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Generalized Cauchy problems for hyperbolic functional differential systems

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Abstract. A generalized Cauchy problem for hyperbolic functional differential systems is considered. The initial problem is transformed into a system of functional integral equations. The existence of solutions of this system is proved by using the method of successive approximations. Differentiability of solutions with respect to initial functions is proved. It is important that functional differential systems considered in this paper do not satisfy the Volterra condition.

1. Introduction. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Suppose that $E=[0,a]\times\mathbb{R}^n$ and $E_0=[-b_0,0]\times\mathbb{R}^n$ where $a>0$ and $b_0\ge 0$. Write $B=[-b_0,0]\times[-b,b]$ where $b\in\mathbb{R}^n_+$, $\mathbb{R}_+=[0,\infty)$, and $E_{0.i}=[-b_0,a_i]\times\mathbb{R}^n$ for $0\le a_i<a$, $1\le i\le k$. Set $D=[-b_0-a,a]\times[-d,d]$ for $d\in\mathbb{R}^n_+$. For $t\in[0,a]$ we write $D_{[t]}=[-b_0-t,a-t]\times[-d,d]$; then $D_{[t]}\subset D$. Given a function $z\colon E_0\cup E\to\mathbb{R}^k$ and a point $(t,x)\in E$, we consider the functions $z_{(t,x)}\colon B\to\mathbb{R}^k$ and $z_{[t,x]}\colon D_{[t]}\to\mathbb{R}^k$ defined by $z_{(t,x)}(\tau,y)=z(t+\tau,x+y)$ for $(\tau,y)\in B$ and $z_{[t,x]}(\tau,y)=z(t+\tau,x+y)$ for $(\tau,y)\in D_{[t]}$.

Put $\Omega=E\times C(B,\mathbb{R}^k)\times C(D,\mathbb{R}^k)$ and suppose that $F\colon\Omega\to\mathbb{R}^k$, $F=(F_1,\dots,F_k)$, is a given function of the variables (t,x,v,w), $v=(v_1,\dots,v_k)$, $w=(w_1,\dots,w_k)$. We denote by $M_{k\times n}$ the set of all $k\times n$ matrices with real elements. If $X\in M_{k\times n}$ then X^T is the transpose matrix. We use the symbol "$\circ$" to denote scalar product in $\mathbb{R}^n$. Suppose that $f\colon E\to M_{k\times n}$, $f=[f_{ij}]_{i=1,\dots,k,\ j=1,\dots,n}$, $\varphi_0\colon[0,a]\to\mathbb{R}$, $\tilde\varphi\colon E\to\mathbb{R}^n$, $\tilde\varphi=(\varphi_1,\dots,\varphi_n)$, $\psi_0\colon[0,a]\to\mathbb{R}$, $\tilde\psi\colon E\to\mathbb{R}^n$, $\tilde\psi=(\psi_1,\dots,\psi_n)$ and $\kappa_i\colon E_{0.i}\to\mathbb{R}$, $i=1,\dots,k$, are given functions. The requirements on φ_0 and ψ_0 are that $0\le\varphi_0(t)\le t$ and

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$0 \le \psi_0(t) \le a$ for $t \in [0,a]$. Write $\varphi(t,x) = (\varphi_0(t), \tilde{\varphi}(t,x))$ and $\psi(t,x) = (\psi_0(t), \tilde{\psi}(t,x))$ for $(t,x) \in E$. For the above $f\colon E \to M_{k\times n}$ we put $f_{[i]} = (f_{i1}, \ldots, f_{in})$, $1 \le i \le k$.

Let us denote by $z = (z_1, \ldots, z_k)$ an unknown function of the variables (t,x). We consider the system of functional differential equations

$$(1.1)\quad \partial_t z_i(t,x) + f_{[i]}(t,x) \circ \partial_x z_i(t,x) = F_i(t,x,z_{\varphi(t,x)}, z_{\psi[t,x]}),\ i = 1, \ldots, k,$$

with the initial conditions

$$(1.2)\qquad z_i(t,x) = \kappa_i(t,x) \quad \text{on } E_{0.i},\ i = 1, \ldots, k,$$

where $z_{\varphi(t,x)} = z_{(\varphi_0(t),\tilde{\varphi}(t,x))}$, $z_{\psi[t,x]} = z_{[\psi_0(t),\tilde{\psi}(t,x)]}$ and $\partial_x z_i = (\partial_{x_1} z_i, \ldots, \partial_{x_n} z_i)$. System (1.1) with initial conditions (1.2) is called a *generalized Cauchy problem*. We consider classical solutions of (1.1), (1.2).

