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Reduction theorems for the Strong Real Jacobian Conjecture

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Abstract. Implementations of known reductions of the Strong Real Jacobian Conjecture (SRJC), to the case of an identity map plus cubic homogeneous or cubic linear terms, and to the case of gradient maps, are shown to preserve significant algebraic and geometric properties of the maps involved. That permits the separate formulation and reduction, though not so far the solution, of the SRJC for classes of nonsingular polynomial endomorphisms of real n-space that exclude the Pinchuk counterexamples to the SRJC, for instance those that induce rational function field extensions of a given fixed odd degree.

1. Introduction. The Jacobian Conjecture (JC) [1, 11] asserts that a polynomial map $F : k^n \to k^n$, where k is a field of characteristic zero, has a polynomial inverse if it is a *Keller map* [16], which means that its Jacobian determinant, $j(F)$, is a nonzero element of k. The JC is still not settled for any $n > 1$ and any specific field k of characteristic zero. It is well known that it would suffice to prove the JC for $k = \mathbb{R}$ and all positive n. There are many generalizations to endomorphisms of $\mathbb{R}^n$ [20, 18]. The most natural is the Strong Real Jacobian Conjecture (SRJC), which asserts that a polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$ has a real analytic inverse if it is *nonsingular*, meaning that $j(F)$, whether constant or not, vanishes nowhere on $\mathbb{R}^n$. However, Sergey Pinchuk [19] exhibited a family of counterexamples for $n = 2$. They are also counterexamples to the Rational Real Jacobian Conjecture (RRJC) [7], which is the extension of the SRJC to include everywhere defined rational nonsingular endomorphisms. Everywhere defined means that each component of the map can be expressed as the quotient of two polynomials with a nowhere vanishing denominator. Any such $F : \mathbb{R}^n \to \mathbb{R}^n$ has finite fibers of size at most the degree of the associated finite algebraic extension of rational function fields.

Let dex denote that extension degree, mfs the maximum fiber size, and sag the size of the automorphism group of the extension. While dex and mfs

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are not generally equal, they are always of the same parity. The conditions dex odd, mfs odd, and sag = 1 are all necessary for invertibility, and mfs = 1 or dex = 1 is sufficient. All the Pinchuk counterexamples satisfy dex = 6, mfs = 2, and sag = 1 [5, 6]. Thus the simplest unproved and unrefuted challenge conjecture in this arena is that dex = 3 and sag = 1 is sufficient in the polynomial case.

Two known reduction procedures for the SRJC, to the case of maps of cubic homogeneous, or even better cubic linear, type [15, 1, 8, 11] and to the case of a symmetric Jacobian matrix [17], are shown here to preserve the numerical attributes dex, mfs, and sag.

In consequence, the two reductions can be applied to conjectures involving those attributes, such as the above challenge conjecture.

2. Background. There are two classic reductions of the ordinary JC to Yagzhev maps [15, 1] and to Drużkowski maps [8]. A *Yagzhev map* is a polynomial map of the form $F = X + H$, where $X = (x_1, \ldots, x_n)$, and each component of H is a cubic homogeneous polynomial in the variables $x_1, \ldots, x_n$. Yagzhev maps are also called *maps of cubic homogeneous type*. A *Drużkowski map* (or *map of cubic linear type*) is a Yagzhev map for which the components of H are cubes of linear forms ($h_i = l_i^3$). In a departure from the convention in some other works, these definitions impose no restriction on $j(F)$, beyond the obvious $j(F)(0) = 1$. Note, however, that a Yagzhev map $F = X + H$ is a Keller map if, and only if, H has a Jacobian matrix, $J(H)$, that is nilpotent, since both assertions are just different ways of saying that the formal power series matrix inverse of $J(F)$ is polynomial.

Reduction theorem proofs use the strategy of transforming an original map into a map of the desired form in a succession of steps that preserve the truth value of certain key properties (and typically increase the number of variables).

For the JC, $\mathbb{C}$ is usually selected as the ground field, and the key properties are the Keller property and the existence of a polynomial inverse. Such proofs then apply over any ground field of characteristic zero, including $\mathbb{R}$. But the strategy and specific steps can be applied more generally than just to polynomial Keller maps and yields, for instance, a reduction of the SRJC to the cubic linear case [8]. Drużkowski noted this explicitly, with the preserved properties being a nowhere vanishing Jacobian determinant and bijectivity.

