpha \equiv 0$ or if $\alpha \neq 0$ then D is involutive. ∎

From Theorems 4.3 and 4.4 we immediately get

Corollary 4.5. *Let* $f\colon M \to \mathbb{R}^{2n+2}$ *be an affine hypersurface with a special vector field* C. *If* C *is locally equiaffine then an almost contact structure induced by the corresponding normalized special vector field is normal.*

In [C] V. Cruceanu gave a local representation of centro-affine hypersurfaces with a J-tangent centro-affine vector field. More precisely he proved

Theorem 4.6 ([C]). *Let* $f\colon M \to \mathbb{R}^{2n+2}$ *be a centro-affine hypersurface with a* J*-tangent centro-affine vector field. Then there exist an open subset*

$U \subset \mathbb{R}^{2n}$, an interval $I \subset \mathbb{R}$ and an immersion $g\colon U \to \mathbb{R}^{2n+2}$ such that f can be locally expressed in the form

$$f(x_1, \dots, x_{2n}, y) = Jg(x_1, \dots, x_{2n}) \cos y + g(x_1, \dots, x_{2n}) \sin y \tag{4.6}$$

for all $(x_1, \dots, x_{2n}, y) \in U \times I$.

Now we can extend this result as follows:

THEOREM 4.7. *Let $f\colon M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with a locally equiaffine special vector field $C_0 = -\alpha_1 f - \alpha_2 Jf$. If $\alpha_2 \equiv 0$ or $\alpha_2 \neq 0$ then f can be locally expressed in the form*

$$f(x_1, \dots, x_{2n}, y) = e^{A(y)} \big(Jg(x_1, \dots, x_{2n}) \cos(B(y)) \\ + g(x_1, \dots, x_{2n}) \sin(B(y))\big),$$

where $A, B\colon I \to \mathbb{R}$ and $g\colon V \to \mathbb{R}^{2n+2}$ are smooth functions defined on an interval I and an open subset $V \subset \mathbb{R}^{2n}$, respectively.

Proof. Assume that the normalized special field for f has the form $C = -\cos\alpha\, f - \sin\alpha\, Jf$. If $\alpha \equiv 0$ then the result follows from Theorem 4.6. Now we assume that $\alpha \neq 0$. In this case Theorem 4.4 implies that the distribution $\mathcal{D}$ is involutive. Since C is locally equiaffine, it follows from Theorem 4.4 and Lemma 4.2 that $S\varphi = \varphi S$ on $\mathcal{D}$. In particular

$$\xi \in \mathcal{D}_2. \tag{4.7}$$

The Frobenius theorem implies that for every $x \in M$ there exist an open neighborhood $U \subset M$ of x and linearly independent vector fields $X_1, \dots, X_{2n}$, $X_{2n+1} = \xi \in \mathcal{X}(U)$ such that $[X_i, X_j] = 0$ for $i, j = 1, \dots, 2n+1$. For every $i = 1, \dots, 2n$ we have

$$X_i = D_i + \alpha_i \xi,$$

where $D_i \in \mathcal{D}$ and $\alpha_i \in C^\infty(U)$. Thus we have

$$0 = [X_i, \xi] = [D_i, \xi] - \xi(\alpha_i)\xi. \tag{4.8}$$

Now (3.8), (4.2) and (4.7) imply that $[D_i, \xi] = 0$ and $\xi(\alpha_i) = 0$ for $i = 1, \dots, 2n$. We also have

$$\begin{aligned} 0 = [X_i, X_j] = [D_i + \alpha_i\xi, D_j + \alpha_j\xi] &= [D_i, D_j] + [D_i, \alpha_j\xi] \\ &\quad + [\alpha_i\xi, D_j] + [\alpha_i\xi, \alpha_j\xi] \\ &= [D_i, D_j] - D_j(\alpha_i)\xi + D_i(\alpha_j)\xi, \end{aligned}$$

where the last equality follows from (4.8). Since $\mathcal{D}$ is involutive, the above equality implies $[D_i, D_j] = 0$ for $i, j = 1, \dots, 2n$. Of course the vector fields $D_1, \dots, D_{2n}, \xi$ are linearly independent, so we can find a map $\psi(x_1, \dots, x_{2n}, y)$ on U such that $\partial/\partial y = \xi$ and $\partial/\partial x_i = D_i$ for $i = 1, \dots, 2n$. In particular f satisfies the differential equation

