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On existence of a unique generalized solution to systems of elliptic PDEs at resonance

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Abstract. The Dirichlet boundary value problem for systems of elliptic partial differential equations at resonance is studied. The existence of a unique generalized solution is proved using a new min-max principle and a global inversion theorem.

1. Introduction. In this paper, we consider the following Dirichlet boundary value problem:

$$Lu + g(x,u) = h(x), \quad x \text{ in } \Omega, \qquad u = 0, \quad x \text{ on } \partial\Omega, \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary, $g : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous and continuously differentiable with respect to u, and $h \in L^2(\Omega)$.

By a global inversion theorem and a non-variational version of the min-max principle, the existence of a unique generalized solution to (1.1) has been proved for $L = \Delta$ in [QL]. The main contribution of this paper is that the existence of a unique solution of (1.1) at resonance can be derived for much more general operators L.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega$ of class C^2, and L be a second order differential operator of the form

$$Lu = \begin{pmatrix} L_1 u_1 \\ L_2 u_2 \\ \vdots \\ L_m u_m \end{pmatrix}, \qquad L_k\varphi = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^{(k)}(x)\frac{\partial \varphi}{\partial x_j}\right), \qquad k = 1,\dots,m,$$

where the operators L_k, $k = 1,\dots,m$, are assumed to be strongly elliptic

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and symmetric, i.e.,

$$a_{ij}^{(k)} = a_{ji}^{(k)}, \quad \exists \mu^{(k)} > 0, \quad \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)}\zeta_i\zeta_j > \sum_{i=1}^{n}\sum_{j=1}^{n} \mu^{(k)}\zeta_i\zeta_j$$

for all $x \in \bar{\Omega}$ and all $\zeta \in \mathbb{R}^n \setminus \{0\}$.

In this paper, we suppose that $a_{ij}^{(k)} \in C^1(\bar{\Omega})$ and

$$A_k = (a_{ij}^{(k)})_{n\times n} \in \mathbb{R}^{n\times n}, \quad A = \begin{pmatrix} A_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_m \end{pmatrix} \in \mathbb{R}^{nm\times nm}.$$

It is easy to see that A_k, $k = 1, \dots, m$, are symmetric matrices. Thus, A is also symmetric. Moreover, L_k, $k = 1, \dots, m$, induce a self-adjoint differential operator in $L^2(\Omega)$ with domain $H_0^1(\Omega) \cap H^2(\Omega)$. Let $0 < \lambda_1^{(k)} < \lambda_2^{(k)} < \cdots$ be all different eigenvalues of the eigenvalue problem $-L_k\varphi = \lambda\varphi$ in Ω with $\varphi = 0$ on $\partial\Omega$, $k = 1, \dots, m$ (see, e.g., Gilbarg and Trudinger [GT]).

Now, we first introduce the following two lemmas which play an important role in this paper. Their detailed proofs are omitted; the interested readers are referred to [L1, LL, L2, L],

Lemma 1.1 ([L1, LL]). *Let X and Y be two closed subspaces of a real Hilbert space H, and $H = X \oplus Y$. Suppose that $T : H \to H$ is a C^1 mapping. If there exist continuous functions $\alpha, \beta : [0,\infty) \to (0,\infty)$ such that*

$$\langle T'(u)v, v\rangle \le -\alpha(\|u\|)\|v\|^2, \tag{1.2}$$

$$\langle T'(u)w, w\rangle \ge \beta(\|u\|)\|w\|^2, \tag{1.3}$$

$$\langle T'(u)v, w\rangle = \langle v, T'(u)w\rangle, \tag{1.4}$$

for all $u \in H$, $v \in X$, $w \in Y$, and

$$\int_1^\infty \min\{\alpha(s), \beta(s)\}\, ds = \infty, \tag{1.5}$$

then T is a diffeomorphism from H onto H.

Lemma 1.2 ([L2, L]). *Let H be a vector space such that $H = Z \oplus Y$ for some subspaces Y and Z. If Z is finite-dimensional and X is a subspace of H such that $X \cap Y = \{0\}$ and $\dim X = \dim Z$, then $H = X \oplus Y$.*

2. Existence of unique solution. To show the existence of a unique solution to equation (1.1), we assume throughout this section that the following condition holds.