Historical note. At the 1997 conference in Lincoln, Nebraska, to honor the mathematical work of Gary H. Meisters, it was suggested by T. Parthasarathy that the SRJC reduction be attempted for the 1994 counterexample of Pinchuk. The challenge was taken up by Engelbert Hubbers, and in 1999 he demonstrated the existence of a counterexample to the SRJC of cubic linear type, coincidentally in dimension 1999. He started with exactly the specific Pinchuk map of total degree 25 circulated by Arno van den Essen in June 1994, which can be found in [11]. He then used a computer algebra system to verify a human guided reduction path to a Yagzhev map in dimension 203, then explicitly computed a Gorni–Zampieri pairing [12] to a Drużkowski map in dimension 1999, using sparse matrix representations as necessary. These details are excerpted from a comprehensive unpublished note by Hubbers, which he made available.

Remark. Drużkowski obviously did not use GZ pairing, since it was unknown at the time. But it also preserves the same two key properties in the SRJC context.

More recently, reductions of the ordinary JC to the symmetric case have been considered, primarily over $\mathbb{R}$ and $\mathbb{C}$. Let k denote a field of characteristic zero. In the JC world a polynomial map $F : k^n \to k^n$ is often called *symmetric*, in a startling abuse of language, if $J(F)$ is a symmetric matrix. In that case, F is the gradient map of a polynomial function $h : k^n \to k$ and $J(F)$ is the Hessian matrix of second order partial derivatives of h. So in the symmetric case, the JC becomes the Hessian Conjecture (HC), namely that gradient maps of polynomials with constant nonzero Hessian determinant have polynomial inverses. In [17], Guowu Meng proves the equivalence of the JC and the HC, using what he refers to as a trick. Meng's trick replaces a map $F = (f_1, \dots, f_n)$ in the variables $x_1, \dots, x_n$ by the map in the $2n$ variables $y_1, \dots, y_n, x_1, \dots, x_n$ obtained by taking the gradient of the scalar function $y_1 f_1 + \dots + y_n f_n$. For $k = \mathbb{R}$, this construction works even for twice continuously differentiable maps. In the SRJC context it provides a one step reduction to the symmetric case that also preserves the Keller property in both directions.

In [2], Michiel de Bondt and Arno van den Essen prove a more targeted reduction over $\mathbb{C}$, namely to symmetric Keller–Yagzhev maps. The reduction process involves the use of $\sqrt{-1}$, and if applied to a real Keller map may yield a Yagzhev map that is not real. Interestingly, it has been shown that all complex symmetric Keller–Drużkowski maps have polynomial inverses [3, 9].

3. Stable and Segre equivalence. Two maps, F and G, from a topological space A to another one B, are called *topologically equivalent* if $F = h_B \circ G \circ h_A$, where h_A and h_B are homeomorphisms, respectively of A to itself and of B to itself. In other words, F and G are the same map up to coordinate changes in the domain and codomain by topological automorphisms. *Topological stable equivalence* for the set of all maps $F : \mathbb{R}^n \to \mathbb{R}^n$ in all dimensions $n > 0$ is the equivalence relation generated by (1) topological equivalences, and (2) the equivalence of any map $F = (f_1, \dots, f_n)$, and its extension by fresh variables to $G = (f_1, \dots, f_n, x_{n+1}, \dots, x_m)$ for any $m > n$. There are many other types of stable equivalence, such as real-analytic or polynomial, each characterized by the type of automorphisms allowed for (global) coordinate changes. *Stable equivalence*, unqualified, will refer to the least restrictive, purely set-theoretic, type, with all bijections allowed as automorphisms.

For brevity, call $F : \mathbb{R}^n \to \mathbb{R}^n$

(1) *nondegenerate* if $j(F)$ is not identically zero,
(2) *nonsingular* if $j(F) \neq 0$ everywhere, and
(3) a *Keller map* if $j(F)$ is a nonzero constant.

These terms are meant to imply that $J(F)$, the Jacobian matrix of F, exists at every point of $\mathbb{R}^n$, and can be applied to any such F if the corresponding restriction on $j(F)$ is satisfied. For polynomial stable equivalence, the applicable automorphisms are polynomial maps with polynomial inverses, making it obvious that such equivalence preserves each of the above three properties. All preservation properties in this paper apply equally well in both directions. Polynomial stable equivalence also clearly preserves each of the properties of being everywhere defined rational, polynomial, injective, surjective, or bijective. If the maps are nonsingular, it preserves the existence of a rational or polynomial inverse.

