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Existence of mild solutions for fractional evolution equations with nonlocal initial conditions

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Abstract. This paper discusses the existence of mild solutions for a class of semilinear fractional evolution equations with nonlocal initial conditions in an arbitrary Banach space. We assume that the linear part generates an equicontinuous semigroup, and the nonlinear part satisfies noncompactness measure conditions and appropriate growth conditions. An example to illustrate the applications of the abstract result is also given.

1. Introduction. In this paper, we discuss the nonlocal Cauchy problem

$$ {}^{c}D^{q}u(t) + Au(t) = f(t, u(t)), \quad t \in J = [0, 1], \tag{1.1} $$

$$ u(0) = \sum_{k=1}^{p} c_k u(t_k) + u_0, \tag{1.2} $$

where ${}^{c}D^{q}$ is the Caputo fractional derivative of order $0 < q < 1$, $-A : D(A) \subset E \to E$ is the infinitesimal generator of a uniformly bounded equicontinuous C_0-semigroup $T(t)$ ($t \geq 0$) on E, $f : J \times E \to E$ is given function satisfying some assumptions, u_0 is an element of the Banach space E, $0 < t_1 < \cdots < t_p < 1$, $p \in \mathbb{N}$, c_k are real numbers, $c_k \neq 0$, $k = 1, \ldots, p$.

Since it has been demonstrated that differential equations involving fractional derivatives in time yield more realistic descriptions of many phenomena in nature than those of integer order in time, the study of fractional differential equations has become an object of extensive study during recent years (see [1], [8], [11], [14], [17]–[19], [25], [26], [29], [30] and the references therein).

On the other hand, nonlocal initial conditions can be applied in physics with better effect than the classical initial condition $u(0) = u_0$. For example,

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in [10], Deng used the nonlocal condition (1.2) to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, condition (1.2) allows additional measurements at t_k, $k = 1, \dots, p$, which is more precise than the measurement just at $t = 0$. In [7], Byszewski pointed out that if $c_k \neq 0$, $k = 1, \dots, p$, then the results can be applied in kinematics to determine the location evolution $t \mapsto u(t)$ of a physical object for which we do not know the positions $u(0), u(t_1), \dots, u(t_p)$, but we know that the nonlocal condition (1.2) holds. The importance of nonlocal conditions have also been discussed in [3]–[6], [15], [21]–[23], [27], [28].

In some articles, fractional evolution equations with nonlocal initial conditions were treated under the hypothesis that the semigroup $T(t)$ $(t \geq 0)$ generated by $-A$ is compact, i.e., the operator $T(t)$ is compact for any $t > 0$ (see Zhou and Jiao [29] and [30], Wang et al. [26] and Diagana et al. [11]). In applications, these results are very convenient for partial differential equations with compact resolvent. However, to the best of our knowledge, no works yet exist for fractional nonlocal problems with noncompact semigroup. In this paper, we are interested in the case where $-A$ generates a uniformly bounded equicontinuous semigroup. Under suitable noncompactness measure conditions and growth conditions on the nonlinear term f, we prove the existence of mild solutions for the fractional nonlocal problem (1.1)–(1.2).

2. Preliminaries. Let E be a Banach space with norm $\|\cdot\|$. We denote by $C(J,E)$ the Banach space of all continuous E-valued functions on the interval J with the norm $\|u\|_C = \max_{t\in J}\|u(t)\|$ and by $L^1(J,E)$ the Banach space of all E-valued Bochner integrable functions defined on J with the norm $\|u\|_1 = \int_0^1 \|u(t)\|\,dt$.

DEFINITION 2.1 ([18]). The *fractional integral* of order $q > 0$ with lower limit 0 for a function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\,ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

DEFINITION 2.2 ([18]). The *Caputo fractional derivative* of order $q > 0$ with the lower limit 0 for a function f is defined as

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)\,ds, \quad t > 0,\ 0 \leq n-1 < q < n.$$

If f is an abstract function with values in E, then the integrals appearing in Definitions 2.1 and 2.2 are taken in Bochner's sense.