(C) $\partial g(x,u)/\partial u$ is symmetric and there exist continuous functions $\alpha, \beta : [0,\infty) \to (0,\infty)$ and constant symmetric $m \times m$ matrices B_1 and B_2 such that

$$B_1 + \alpha(\|u\|)I \le \frac{\partial g(x,u)}{\partial u} \le B_2 - \beta(\|u\|)I, \tag{2.1}$$

on $\mathbb{R}^n \times \mathbb{R}^m$ and the eigenvalues of B_1 and B_2 are $\lambda_{N_k}^{(k)}$ and $\lambda_{N_k+1}^{(k)}$, $k = 1, \ldots, m$, respectively. Here $\lambda_{N_k}^{(k)}$ and $\lambda_{N_k+1}^{(k)}$ are two consecutive eigenvalues of $-L_k\varphi = \lambda\varphi$ in Ω with $\varphi = 0$ on $\partial\Omega$, $k = 1, \ldots, m$. Moreover

$$\int_1^\infty \min\{\alpha(s), \beta(s)\}\, ds = \infty. \tag{2.2}$$

For convenience, we introduce the following notations:
If $f, g \in L^2(\Omega)$, let

$$B[\phi, \psi] = \int_\Omega \sum_{k=1}^m \sum_{i=1}^n \sum_{j=1}^n a_{ij}^{(k)} \frac{\partial \phi_k}{\partial x_i} \frac{\partial \psi_k}{\partial x_j}\, dx = \int_\Omega \sum_{k=1}^m \nabla \phi_k^T A_k \nabla \psi_k\, dx,$$

$$\langle f, g\rangle_0 = \int_\Omega f^T g\, dx, \qquad \|f\|_0^2 = \int_\Omega f^T f dx,$$

$$\langle u, v\rangle = \int_\Omega \Big[u^T(x)v(x) + \sum_{k=1}^m \sum_{i=1}^n \sum_{j=1}^n a_{ij}^{(k)} \frac{\partial u_k}{\partial x_i} \frac{\partial v_k}{\partial x_j}\Big]\, dx,$$

$$\|u\|^2 = \int_\Omega \Big[u^T(x)u(x) + \sum_{k=1}^m \sum_{i=1}^n \sum_{j=1}^n a_{ij}^{(k)} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j}\Big]\, dx.$$

By (1.1), we have

$$B[u, v] - \langle g(x, u), v\rangle_0 = -\langle h(x), v\rangle_0, \qquad \forall v \in H_0^1(\Omega). \tag{2.3}$$

MAIN THEOREM 2.1. *If $g(x, u)$ satisfies condition* (C) *for all $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, then there exists a unique generalized solution to equation* (1.1) *for every $h \in L^2(\Omega)$.*

Proof. By the Riesz representation theorem and the above assumptions, we can define a mapping $T(u) : H_0^1(\Omega) \to H_0^1(\Omega)$ as follows:

$$\langle T(u), v\rangle = B[u, v] - \langle g(x, u), v\rangle_0. \tag{2.4}$$

Obviously, $T(u)$ is continuously differentiable for all $u \in H_0^1(\Omega)$ and

$$\langle T'(u)w, v\rangle = B[w, v] - \int_\Omega v^T(x) \frac{\partial g(x, u)}{\partial u} w(x)\, dx. \tag{2.5}$$

Invoking the Riesz representation theorem again, there exists a function $d \in H_0^1(\Omega)$ such that

$$\langle d, v\rangle = -\int_\Omega v^T(x)h(x)\, dx. \tag{2.6}$$

By (2.3), we just need to prove that there exists a unique u satisfying

$$T(u) = d. \tag{2.7}$$

We assume that $e_1, e_2, \ldots, e_m$ are eigenvectors corresponding to the eigenvalues $\lambda_{N_1}^{(1)}, \lambda_{N_2}^{(2)}, \ldots, \lambda_{N_m}^{(m)}$ of B_1, and $\xi_1, \xi_2, \ldots, \xi_m$ are eigenvectors corresponding to the eigenvalues $\lambda_{N_1+1}^{(1)}, \lambda_{N_2+1}^{(2)}, \ldots, \lambda_{N_m+1}^{(m)}$ of B_2, with $\|e_i\|_0 = 1$ and $\|\xi_i\|_0 = 1$ for all $i = 1, \ldots, m$. And let $\tau_{i1}^{(k)}(x), \tau_{i2}^{(k)}(x), \ldots, \tau_{il_i}^{(k)}(x)$ be eigenfunctions of the problem $-L_k\varphi = \lambda_i^{(k)}\varphi$, $x \in \Omega$, and $\varphi|_{\partial\Omega} = 0$.