$$f_y = -\cos\alpha\, Jf + \sin\alpha\, f.$$

We obtain (using Lemma 2.3)

$$f = e^{\widehat{\sin\alpha}}\big(Jg\cos(\widehat{\cos\alpha}) + g\sin(\widehat{\cos\alpha})\big),$$

where $\widehat{\sin\alpha}$ and $\widehat{\cos\alpha}$ are any primitives relative to y variable of $\sin\alpha$ and $\cos\alpha$ respectively. Since α is constant in the direction of $\mathcal{D}$ (Theorem 4.4) and $\{\partial/\partial x_i\}$ span $\mathcal{D}$, the function α does not depend on $x_1,\dots,x_{2n}$. Now we can set $A := \widehat{\sin\alpha}$ and $B := \widehat{\cos\alpha}$. ∎

The next theorem characterizes special centro-affine hypersurfaces with an involutive distribution $\mathcal{D}$.

THEOREM 4.8. *Let $f\colon M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with a centro-affine J-tangent vector field. The distribution $\mathcal{D}$ is involutive if and only if for every $x \in M$ there exists a Kählerian immersion $g\colon V \to \mathbb{R}^{2n+2}$ defined on an open subset $V \subset \mathbb{R}^{2n+2}$ such that f can be expressed in the neighborhood of x in the form*

$$f(x_1,\dots,x_{2n},y) = Jg(x_1,\dots,x_{2n})\cos y + g(x_1,\dots,x_{2n})\sin y.$$

Proof. Let C be a centro-affine J-tangent vector field for f and let (φ,ξ,η) be the almost contact structure induced by C. Assume that $\mathcal{D}$ is involutive. Since $\tau = 0$ and $\xi \in \mathcal{D}_2$ for every $x \in M$, we can find (in a similar way to the proof of Theorem 4.7) a neighborhood U of x and a map $\psi(x_1,\dots,x_{2n},y)$ on U such that $\partial/\partial y = \xi$ and $\partial/\partial x_i \in \mathcal{D}$ for $i = 1,\dots,2n$. Now applying Lemma 2.3 we find that f can be expressed locally in the form

$$f(x_1,\dots,x_{2n},y) = Jg(x_1,\dots,x_{2n})\cos y + g(x_1,\dots,x_{2n})\sin y,$$

where $g\colon V \to \mathbb{R}^{2n+2}$ is an immersion defined on an open subset $V \subset \mathbb{R}^{2n}$. Moreover, we have

$$f_{x_i} = Jg_{x_i}\cos y + g_{x_i}\sin y \in f_*(\mathcal{D})$$

for $i = 1,\dots,2n$. Since $f_*(D)$ is J-invariant we also have

$$Jf_{x_i} = -g_{x_i}\cos y + Jg_{x_i}\sin y \in f_*(\mathcal{D})$$

for $i = 1,\dots,2n$. The above formulas imply that

$$Jg_{x_i} \in \operatorname{span}\{g_{x_1},\dots,g_{x_{2n}}\}$$

for $i = 1,\dots,2n$, that is, g is a Kählerian immersion. To prove the converse note that

$$f_y = -Jf,$$

i.e. the centro-affine vector field $-\overrightarrow{of}$ is J-tangent. It is sufficient to prove that $\mathcal{D}$ is involutive. Consider the fields $f_{x_1},\dots,f_{x_{2n}}$ and $Jf_{x_1},\dots,Jf_{x_{2n}}$.

Since g is Kählerian, we have

$$Jg_{x_i} = \sum_{j=1}^{2n} \alpha_{ij} g_{x_j},$$

and consequently

$$\begin{aligned}\sum_{j=1}^{2n} \alpha_{ij} f_{x_j} &= \sum_{j=1}^{2n} \alpha_{ij} J g_{x_j} \cos y + \sum_{j=1}^{2n} \alpha_{ij} g_{x_j} \sin y \\ &= J^2 g_{x_i} \cos y + J g_{x_i} \sin y = J f_{x_i}\end{aligned}$$

for $i = 1, \ldots, 2n$. The last formula implies that the space spanned by $f_{x_1}, \ldots, f_{x_{2n}}$ is J-invariant. We also have $\dim \operatorname{span}\{f_{x_1}, \ldots, f_{x_{2n}}\} = 2n$. Therefore, it is the largest J-invariant subspace of $f_*(TM)$. Thus the vector fields $\{\partial/\partial x_i\}_{i=1,\ldots,2n}$ span the distribution $\mathcal{D}$, which completes the proof. ∎

Theorem 3.3 gives a necessary and sufficient condition for the affine normal to be J-tangent. Now using this theorem we will prove that in the case of centro-affine hypersurfaces with a J-tangent centro-affine vector field the affine normal is always J-tangent.