To verify this last assertion in the case of extension by fresh variables, one checks the Jacobian matrices to see that the appropriate part of an inverse in the larger number of variables is independent of the fresh variables and restricts to an inverse in the smaller number of variables.

The slightly more general and less familiar concept of *birational stable equivalence* allows the use of automorphisms that are everywhere defined rational maps with everywhere defined rational inverses. By inspection of the arguments in the polynomial case, one sees easily that birational stable equivalence has all the preservation properties listed above for the polynomial case, except that it need not preserve polynomial maps, polynomial inverses, or the Keller property.

If $F = (f_1, \dots, f_n)$ is an everywhere defined rational map and is nondegenerate, then its components are algebraically independent over $\mathbb{R}$ in the field $\mathbb{R}(X)$ of rational functions in the coordinate variables $X = x_1, \dots, x_n$, and so they generate a subfield $\mathbb{R}(F) \subseteq \mathbb{R}(X)$ over $\mathbb{R}$, that is also a rational function field in n variables over $\mathbb{R}$. Even without nonsingularity, the extension $\mathbb{R}(X)/\mathbb{R}(F)$ permits the definition of dex and sag as in the introduction. The extension degree $d = \text{dex}$ is finite and equal to the degree of the minimal polynomial over $\mathbb{R}(F)$ of any $h \in \mathbb{R}(X)$ that is primitive, meaning that h generates $\mathbb{R}(X)$ as a field over $\mathbb{R}(F)$. For such an h, the powers h^i for $i = 0, \ldots, d-1$ are a basis for $\mathbb{R}(X)$ as a vector space over $\mathbb{R}(F)$. An automorphism of the extension is, by definition, a field automorphism of $\mathbb{R}(X)$ that fixes every element of $\mathbb{R}(F)$. So it is linear over $\mathbb{R}(F)$, and a multiplicative homomorphism, hence completely determined by its value on h. That value must be a root of the minimal polynomial of h and must lie in $\mathbb{R}(X)$, and any such root determines a unique automorphism of the extension. Thus sag is the number of such roots, which is therefore the same for any choice of h.

In another relaxation of assumptions, it suffices to assume that F is an everywhere defined nondegenerate rational map and an open map in order to conclude that it is quasifinite and that the maximum fiber size is at most dex.

The main concern here is the case of everywhere defined rational nonsingular maps, for which all the assertions in the introduction are proved in [7].

THEOREM 1. *Assume $F : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^m \to \mathbb{R}^m$ are birationally stably equivalent. If either one is an everywhere defined rational nonsingular map, then so is the other, and each of the numerical attributes* dex, sag, *and* mfs *has the same value for both maps.*

Proof. Only the equality of the numerical attributes needs checking. And it needs to be checked only for the generating equivalences.

Suppose first that $m > n$ and $G = (F, Z)$, with Z a list of fresh variables. For any $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^{m-n}$, the fiber of F over y is the same size as the fiber of G over (y, z), so mfs is preserved. A primitive element for $\mathbb{R}(X)/\mathbb{R}(F)$ is clearly also a primitive element for $\mathbb{R}(X, Z)/\mathbb{R}(G)$. Since the tensor product over $\mathbb{R}$ with $\mathbb{R}(Z)$ is an exact functor, the associated power basis for the first extension is also one for the second, and so dex is preserved. An automorphism of the extension $\mathbb{R}(X, Z)/\mathbb{R}(G)$ is determined by the image of the primitive element. That image is a root of the minimal polynomial for the primitive element. That polynomial has coefficients independent of the fresh variables in Z. On a Zariski open subset of $\mathbb{R}^m$, where the root is a real-analytic function of the coefficients, the root is independent of the fresh variables. So the first order partials with respect to those variables are identically zero and the root lies in $\mathbb{R}(X)$. Thus the automorphism is uniquely the natural lift of an automorphism of $\mathbb{R}(X)/\mathbb{R}(F)$, and so sag is preserved.

Second, suppose that $m = n$ and $G = A \circ F \circ B$, with A and B everywhere defined birational automorphisms. Viewing them as coordinate changes makes it clear that mfs is preserved. It suffices to consider further only the special cases (i) $G = A \circ F$ and (ii) $G = F \circ B$, and A, B and their inverses do not need to be defined everywhere.