Let

$$X = \Big\{ v \in H_0^1(\Omega) \;\Big|\; v = \sum_{k=1}^m b_k(x) e_k,\ b_k(x) = \sum_{i=1}^{N_k} \sum_{j=1}^{l_i} p_{ij}^{(k)} \tau_{ij}^{(k)}(x) \Big\},$$

$$Y = \Big\{ w \in H_0^1(\Omega) \;\Big|\; w = \sum_{k=1}^m r_k(x) \xi_k,\ r_k(x) = \sum_{i=N_k+1}^{\infty} \sum_{j=1}^{l_i} q_{ij}^{(k)} \tau_{ij}^{(k)}(x) \Big\},$$

$$Z = \Big\{ z \in H_0^1(\Omega) \;\Big|\; z = \sum_{k=1}^m s_k(x) \xi_k,\ s_k(x) = \sum_{i=1}^{N_k} \sum_{j=1}^{l_i} q_{ij}^{(k)} \tau_{ij}^{(k)}(x) \Big\},$$

where N_i, $i = 1, \ldots, m$, are as in condition (C) and $p_{ij}^{(k)}, q_{ij}^{(k)}$ are constants. Obviously, $H_0^1(\Omega) = Z \oplus Y$.

For all $v \in X$ and $u \in H$, we have

$$\begin{aligned}
\langle T'(u)v, v\rangle &= B[v,v] - \int_\Omega v^T(x) \frac{\partial g(x,u)}{\partial u} v(x)\,dx \\
&= \int_\Omega \sum_{k=1}^m \sum_{i=1}^n \sum_{j=1}^n a_{ij}^{(k)} \frac{\partial v_k}{\partial x_i} \frac{\partial v_k}{\partial x_j}\,dx - \int_\Omega v^T(x) \frac{\partial g(x,u)}{\partial u} v(x)\,dx \\
&\le \int_\Omega \langle -Lv, v\rangle_0\,dx - \int_\Omega v^T(x)(B_1 + \alpha(\|u\|)I) v(x)\,dx \\
&= \int_\Omega \langle -Lv, v\rangle_0\,dx - \int_\Omega v^T(x) B_1 v(x)\,dx - \alpha(\|u\|)\|v\|_0^2 \\
&\le \sum_{k=1}^m \lambda_{N_k}^{(k)} \int_\Omega b_k^2(x)\,dx - \int_\Omega \sum_{i=1}^m \sum_{j=1}^m b_i(x) b_j(x) e_i^T B_1 e_j\,dx - \alpha(\|u\|)\|v\|_0^2 \\
&\le -\alpha(\|u\|)\|v\|_0^2.
\end{aligned} \tag{2.8}$$

Since

$$\begin{aligned}
\|v\|^2 = \|v\|_0^2 + B[v,v] &= \|v\|_0^2 - \langle Lv, v\rangle_0 \\
&\le \|v\|_0^2 + \sum_{k=1}^m \lambda_{N_k}^{(k)} b_k^2(x) \langle e_k, e_k\rangle_0 \le (1+M)\|v\|_0^2,
\end{aligned} \tag{2.9}$$

where $M = \max\{\lambda_{N_k}^{(k)} : k = 1, \ldots, m\}$, we have

$$\langle T'(u)v, v\rangle \le -\frac{\alpha(\|u\|)}{M+1}\|v\|^2. \tag{2.10}$$

Similarly, denoting $N = \max\{\lambda_{N_k+1}^{(k)} : k = 1, \ldots, m\}$, we have

$$\langle T'(u)w, w\rangle \ge \frac{\beta(\|u\|)}{N+1}\|w\|^2, \qquad \forall w \in Y,\ \forall u \in H. \tag{2.11}$$

By the symmetry of B, it is easy to see that

$$\langle T'(u)v, w\rangle = \langle v, T'(u)w\rangle. \tag{2.12}$$

Let $\alpha_1(s) = \frac{\alpha(s)}{M+1}$ and $\beta_1(s) = \frac{\beta(s)}{N+1}$; then

$$\delta(s) = \min\{\alpha_1(s), \beta_1(s)\} \ge \frac{\min\{\alpha(s), \beta(s)\}}{N+1}. \tag{2.13}$$