THEOREM 4.9. *Let $f\colon M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with a centro-affine J-tangent vector field. Then the affine normal is J-tangent.*

Proof. It follows from Theorem 4.6 that there exist an open subset $U \subset \mathbb{R}^{2n}$ and an interval $I \subset \mathbb{R}$ such that f can be locally expressed in the form

$$f(x_1, \ldots, x_{2n}, y) = Jg(x_1, \ldots, x_{2n}) \cos y + g(x_1, \ldots, x_{2n}) \sin y,$$

on $U \times I$, where $g\colon U \to \mathbb{R}^{2n+2}$ is an immersion. Since f is an immersion, we also see that Jg is linearly independent of $g_{x_1}, \ldots, g_{x_{2n}}$. Thus there exists a function $W\colon U \to \mathbb{R}^{2n+2}$ such that

$$W \cdot g_{x_i} = 0, \qquad W \cdot Jg = 0, \qquad \|W\| = 1,$$

where $\cdot$ and $\|\cdot\|$ are the standard inner product and the standard norm on $\mathbb{R}^{2n+2}$, respectively. We define a new function $N\colon U \times I \to \mathbb{R}^{2n+2}$ by the formula

$$N(x_1, \ldots, x_{2n}, y) = JW(x_1, \ldots, x_{2n}) \cos y + W(x_1, \ldots, x_{2n}) \sin y.$$

It is not difficult to see that $\|N\| = 1$, $N \cdot f_{x_i} = 0$ for $i = 1, \ldots, 2n$ and $N \cdot f_y = 0$. This means that N is a metric normal for f. The equality $W \cdot W = 1$ implies that $W_{x_i} \cdot W = 0$. Thus we have

$$W_{x_i} \in \operatorname{span}\{g_{x_1}, \ldots, g_{x_{2n}}, Jg\}$$

for every $i = 1, \ldots, 2n$. Also for every $i = 1, \ldots, 2n$ we can find functions

α_{ij}, $j = 1, \ldots, 2n+1$, (independent of y) such that

$$W_{x_i} = \sum_{j=1}^{2n} \alpha_{ij} g_{x_j} + \alpha_{i,2n+1} Jg.$$

Using the above formula we have

$$\begin{aligned} N_{x_i} &= JW_{x_i} \cos y + W_{x_i} \sin y \\ &= \sum_{j=1}^{2n} \alpha_{ij} (Jg_{x_j} \cos y + g_{x_j} \sin y) + \alpha_{i,2n+1}(-g \cos y + Jg \sin y) \\ &= \sum_{j=1}^{2n} \alpha_{ij} f_{x_j} - \alpha_{i,2n+1} f_y. \end{aligned}$$

Since $JW \cdot W = 0$, we also have

$$JW \in \operatorname{span}\{g_{x_1}, \ldots, g_{x_{2n}}, Jg\},$$

so there exist functions $\beta_1, \ldots, \beta_{2n+1}$ (independent of y) such that

$$JW = \sum_{j=1}^{2n} \beta_j g_{x_j} + \beta_{2n+1} Jg.$$

The last equality implies

$$\begin{aligned} N_y &= -JW \sin y + W \cos y \\ &= -\sum_{j=1}^{2n} \beta_j (g_{x_j} \sin y + Jg_{x_j} \cos y) + \beta_{2n+1}(-Jg \sin y + g \cos y) \\ &= -\sum_{j=1}^{2n} \beta_j f_{x_j} + \beta_{2n+1} f_y. \end{aligned}$$

In this way, we have calculated that the Gauss–Kronecker curvature K for f can be locally expressed in the form

$$K = \det \begin{bmatrix} -\alpha_{11} & \cdots & -\alpha_{2n,1} & \beta_1 \\ \vdots & \vdots & \vdots & \vdots \\ -\alpha_{1,2n} & \cdots & -\alpha_{2n,2n} & \beta_{2n} \\ \alpha_{1,2n+1} & \cdots & \alpha_{2n,2n+1} & -\beta_{2n+1} \end{bmatrix}.$$

Since α_{ij} as well as β_i do not depend on y, the curvature K does not depend on y either. But $\partial/\partial y \in \mathcal{D}_2$ and consequently the Gauss–Kronecker curvature for f is constant in the direction of $\mathcal{D}_2$. The theorem now follows from Theorem 3.3. ∎

Here is an example of an affine J-tangent hypersphere.