The following problems are considered in this paper. We prove that under natural assumptions on given functions there exists exactly one solution to (1.1), (1.2) and the solution is defined on $E_0 \cup E$. Let us denote by X the class of all functions $\kappa = (\kappa_1, \ldots, \kappa_k)$, $\kappa_i\colon E_{0.i} \to \mathbb{R}$, $1 \le i \le k$, such that there exists exactly one solution $\Xi[\kappa]$ of problem (1.1), (1.2). We give a construction of the space X and we prove that under natural assumptions on f, F and φ, ψ the operator $\Xi\colon X \to C(E_0 \cup E, \mathbb{R}^k)$ has the following property: for each $\kappa \in X$ the Fréchet derivative $\partial\Xi[\kappa]$ exists. Moreover, if $\kappa, \chi \in X$ and $z_* = \partial\Xi[\kappa]\chi$ then z_* is the solution of an integral functional equation generated by (1.1), (1.2) and this equation is linear.

There is a wide literature on first order partial functional differential problems; we wish to mention here just some existence results. There are various concepts of solution to initial value or mixed problems for functional differential equations. Continuous functions satisfying integral systems obtained by integrating original equations along bicharacteristics were considered in [1], [16]. Generalized solutions in the Carathéodory sense were investigated in [5], [15]. Results on the existence of solutions are obtained in those papers by using the method of bicharacteristics. Classical solutions in the functional setting were studied in [2], [7], [13], [14]. Cinquini Cibrario solutions of nonlinear differential functional equations were first treated in [3]. This class of solutions lies between classical solutions and solutions in the Carathéodory sense and both inclusions are strict. Existence results for mixed problems for nonlinear equations can be found in [4]. They are obtained by a linearization procedure and by constructing functional integral systems for unknown functions and for their derivatives with respect to spatial variables. Sufficient conditions for the existence of classical solutions defined on the Haar pyramid are given in [12], [9]. Classical solutions and differentiability with respect to initial data for Volterra type of equations were studied in [11]. Existence and uniqueness of solutions on the Haar pyramid were investigated in [10].

All the above results have the following property: solutions of initial or mixed problems exist locally with respect to the variable t. The aim of this paper is to prove a theorem on the existence of solutions of the problem and a theorem on the differentiability of solutions with respect to initial functions. Theorems on the continuous dependence of solutions on initial or initial boundary conditions are given in [8, Chapters 4 and 5]. In this paper we start investigations of the differentiability with respect to initial functions for partial functional differential equations. The monograph [6] contains results on differentiability with respect to initial functions of solutions for ordinary functional differential equations.

Until now there have been no results on existence and differentiability of solutions with respect to initial functions for partial functional differential systems with arguments of both Volterra and Fredholm type in an unbounded domain.

Suppose that $G\colon E \times C(E_0 \cup E, \mathbb{R}^k) \to \mathbb{R}^k$, $G = (G_1, \ldots, G_k)$, is a given function. Let us consider the system of functional differential equations

$$\partial_t z_i(t,x) + f_{[i]}(t,x) \circ \partial_x z_i(t,x) = G_i(t,x,z), \qquad i = 1,\ldots,k, \tag{1.3}$$

where z is the functional variable. It is clear that (1.1) is a particular case of (1.3).

We will say that G satisfies the *Volterra condition* if for each $(t,x) \in E$ and for $z, \tilde{z} \in C(E_0 \cup E, \mathbb{R}^k)$ such that $z(\tau,y) = \tilde{z}(\tau,y)$ for $(\tau,y) \in (E_0 \cup E) \cap ([-b_0,t] \times \mathbb{R}^n)$ we have $G(t,x,z) = G(t,x,\tilde{z})$. The Volterra condition means that the value of G at $(t,x,z) \in E \times C(E_0 \cup E, \mathbb{R}^k)$ depends on (t,x) and on restrictions of z to the set $(E_0 \cup E) \cap ([-b_0,t] \times \mathbb{R}^n)$ only. Note that system (1.1) fails to satisfy the Volterra condition.

Note that functional differential equations or systems considered in [1]–[5], [7]–[8], [11]–[16] satisfy the Volterra condition. Until now there have been no results on functional differential equations of the form (1.3) which do not satisfy the Volterra condition.

With the above motivation we consider the initial value problem (1.1), (1.2).

We give examples of functional differential systems which can be obtained from (1.1) by specifying the function F.