For $u \in E$, define two operators $\mathcal{T}(t)$ $(t \geq 0)$ and $\mathcal{S}(t)$ $(t \geq 0)$ by

$$\mathcal{T}(t)u = \int_0^\infty h_q(s)T(t^q s)u\,ds, \qquad \mathcal{S}(t)u = q\int_0^\infty sh_q(s)T(t^q s)u\,ds,$$

where

$$h_q(s) = \frac{1}{\pi q}\sum_{n=1}^{\infty}(-s)^{n-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \qquad s \in (0,\infty),$$

is a function of Wright type defined on $(0,\infty)$ which satisfies

$$h_q(s) \geq 0, \quad s \in (0,\infty), \qquad \int_0^\infty h_q(s)\,ds = 1,$$

and

$$\int_0^\infty s^v h_q(s)\,ds = \frac{\Gamma(1+v)}{1+qv}, \qquad v \in [0,1].$$

Let $M = \sup_{t\in[0,\infty)} \|T(t)\|_{\mathcal{L}(E)}$, where $\mathcal{L}(E)$ stands for the Banach space of all bounded linear operators on E. The following lemma follows from the results in [12] and [13].

Lemma 2.3. *The operators $\mathcal{T}(t)$ $(t \geq 0)$ and $\mathcal{S}(t)$ $(t \geq 0)$ have the following properties:*

(1) *For any fixed $t \geq 0$, $\mathcal{T}(t)$ and $\mathcal{S}(t)$ are bounded linear operators, i.e., for any $u \in E$,*

$$\|\mathcal{T}(t)u\| \leq M\|u\|, \qquad \|\mathcal{S}(t)u\| \leq \frac{qM}{\Gamma(1+q)}\|u\|.$$

(2) *For every $u \in E$, $t \mapsto \mathcal{T}(t)u$ and $t \mapsto \mathcal{S}(t)u$ are continuous functions from $[0,\infty)$ into E.*

(3) *$\mathcal{T}(t)$ $(t \geq 0)$ and $\mathcal{S}(t)$ $(t \geq 0)$ are strongly continuous, which means that for all $u \in E$ and $0 \leq t' < t'' \leq 1$, we have*

$$\|\mathcal{T}(t'')u - \mathcal{T}(t')u\| \to 0 \quad \text{and} \quad \|\mathcal{S}(t'')u - \mathcal{S}(t')u\| \to 0 \quad \text{as } t'' \to t'.$$

Throughout this paper, we assume that

$$\sum_{k=1}^{p} |c_k| < \frac{1}{M}. \tag{2.1}$$

Thus, $\|\sum_{k=1}^{p} c_k\mathcal{T}(t_k)\| \leq M\sum_{k=1}^{p}|c_k| < 1$. By the operator spectral theorem, we know that the operator

$$\mathcal{B} := \Big(I - \sum_{k=1}^{p} c_k\mathcal{T}(t_k)\Big)^{-1} \tag{2.2}$$

exists, is bounded and $D(\mathcal{B}) = E$. Furthermore, by Neumann's formula, $\mathcal{B}$ can be expressed as

$$\mathcal{B} = \sum_{n=0}^{\infty} \Big(\sum_{k=1}^{p} c_k \mathcal{T}(t_k) \Big)^n. \tag{2.3}$$

Therefore

$$\begin{aligned} \|\mathcal{B}\| &\le \sum_{n=0}^{\infty} \Big\| \sum_{k=1}^{p} c_k \mathcal{T}(t_k) \Big\|^n = \frac{1}{1 - \| \sum_{k=1}^{p} c_k \mathcal{T}(t_k) \|} \\ &\le \frac{1}{1 - M \sum_{k=1}^{p} |c_k|}. \end{aligned} \tag{2.4}$$

DEFINITION 2.4. A function $u \in C(J, E)$ is said to be a *mild solution* of the problem (1.1)–(1.2) if it satisfies

$$\begin{aligned} u(t) = \mathcal{T}(t)\mathcal{B}u_0 &+ \sum_{k=1}^{p} c_k \mathcal{T}(t)\mathcal{B} \int_0^{t_k} (t_k - s)^{q-1} \mathcal{S}(t_k - s) f(s, u(s))\, ds \\ &+ \int_0^t (t - s)^{q-1} \mathcal{S}(t - s) f(s, u(s))\, ds. \end{aligned} \tag{2.5}$$

Since no confusion can occur, we denote by $\alpha(\cdot)$ the Kuratowski measure of noncompactness on both E and $C(J, E)$. For the details of the definition and properties of the measure of noncompactness, we refer to the monographs [2] and [9].