EXAMPLE 4.10. Let us consider the affine immersion defined by

$$f\colon \mathbb{R}^3 \ni (x,y,z) \mapsto \begin{bmatrix} \sin x \sinh y \\ -\cos x \sinh y \\ \cos x \cosh y \\ \sin x \cosh y \end{bmatrix} \cos z + \begin{bmatrix} \cos x \cosh y \\ \sin x \cosh y \\ -\sin x \sinh y \\ \cos x \sinh y \end{bmatrix} \sin z \in \mathbb{R}^4$$

with the transversal vector field

$$C\colon \mathbb{R}^3 \ni (x,y,z) \mapsto -f(x,y,z) \in \mathbb{R}^4.$$

It is not difficult to see that C is J-tangent. Moreover, we have

$$\tau = 0, \qquad S = \mathrm{id},$$

and

$$h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in the canonical base on $\mathbb{R}^3$. It can be shown that C is the affine normal vector field, so f is a J-tangent affine sphere.

The next example shows that not every affine hypersurface with a centro-affine J-tangent vector field must be an affine sphere.

EXAMPLE 4.11. Let an affine immersion be defined as follows:

$$f\colon (0,\infty)^2 \times \mathbb{R} \ni (x,y,z) \mapsto \begin{bmatrix} -xy \\ 0 \\ x \\ y \end{bmatrix} \cos z + \begin{bmatrix} x \\ y \\ xy \\ 0 \end{bmatrix} \sin z \in \mathbb{R}^4.$$

It is easy to see that

$$C\colon (0,\infty)^2 \times \mathbb{R} \ni (x,y,z) \mapsto -f(x,y,z) \in \mathbb{R}^4$$

is a transversal centro-affine J-tangent vector field. Obviously

$$\tau = 0, \qquad S = \mathrm{id}\,.$$

We can also compute that

$$h = \begin{bmatrix} 0 & \frac{1}{xy} & -\frac{y^2+1}{xy} \\ \frac{1}{xy} & 0 & \frac{1}{y^2} \\ -\frac{y^2+1}{xy} & \frac{1}{y^2} & 1 \end{bmatrix}$$

in the canonical base $\{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$ on $(0,\infty)^2 \times \mathbb{R}$, so f is nondegenerate. Moreover, by straightforward computations we find that the vector fields

$$X := y^2 \frac{\partial}{\partial y} - \frac{\partial}{\partial z}, \qquad Y := JX = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$

span the distribution $\mathcal{D}$. It follows that $h(X,X) = -1$, $h(X,Y) = h(Y,X) = -2y$ and $h(Y,Y) = -y^2 + 2$, so h is nondegenerate on $\mathcal{D}$. We also have

$$\theta = \frac{x^2y^4}{\sqrt{3y^2+2}} \cdot \omega_h,$$

thus C is not the affine normal (it is not even proportional to it).

References

[B] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progr. Math. 203, Birkhäuser, 2010.

[C] V. Cruceanu, *Real hypersurfaces in complex centro-affine spaces*, Results Math. 13 (1988), 224–234.

[NS] K. Nomizu and T. Sasaki, *Affine Differential Geometry*, Cambridge Univ. Press, 1994.

[SS] M. Szancer and Z. Szancer, *Real hypersurfaces with an induced almost contact structure*, Colloq. Math. 114 (2009), 41–51.

[YI] K. Yano and S. Ishihara, *Real hypersurfaces of a complex manifold and distributions with complex structure*, Kodai Math. J. 1 (1978), 289–303.

Zuzanna Szancer
Department of Applied Mathematics
University of Agriculture in Kraków
253c Balicka Street
30-198 Kraków, Poland
E-mail: Zuzanna.Szancer@ur.krakow.pl

Received 29.10.2011
and in final form 21.3.2012 (2587)