In case (i), A induces an automorphism of $\mathbb{R}(F)$, so $\mathbb{R}(F)$ and $\mathbb{R}(A \circ F)$ are the same subfield of $\mathbb{R}(X)$, hence the two extensions are the same and so have the same properties. In case (ii) the two extensions are $\mathbb{R}(X)/\mathbb{R}(F)$ and $\mathbb{R}(X)/\mathbb{R}(F \circ B)$, which are generally different extensions. Take a primitive element h for the first extension, then apply the automorphism of $\mathbb{R}(X)$ induced by B to h, its minimal polynomial over $\mathbb{R}(F)$, and the roots of that polynomial in $\mathbb{R}(X)$. The image of h is primitive over $\mathbb{R}(F \circ B)$, the new polynomial is irreducible there, and the roots of the two polynomials correspond. So dex and sag are preserved. ∎

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map satisfying $F(0) = 0$. Then $H(x,t) = (1/t)F(tx)$ is polynomial and provides, for $0 \le t \le 1$, a homotopy between the linear part of F and F itself.

This *Segre homotopy* [21] can be generalized in many ways, e.g. to the case of a complex map or parameter t and to analytic or rational maps, not to mention formal and convergent power series. It is used here to define the concept of *Segre equivalence* on the set of real-analytic endomorphisms of $\mathbb{R}^n$ ($n > 0$) that fix 0. It is the equivalence relation generated by declaring $F : \mathbb{R}^n \to \mathbb{R}^n$ equivalent to $G : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $G(x,t) = (F(tx)/t, t)$. In that case, for $t \neq 0$ consideration of the Jacobian matrix of G shows that $j(G)(x,t) = j(F)(tx)$, a result that then also holds for $t = 0$, by continuity. So Segre equivalence preserves nondegeneracy, nonsingularity, and the Keller property. Again all preservation properties apply in both directions. It also preserves polynomial maps and everywhere defined rational maps, because $G(x,1) = (F(x),1)$. For $t \neq 0$, the set $G^{-1}(y,t)$ is $\{(x/t,t) \mid x \in F^{-1}(ty)\}$ for any $y \in \mathbb{R}^n$. This implies that injectivity and surjectivity are preserved provided that G is bijective on the set of points $(x,0)$, a condition equivalent to $j(F)(0) \neq 0$. In particular, for bijective nonsingular F one has $G^{-1}(y,t) = (F^{-1}(ty)/t, t)$, and so polynomial and everywhere defined rational inverses are also preserved.

THEOREM 2. *Assume $F : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^m \to \mathbb{R}^m$ are real-analytic maps sending* 0 *to* 0 *and that they are Segre equivalent. If either one is an everywhere defined rational nonsingular map, then so is the other, and each of the numerical attributes* dex, sag, *and* mfs *has the same value for both maps.*

Proof. Only the equality of the attributes needs checking, and only for the case $G(x,t) = (F(tx)/t, t)$.

Fibers over points $(y,0)$ are of size 1 and for $t \neq 0$ the fiber of G over (y,t) has the same size as the fiber of F over ty, by the formula given above for the set $G^{-1}(y,t)$. So mfs is preserved.

Now consider the automorphism of the field $\mathbb{R}(X,t)$ that sends x_i to tx_i and t to itself. It restricts to an isomorphism of $\mathbb{R}(F,t)$ onto $\mathbb{R}(G)$.

This is just an instance of the special case (ii) in the proof of the previous theorem. So dex and sag have the same value for G as for (F, t), and hence, by the previous theorem, as for F. ■

The *coimage* of a map $F : \mathbb{R}^n \to \mathbb{R}^n$ is the set $\mathbb{R}^n \setminus F(\mathbb{R}^n)$ of points in the codomain that are not in the image of F. In the complex JC context, it is well known that the coimage has complex codimension at least two. Briefly, the reasoning is as follows. Since the coimage is closed and constructible, if it has codimension less than two it contains an irreducible hypersurface $h = 0$, $h \circ F$ vanishes nowhere and so is constant, contradicting the algebraic independence of the components of F. In the SRJC and RRJC contexts, there are no parallel results for the real codimension of the coimage, even if the map has dense image. The Pinchuk maps, however, do have finite coimages, which are indeed of codimension two in $\mathbb{R}^2$.