The following lemmas will be used in the proofs of our main results.

LEMMA 2.5 ([2]). *Let E be a Banach space, and $D \subset C(J, E)$ be bounded and equicontinuous set. Then $\alpha(D(t))$ is continuous on J, and*

$$\alpha(D) = \max_{t \in J} \alpha(D(t)) = \alpha(D(J)).$$

LEMMA 2.6 ([16]). *Let E be a Banach space, and $D = \{u_n\} \subset C(J, E)$ be a bounded and countable set. Then $\alpha(D(t))$ is Lebesgue integrable on J, and*

$$\alpha\Big(\Big\{ \int_J u_n(t)\, dt \;\Big|\; n \in \mathbb{N} \Big\} \Big) \le 2 \int_J \alpha(D(t))\, dt.$$

LEMMA 2.7 ([20]). *Let E be a Banach space, and $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$ such that $\alpha(D) \le 2\alpha(D_0)$.*

Proof. We give the proof for the convenience of the reader. Without loss of generality, we assume that $\alpha(D) > 0$. Let $r_n = (1 - 1/2^n)\alpha(D)$; then $0 < r_n < \alpha(D)$. Choose $x_1^{(n)} \in D$; then $D \setminus B(x_1^{(n)}, r_n/2) \ne \emptyset$: indeed, if $D \subset B(x_1^{(n)}, r_n/2)$, then by the definition of noncompactness measure, $\alpha(D) \le r_n$, which is a contradiction.

Choose $x_2^{(n)} \in D \setminus B(x_1^{(n)}, r_n/2)$; then similarly, $D \setminus \big(B(x_1^{(n)}, r_n/2) \cup B(x_2^{(n)}, r_n/2)\big) \neq \emptyset$. Therefore, we can choose $x_3^{(n)} \in D \setminus (B(x_1^{(n)}, r_n/2) \cup B(x_2^{(n)}, r_n/2))$. Continuing, we obtain a sequence $\{x_k^{(n)} \mid k = 1, 2, \ldots\}$ such that $x_{k+1}^{(n)} \in D \setminus \bigcup_{i=1}^{k} B(x_i^{(n)}, r_n/2)$, $k = 1, 2, \ldots$. Let $D_n = \{x_k^{(n)} \mid k = 1, 2, \ldots\}$. Combining this with the definition of noncompactness measure, we find that $\alpha(D_n) \geq r_n/2$. Let $D_0 = \bigcup_{n=1}^{\infty} D_n$; then D_0 is a countable set. Since $\alpha(D_0) \geq \alpha(D_n) \geq r_n/2 \to \alpha(D)/2\,(n \to \infty)$, we conclude that $\alpha(D) \leq 2\alpha(D_0)$. ■

LEMMA 2.8 ([9]). *Let E be a Banach space, $D \subset E$ be a bounded closed and convex subset, and $Q : D \to D$ be condensing. Then Q has a fixed point in D.*

LEMMA 2.9. *For $\sigma \in (0, 1]$ and $0 < a \leq b$, we have*

$$|a^\sigma - b^\sigma| \leq (b - a)^\sigma.$$

For any $R > 0$, let

$$\Omega_R = \{u \in C(J, E) \mid \|u(t)\| \leq R,\ t \in J\};$$

then Ω_R is a bounded closed and convex set in $C(J, E)$.

3. Main result. To state the main result, we introduce the following hypotheses:

(H1) $f : J \times E \to E$ is such that $f(\cdot, u)$ is measurable for all $u \in E$, $f(t, \cdot)$ is continuous for each $t \in J$, and there exist a constant $q_1 \in [0, q)$ and a function $m \in L^{1/q_1}(J, \mathbb{R}^+)$ such that $\|f(t, u)\| \leq m(t)$ for all $u \in E$ and $t \in J$.