By (2.2), it is easy to see that

$$\int_1^\infty \delta(s)\, ds = \infty. \tag{2.14}$$

Obviously, $X \cap Y = \{0\}$ and $\dim X = \dim Z = \sum_{k=1}^m \sum_{i=1}^{N_k} l_i$. From $H = Z \oplus Y$ and Lemma 1.2, we get $H = X \oplus Y$. Thus, $T(u)$ is a diffeomorphism from $H_0^1(\Omega)$ onto itself by Lemma 1.1. Therefore, there exists a unique u satisfying $T(u) = d$, that is, u is the unique generalized solution to (1.1). ■

3. An example. Consider the following coupled partial differential equations:

$$\begin{cases} 2\dfrac{\partial^2 u_1}{\partial x^2} + \dfrac{\partial^2 u_1}{\partial y^2} + \dfrac{5}{2}u_1 + 3\sin^2\gamma(x,y)u_1 - \dfrac{1}{2}u_2 \\ \qquad + \dfrac{1}{2}\ln(u_1 + u_2 + \sqrt{1 + (u_1+u_2)^2}) = h_1(x,y), \\ \dfrac{\partial^2 u_2}{\partial x^2} + \dfrac{\partial^2 u_2}{\partial y^2} + \dfrac{5}{2}u_2 + 2\cos^2\gamma(x,y)u_2 - \dfrac{1}{2}u_1 \\ \qquad + \dfrac{1}{2}\ln(u_1 + u_2 + \sqrt{1 + (u_1+u_2)^2}) = h_2(x,y), \end{cases} \tag{3.1}$$

for $(x,y) \in \Omega = [0,\pi] \times [0,\pi] \subseteq \mathbb{R}^2$, and $u(x,y) = (u_1(x,y), u_2(x,y)) \in \mathbb{R}^2$, with boundary conditions $u(0,y) = u(\pi,y) = u(x,0) = u(x,\pi) = u(0,0) = 0$, $h_1, h_2 \in L^2(\Omega)$. Since

$$\frac{\partial g(x,u)}{\partial u} = \begin{pmatrix} \frac{5}{2} + 3\sin^2\gamma + \frac{1}{2\sqrt{1+(u_1+u_2)^2}} & -\frac{1}{2} + \frac{1}{2\sqrt{1+(u_1+u_2)^2}} \\ -\frac{1}{2} + \frac{1}{2\sqrt{1+(u_1+u_2)^2}} & \frac{5}{2} + 2\cos^2\gamma + \frac{1}{2\sqrt{1+(u_1+u_2)^2}} \end{pmatrix},$$

we have $\gamma = \gamma(x,y)$. It is easy to see that

$$\begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2\sqrt{1+(u_1+u_2)^2}} & 0 \\ 0 & \frac{1}{2\sqrt{1+(u_1+u_2)^2}} \end{pmatrix} \le \frac{\partial g(x,u)}{\partial u} \le \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} - \frac{1}{2\sqrt{1+(u_1+u_2)^2}} & 0 \\ 0 & \frac{1}{2} - \frac{1}{2\sqrt{1+(u_1+u_2)^2}} \end{pmatrix}.$$

Here, the set of all eigenvalues of L_1 under the given boundary conditions is

$$C_1 = \{2m^2 + n^2 : n \text{ and } m \text{ are natural numbers}\} = \{3, 6, 9, 11, 12, 17, \dots\},$$

and the set of all eigenvalues of L_2 under the given boundary conditions is

$$C_2 = \{m^2 + n^2 : n \text{ and } m \text{ are natural numbers}\} = \{2, 5, 8, 10, 13, 17, \dots\}.$$

The eigenvalues of B_1 are 3 and 2, and the eigenvalues of B_2 are 6 and 5. Obviously, the integral is divergent, $\int_0^\infty ds/\sqrt{1+s^2} = \infty$. If $\alpha(s) = \frac{1}{2}\sqrt{1+s^2}$ and $\beta(s) = \frac{1}{2}(1 - 1/\sqrt{1+s^2})$, then the conditions of Theorem 2.1 are satisfied. Thus, there exists a unique generalized solution to equation (3.1) for arbitrary $h_1(x,y), h_2(x,y) \in L^2(\Omega)$.

Remark. It should be highlighted that the existence of a unique generalized solution to our system cannot be deduced from the previously known results [AP, BF, HN, INW, IN, LS].

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