THEOREM 3. *Assume $F : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^m \to \mathbb{R}^m$ are everywhere defined rational nonsingular maps and that they are birationally stably or Segre equivalent. Then the codimension of the coimage is the same for both maps.*

Proof. Only the Segre case is not totally trivial, and in the base Segre case the coimage of G consists of the point $(a/t, t)$ for a in the coimage of F and $t \neq 0$. ■

4. Gorni–Zampieri pairing. Let $f_i = x_i + l_i^3$ be the components of a map F of cubic linear type in dimension n. It is customary to write F in the compact form $F(x) = x + (Ax)^{*3}$, where A is the matrix of coefficients of the linear forms, and the exponent indicates componentwise cubing. Let G be a map of cubic homogeneous type in dimension $m < n$. A *GZ pairing* between G and F is given by two matrices B and C, respectively of sizes $m \times n$ and $n \times m$, satisfying $BC = I$, $\ker B = \ker A$, and $G(x) = BF(Cx)$ for all $x \in \mathbb{R}^m$. The original definition [12] writes F as $F(x) = x - (Ax)^{*3}$, but the different sign affects only some formulas not used here.

THEOREM 4. *If G and F are GZ paired, then they are polynomially stably equivalent.*

Proof. Note that $\ker C = 0$, $\operatorname{Im} B = \mathbb{R}^m$, and that $\mathbb{R}^n$ is the direct sum of $\operatorname{Im} C$ and $\ker B$. Choose a linear isomorphism D from $\mathbb{R}^{n-m}$ to $\ker B$. Let E be its inverse. Consider the extension of G by fresh variables to $G' = (g_1, \ldots, g_m, z_{m+1}, \ldots, z_n)$. Let $C'(x, z) = Cx + D(z) \in \mathbb{R}^n$ and $B'(x) = (Bx, E'(x))$, where E' is the linear extension of E to $\mathbb{R}^n$ that is 0 on $\operatorname{Im} C$.

Note that both B' and C' are linear automorphisms of $\mathbb{R}^n$. Observe that $F(Cx + D(z)) = Cx + D(z) + (ACx)^{*3}$. Consequently, $(B' \circ F \circ C')(x, z) = \big(G(x), z + E'((ACx)^{*3})\big) = G' \circ (x, z + H(x))$, where H is cubic homogeneous.

Since $(x, z + H(x))$ has the obvious inverse $(x, z - H(x))$, it follows that G and F are polynomially stably equivalent. ∎

Remarks. The same reasoning works over any ground field k. There is also nothing special about the use of 3 as the exponent. All works just as well for power homogeneous and power linear maps of the same degree $d > 1$.

In a GZ pairing the rank of the map of cubic linear type, meaning the rank of the coefficient matrix A, is the same as the dimension m of the map of cubic homogeneous type. In [10] the SRJC is proved for all maps of cubic linear type and rank 2. The heart of the proof is a theorem proving that the SRJC is true for all maps of cubic homogeneous type in dimension 2. These facts are of interest when considering structured counterexamples in higher dimensions.

Remarks. For reasons not clear to me, [10] presents the results mentioned above for maps with an everywhere positive Jacobian determinant, which is automatically true for nonsingular maps of cubic homogeneous type. Other results include the SRJC for all maps of cubic linear type in dimension 3.

The dimension 2 results were later [13] improved to cover polynomial maps with components of degree at most 3, and then [4] to polynomial maps with one component of degree at most 3.

5. Main results

Theorem 5. *There is an algorithm that transforms a nondegenerate, polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$ into a map $G : \mathbb{R}^m \to \mathbb{R}^m$ of cubic homogeneous type, where m is generally much larger than n, using polynomially stable equivalences and a single Segre equivalence.*

Proof. In each step below, a map F is replaced by a map G, which becomes the new F for the next step. At each step, both F and G are nondegenerate, since that property is preserved by the equivalences.

Step 1: *Lower the degree.* Suppose $F = (f_1, \ldots, f_n)$. Then F is polynomially stably equivalent to $(f_1 - (y + a)(z + b), f_2, \ldots, f_n, y + a, z + b)$, where a, b are polynomials that depend only on $x_1, \ldots, x_n$. Thus, if a term of f_1 has the form ab, with $\deg(a) > 1$ and $\deg(b) > 1$, it can be removed at the cost of introducing two new variables and some terms of degree less than $\deg(ab)$. Repeating this for terms of maximum degree until there are no more maximal degree terms of the specified form in any component, one finally obtains a polynomial map G (in a generally much higher dimension) all of whose terms are of degree no more than three. This is a standard algorithm [1, 11]. There is flexibility in the choice of term to remove next, and one can opportunistically remove a product ab that is not a single term, making choices to reach a cubic map more quickly. This step is a polynomial stable equivalence.