(H2) There exists a constant $L > 0$ with
$$L < \frac{\Gamma(1 + q)(1 - M \sum_{k=1}^{p} |c_k|)}{4M}$$
such that for any bounded $D \subset E$,
$$\alpha(f(t, D)) \leq L(\alpha(D)) \quad \text{for any } t \in J.$$

THEOREM 3.1. *If the hypotheses* (H1) *and* (H2) *are satisfied, then the problem* (1.1)–(1.2) *has a mild solution.*

Proof. We consider the operator F on $C(J, E)$ defined by

$$(Fu)(t) = \mathcal{T}(t)\mathcal{B}u_0 + \sum_{k=1}^{p} c_k \mathcal{T}(t)\mathcal{B} \int_0^{t_k} (t_k - s)^{q-1} \mathcal{S}(t_k - s) f(s, u(s))\, ds + \int_0^{t} (t - s)^{q-1} \mathcal{S}(t - s) f(s, u(s))\, ds, \quad t \in J. \tag{3.1}$$

By direct calculation, we find that F is well defined. From Definition 2.4, it is easy to see that the fixed points of F are the mild solutions of problem (1.1)–(1.2). In the following, we will prove that F has a fixed point by applying Sadovskiĭ's famous fixed point theorem.

First, we prove that F is continuous on $C(J,E)$. To this end, let $\{u_n\}_{n=1}^{\infty} \subset C(J,E)$ be a sequence such that $\lim_{n\to\infty} u_n = u$ in $C(J,E)$. By the continuity of f with respect to the second variable, for each $s \in J$ we have $\lim_{n\to\infty} f(s,u_n(s)) = f(s,u(s))$. Therefore,

$$\sup_{s\in J} \|f(s,u_n(s)) - f(s,u(s))\| \to 0 \quad \text{as } n\to\infty. \tag{3.2}$$

On the other hand, for $t \in J$, we have

$$\begin{aligned}
&\|(Fu_n)(t) - (Fu)(t)\| \\
&\quad\le \frac{M\sum_{k=1}^{p}|c_k|}{1-M\sum_{k=1}^{p}|c_k|}\,\frac{qM}{\Gamma(1+q)}\int_0^{t_k}(t_k-s)^{q-1}\|f(s,u_n(s))-f(s,u(s))\|\,ds \\
&\qquad + \frac{qM}{\Gamma(1+q)}\int_0^{t}(t-s)^{q-1}\|f(s,u_n(s))-f(s,u(s))\|\,ds \\
&\quad\le \frac{M}{\Gamma(1+q)(1-M\sum_{k=1}^{p}|c_k|)}\sup_{s\in J}\|f(s,u_n(s))-f(s,u(s))\|,
\end{aligned}$$

which implies that

$$\|(Fu_n)-(Fu)\|_C \le \frac{M}{\Gamma(1+q)(1-M\sum_{k=1}^{p}|c_k|)}\sup_{s\in J}\|f(s,u_n(s))-f(s,u(s))\|.$$

From (3.2), we infer that

$$\|(Fu_n)-(Fu)\|_C \to 0 \quad \text{as } n\to\infty.$$

That is, F is continuous on $C(J,E)$.

Next, we prove that there exists a positive constant R_0 such that $F(\Omega_{R_0}) \subset \Omega_{R_0}$. In fact, choose

$$R_0 = \frac{M}{1-M\sum_{k=1}^{p}|c_k|}\left[\|u_0\| + \frac{qN}{\Gamma(1+q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\right],$$

where $N = \|m\|_{L^{1/q_1}[0,1]}$. For any $u \in \Omega_{R_0}$, we have

$$\begin{aligned}
\|(Fu)(t)\| &\le M\|\mathcal{B}\|\cdot\|u_0\| + M\sum_{k=1}^{p}|c_k| \\
&\qquad\times \|\mathcal{B}\|\frac{qM}{\Gamma(1+q)}\int_0^{t_k}(t_k-s)^{q-1}\|f(s,u(s))\|\,ds \\
&\qquad+ \frac{qM}{\Gamma(1+q)}\int_0^{t}(t-s)^{q-1}\|f(s,u(s))\|\,ds
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{M\|u_0\|}{1-M\sum_{k=1}^{p}|c_k|}+\frac{M\sum_{k=1}^{p}|c_k|}{1-M\sum_{k=1}^{p}|c_k|}\\
&\quad\times \frac{qM}{\Gamma(1+q)}\Big(\int_0^{t_k}(t_k-s)^{\frac{q-1}{1-q_1}}\,ds\Big)^{1-q_1}\|m\|_{L^{1/q_1}[0,t_k]}\\
&\quad+\frac{qM}{\Gamma(1+q)}\Big(\int_0^{t}(t-s)^{\frac{q-1}{1-q_1}}\,ds\Big)^{1-q_1}\|m\|_{L^{1/q_1}[0,t]}\\
&\le \frac{M}{1-M\sum_{k=1}^{p}|c_k|}\bigg[\|u_0\|+\frac{qN}{\Gamma(1+q)}\Big(\frac{1-q_1}{q-q_1}\Big)^{1-q_1}\bigg]=R_0.
\end{aligned}
$$

Therefore, $F(\Omega_{R_0})\subset\Omega_{R_0}$.

Now, we demonstrate that $F(\Omega_{R_0})$ is equicontinuous. For any $u\in\Omega_{R_0}$ and $0\le t_1<t_2\le 1$, we get

$$
\begin{aligned}
(Fu)(t_2)-(Fu)(t_1)&=\mathcal{T}(t_2)\mathcal{B}u_0-\mathcal{T}(t_1)\mathcal{B}u_0\\
&\quad+\sum_{k=1}^{p}c_k(\mathcal{T}(t_2)-\mathcal{T}(t_1))\mathcal{B}\int_0^{t_k}(t_k-s)^{q-1}\mathcal{S}(t_k-s)f(s,u(s))\,ds\\
&\quad+\int_{t_1}^{t_2}(t_2-s)^{q-1}\mathcal{S}(t_2-s)f(s,u(s))\,ds\\
&\quad+\int_0^{t_1}((t_2-s)^{q-1}-(t_1-s)^{q-1})\mathcal{S}(t_2-s)f(s,u(s))\,ds\\
&\quad+\int_0^{t_1}(t_1-s)^{q-1}(\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s))f(s,u(s))\,ds\\
&=: I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

It is obvious that

$$
\|(Fu)(t_2)-(Fu)(t_1)\|\le\sum_{i=1}^{5}\|I_i\|.
$$

Now, we only need to check $\|I_i\|\to 0$ independently of $u\in\Omega_{R_0}$ when $t_2\to t_1$, $i=1,\dots,5$.

By Lemma 2.3(3), $\|I_1\|\to 0$ as $t_2\to t_1$.

By Lemma 2.3, (2.4), the Hölder inequality, and the equicontinuity of the semigroup $T(t)$ $(t\ge 0)$, we have

$$
\|I_2\|\le\frac{qM\sum_{k=1}^{p}|c_k|\cdot\|\mathcal{B}\|}{\Gamma(1+q)}\int_0^{t_k}(t_k-s)^{q-1}m(s)\,ds\,\|\mathcal{T}(t_2)-\mathcal{T}(t_1)\|
$$

$$\leq \frac{qMN\sum_{k=1}^{p}|c_k|\cdot\|\mathcal{B}\|}{\Gamma(1+q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\|\mathcal{T}(t_2)-\mathcal{T}(t_1)\|$$
$$\to 0 \quad \text{as } t_2\to t_1.$$

For I_3, by Lemma 2.3 and the Hölder inequality,

$$\|I_3\| \leq \frac{qMN(t_2-t_1)^{q-q_1}}{\Gamma(1+q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1} \to 0 \quad \text{as } t_2\to t_1.$$