Step 2: *Normalize.* F is now cubic and (still) nondegenerate. Let n be the current dimension. Choose $x_0 \in \mathbb{R}^n$ with $j(F)(x_0) \neq 0$. After suitable translations, $(J(F)(x_0))^{-1}F$ becomes a cubic map G such that $G(0) = 0$ and $G'(0) = J(G)(0)$ is the identity matrix I. This step is an affine (in the vector space sense) equivalence.

Step 3: *Segre equivalence.* Now $F = X + Q + C$, where Q and C are, respectively, the quadratic and cubic homogeneous components of F. Let t be a new variable, and put $G = (X + tQ + t^2C, t)$. This is a polynomial Segre equivalence, as defined previously.

Step 4: *Final step.* Now $F = (X + tQ + t^2C, t)$, with Q quadratic homogeneous and C cubic homogeneous, and both independent of t. Define two polynomial automorphisms A_1, A_2 in X, Y, t, where Y is a sequence of n additional variables, by $A_1 = (X - t^2Y, Y, t)$ and $A_2 = (X, Y + C, t)$. Then $G = A_1 \circ (X + tQ + t^2C, Y, t) \circ A_2$ is the map of cubic homogeneous type $(X - t^2Y + tQ, Y + C, t)$. This step is a polynomial stable equivalence. ∎

The theorem and proof are valid over $\mathbb{C}$ as well as over $\mathbb{R}$, and, indeed, more generally for Keller maps. All proofs of reduction to cubic homogeneous type start with reduction to degree 3, followed by elimination of the quadratic terms. The given proof most closely follows that of Drużkowski in [8], which explicitly allows for nonconstant Jacobian determinants.

The main point of the given proof is that for nonsingular polynomial maps, by the preservation results previously proved, the reduction preserves (in both directions) not only bijectivity, but also dex, mfs, sag, the Keller property, and the codimension of the coimage.

So if it is applied to a Pinchuk map, it yields a Yagzhev map G for which $j(G)$ is not constant and $J(G)$ is not unipotent. Up to inessential details, Hubbers follows the above steps in the first part of his 1999 reduction, obtaining a cubic map in dimension $n = 101$ and then a map of cubic homogeneous type in dimension $2n + 1 = 203$. Hubbers' Yagzhev map in dimension 203 is thus not Keller and satisfies dex $= 6$, mfs $= 2$, sag $= 1$, and has a coimage of codimension 2.

A further reduction to a map of cubic linear type can be effected using the method of Drużkowski in [8] or the method of GZ pairing developed by Gianluca Gorni and Gaetano Zampieri in [12]. Since GZ pairing has been shown to be a polynomial stable equivalence, Hubbers' final Drużkowski map in dimension 1999 has the same properties as those stated for his Yagzhev map in dimension 203.

Theorem 6. *Any nonsingular $\mathcal{C}^2$ map $F : \mathbb{R}^n \to \mathbb{R}^n$ is stably equivalent to a $\mathcal{C}^1$ nonsingular map $G : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with a symmetric Jacobian matrix. The equivalence is birationally stable if F is everywhere defined rational, and polynomially stable if F is a polynomial Keller map.*

Proof. This is Meng's trick with a reordering of the variables. Let $z = (x, v)$ be any point of $\mathbb{R}^{2n}$, with $x = (x_1, \ldots, x_n)$ and $v = (x_{n+1}, \ldots, x_{2n})$. Let $h = x \cdot F(v)$, where the dot denotes the standard inner product of n-vectors. Let G be the gradient of the scalar function h. Then G has a symmetric Jacobian matrix and $G = (F(v), x \cdot J(F)(v))$, with the dot now denoting a vector matrix product and $J(F)$ the Jacobian matrix of F. But $(F(v), x \cdot J(F)(v)) = (F(x), v \cdot J(F)(x)) \circ (v, x)$ and $(F(x), v \cdot J(F)(x)) = (F(x), v) \circ (x, v \cdot J(F)(x))$. Since $(x, v \cdot J(F)(x))$ has the $\mathcal{C}^1$ inverse $(x, v \cdot J(F)^{-1}(x))$, the composition $A = (x, v \cdot J(F)(x)) \circ (v, x)$ is a $\mathcal{C}^1$ automorphism. Moreover, if F is an everywhere defined rational or polynomial Keller map, it is clear that A has the claimed properties. ∎

This theorem reduces the entire RRJC, not just the SRJC, to the case of a symmetric Jacobian matrix and preserves dex, sag, and mfs.

It is natural to attempt to combine the two main results by applying Theorem 6 to a Yagzhev map. The resulting map is polynomial, with only its linear and cubic homogeneous components nonzero. But its linear part is not the identity.

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