For I_4, by elementary computation, using Lemmas 2.3 and 2.9 and the Hölder inequality,

$$\|I_4\| \leq \frac{qM}{\Gamma(1+q)}\int_0^{t_1}\big((t_1-s)^{q-1}-(t_2-s)^{q-1}\big)m(s)\,ds \leq \frac{qM}{\Gamma(1+q)}$$
$$\times\Big(\int_0^{t_1}\big((t_1-s)^{q-1}-(t_2-s)^{q-1}\big)^{\frac{1}{1-q_1}}\,ds\Big)^{1-q_1}\|m\|_{L^{1/q_1}[0,t_1]}$$
$$\leq \frac{qMN}{\Gamma(1+q)}\Big(\int_0^{t_1}\big((t_1-s)^{\frac{q-1}{1-q_1}}-(t_2-s)^{\frac{q-1}{1-q_1}}\big)\,ds\Big)^{1-q_1}$$
$$\leq \frac{qMN}{\Gamma(1+q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\big(t_1^{\frac{q-q_1}{1-q_1}}-t_2^{\frac{q-q_1}{1-q_1}}+(t_2-t_1)^{\frac{q-q_1}{1-q_1}}\big)^{1-q_1}$$
$$\leq \frac{qMN}{\Gamma(1+q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}(2(t_2-t_1))^{q-q_1}$$
$$\to 0 \quad \text{as } t_2\to t_1.$$

For $t_1=0$, $0<t_2\leq 1$, it is easy to see that $\|I_5\|=0$. For $t_1>0$ and $\epsilon>0$ small enough, by the equicontinuity of the semigroup $T(t)$ $(t\geq 0)$ we have

$$\|I_5\| \leq \int_0^{t_1-\epsilon}(t_1-s)^{q-1}\|\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\|\cdot\|f(s,u(s))\|\,ds$$
$$+\int_{t_1-\epsilon}^{t_1}(t_1-s)^{q-1}\|\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\|\cdot\|f(s,u(s))\|\,ds$$
$$\leq \sup_{s\in[0,t_1-\epsilon]}\|\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\|\int_0^{t_1-\epsilon}(t_1-s)^{q-1}m(s)\,ds$$
$$+\frac{2qM}{\Gamma(1+q)}\int_{t_1-\epsilon}^{t_1}(t_1-s)^{q-1}m(s)\,ds$$

$$\leq \sup_{s\in[0,t_1-\epsilon]} \|\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\| \cdot N\big(t_1^{\frac{q-q_1}{1-q_1}} - \epsilon^{\frac{q-q_1}{1-q_1}}\big)^{1-q_1}$$
$$\times \left(\frac{1-q_1}{q-q_1}\right)^{1-q_1} + \frac{2qMN}{\Gamma(1+q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\epsilon^{q-q_1}$$
$$\to 0 \quad \text{as } t_2 \to t_1.$$

As a result, $\|(Fu)(t_2)-(Fu)(t_1)\|$ tends to zero independently of $u \in \Omega_{R_0}$ as $t_2 \to t_1$, which means that $F(\Omega_{R_0})$ is equicontinuous.

Next we prove that $F : \Omega_{R_0} \to \Omega_{R_0}$ is a condensing operator. For any $D \subset \Omega_{R_0}$, by Lemma 2.7, there exists a countable set $D_1 = \{u_n\} \subset D$ such that

$$\alpha(F(D)) \leq 2\alpha(F(D_1)). \tag{3.3}$$

Since $F(D_1) \subset F(\Omega_{R_0})$ is equicontinuous, by Lemma 2.5, $\alpha(F(D_1)) = \max_{t\in J}\alpha(F(D_1)(t))$. From (3.1), using Lemma 2.6 and assumption (H2), we obtain

$$\begin{aligned}
&\alpha(F(D_1)(t))\\
&\quad= \alpha\Big(\Big\{\mathcal{T}(t)\mathcal{B}u_0 + \sum_{k=1}^{p} c_k\mathcal{T}(t)\mathcal{B}\int_0^{t_k}(t_k-s)^{q-1}\mathcal{S}(t_k-s)f(s,u_n(s))\,ds\\
&\qquad+\int_0^t (t-s)^{q-1}\mathcal{S}(t-s)f(s,u_n(s))\,ds\Big\}\Big)\\
&\quad\leq \frac{2\sum_{k=1}^p |c_k|\cdot\|\mathcal{B}\|qM^2}{\Gamma(1+q)}\int_0^{t_k}(t_k-s)^{q-1}\alpha(\{f(s,u_n(s))\})\,ds\\
&\qquad+\frac{2qM}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\alpha(\{f(s,u_n(s))\})\,ds\\
&\quad\leq \frac{2qM^2L\sum_{k=1}^p |c_k|}{\Gamma(1+q)(1-M\sum_{k=1}^p |c_k|)}\int_0^{t_k}(t_k-s)^{q-1}\alpha(D_1(s))\,ds\\
&\qquad+\frac{2qML}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\alpha(D_1(s))\,ds\\
&\quad\leq \frac{2ML}{\Gamma(1+q)(1-M\sum_{k=1}^p |c_k|)}\alpha(D).
\end{aligned}$$

Therefore, from (3.3) and assumption (H2), we deduce that

$$\alpha(F(D)) \leq \frac{4ML}{\Gamma(1+q)(1-M\sum_{k=1}^p |c_k|)}\alpha(D) < \alpha(D).$$

Thus, $F : \Omega_{R_0} \to \Omega_{R_0}$ is a condensing operator. From Lemma 2.8, we infer that F has at least one fixed point in Ω_{R_0}, which is just a mild solution of the problem (1.1)–(1.2). ∎

REMARK. Analytic semigroups and differentiable semigroups are equicontinuous semigroups [24]. In the application to partial differential equations, such as parabolic and strongly damped wave equations, the corresponding solution semigroups are analytic semigroups. Therefore, Theorem 3.1 in this paper has broad applicability.

4. An example. Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded domain with smooth boundary $\partial\Omega$, $E = L^2(\Omega)$. Consider the following fractional parabolic partial differential equation with nonlocal initial condition:

$$(4.1)\qquad \begin{cases} \dfrac{\partial^q}{\partial t^q} u(x,t) + A(x,D)u(x,t) = f(x,t,u(x,t)), & x \in \Omega, t \in J, \\ D^\alpha u(x,t) = 0, \quad (x,t) \in \partial\Omega \times J,\ |\alpha| \leq m, & \\ u(x,0) = \displaystyle\sum_{k=1}^{p} c_k u(x,t_k) + u_0(x), & x \in \Omega, \end{cases}$$

where $\partial^q/\partial t^q$ is the Caputo fractional partial derivative of order $0 < q < 1$, $A(x,D)u = \sum_{|\alpha|\leq 2m} a_\alpha(x) D^\alpha u$ is a strongly elliptic operator, the coefficient function $a_\alpha(x)$ is in $C^{2m}(\overline{\Omega})$, $J = [0,1]$, $0 < t_1 < \cdots < t_p < 1$, c_k are real numbers, $c_k \neq 0$, $k = 1, \ldots, p$, $f : \Omega \times J \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exist a constant $q_1 \in [0,q)$ and a function $m \in L^{1/q_1}(J, \mathbb{R}^+)$ such that $|f(x,t,u(x,t))| \leq m(t)$ for all $x \in \Omega$, $u \in \mathbb{R}$ and $t \in J$; finally, the partial derivative $f'_u(x,t,u)$ is continuous on any bounded domain.

We define an operator A by $Au = A(x,D)u$ with domain

$$D(A) = H^{2m}(\Omega) \cap H_0^m(\Omega).$$

From [24, Theorem 7.2.7], we know that $-A$ generates a uniformly bounded equicontinuous C_0-semigroup $T(t)$ $(t \geq 0)$ on E. Let $f(t,u(t)) = f(t.u(\cdot,t))$, $u_0 = u_0(\cdot)$. Then the problem (4.1) can be rewritten in the form of (1.1)–(1.2).

Let $\sum_{k=1}^{p} |c_k| < 1/M$, where $M = \sup_{t\in[0,\infty)} \|T(t)\|_{\mathcal{L}(E)}$. From the properties of f, it is easy to see that the hypotheses (H1) and (H2) are satisfied. By using Theorem 3.1, the problem (4.1) has a mild solution $u \in C([0,1], L^2(\Omega))$.

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