Example 1.1. Suppose that $G\colon E \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$, $G = (G_1, \ldots, G_k)$, is a given function and F is defined by

$$F(t,x,v,w) = G\big(t,x,v(0,0_{[n]}), w(0,0_{[n]})\big) \quad \text{on } \Omega \tag{1.4}$$

where $0_{[n]} = (0,\ldots,0) \in \mathbb{R}^n$. Then (1.1) reduces to the system of differential equations with deviated variables

$$\partial_t z_i(t,x) + f_{[i]}(t,x) \circ \partial_x z_i(t,x) = G_i\big(t,x,z(\varphi(t,x)), z(\psi[t,x])\big). \tag{1.5}$$

Example 1.2. Suppose that $\varphi(t,x) = (t,x)$ and $\psi[t,x] = [t,x]$ for $(t,x) \in E$, and for the above G put

$$F(t,x,v,w) = G\Big(t,x, \int_B v(\tau,s)\,ds\,d\tau, \int_{D[t]} w(\tau,s)\,ds\,d\tau\Big) \quad \text{on } \Omega. \tag{1.6}$$

Then (1.1) reduces to the differential integral system

$$\begin{aligned}\partial_t z_i(t,x) + f_{[i]}(t,x) \circ \partial_x z_i(t,x) \\ = G\Big(t,x, \int_B z_{(t,x)}(\tau,s)\,ds\,d\tau, \int_{D[t]} z_{[t,x]}(\tau,s)\,ds\,d\tau\Big),\end{aligned} \tag{1.7}$$

for $i = 1,\ldots,k$.

It is clear that more complicated examples of differential systems with deviated variables and differential integral systems can be obtained from (1.1) by specializing the operator F. Note that systems (1.5) and (1.7) do not satisfy the Volterra condition.

2. Sequences of successive approximations. Write $E_t = [-b_0,t] \times \mathbb{R}^n$ for $0 \le t \le a$. For $t \in [0,a]$ and $z \in C(E_0 \cup E, \mathbb{R}^k)$, $v \in C(E_0 \cup E, \mathbb{R}^n)$, $u \in C(E_0 \cup E, M_{k\times n})$ we define the seminorms $\|z\|_{(t,\mathbb{R}^k)} = \max\{\|z(\tau,x)\|_\infty : (\tau,x) \in E_t\}$, $\|v\|_{(t,\mathbb{R}^n)} = \max\{\|v(\tau,x)\| : (\tau,x) \in E_t\}$, $\|u\|_{(t,M_{k\times n})} = \max\{\|u(\tau,x)\|_{k\times n} : (\tau,x) \in E_t\}$. We denote by $CL(B,\mathbb{R})$ the class of continuous linear operators from $C(B,\mathbb{R})$ taking values in $\mathbb{R}$. In a similar way we define the space $CL(D,\mathbb{R})$. The norms in $CL(B,\mathbb{R})$ and $CL(D,\mathbb{R})$ generated by the maximum norms in $C(B,\mathbb{R})$ and $C(D,\mathbb{R})$ will be denoted by $\|\cdot\|_{B\star}$ and $\|\cdot\|_{D\star}$ respectively. For $V = [V_{ij}]_{i,j=1}^k$ where $V_{ij} \in CL(B,\mathbb{R})$, and $\tilde{V} = [\tilde{V}_{ij}]_{i,j=1}^k$ where $\tilde{V}_{ij} \in CL(D,\mathbb{R})$, we denote $\|V\|_{k\times k;*} = \max\{\sum_{j=1}^k \|V_{ij}\|_{B\star} : 1 \le i \le k\}$, $\|\tilde{V}\|_{k\times k;*} = \max\{\sum_{j=1}^k \|\tilde{V}_{ij}\|_{D\star} : 1 \le i \le k\}$.

Assumption $H[\varphi,\psi]$. The functions $\varphi_0, \psi_0 \colon [0,a] \to \mathbb{R}$, and $\tilde{\varphi}, \tilde{\psi} \colon E \to \mathbb{R}^n$ are continuous and

1) $0 \le \varphi_0(t) \le t$ and $0 \le \psi_0(t) \le a$ for $t \in [0,a]$,
2) the derivatives $\partial_x \tilde{\varphi} = [\partial_{x_j} \varphi_i]_{i,j=1}^n$, $\partial_x \tilde{\psi} = [\partial_{x_j} \psi_i]_{i,j=1}^n$ exist and $\partial_x \tilde{\varphi} \in C(E, M_{n\times n}), \partial_x \tilde{\psi} \in C(E, M_{n\times n})$,
3) there are $\tilde{Q}, \bar{Q} \in \mathbb{R}_+$ such that on E we have

$$\|\partial_x \tilde{\varphi}(t,x)\|_{n\times n} \le \tilde{Q}, \qquad \|\partial_x \tilde{\psi}(t,x)\|_{n\times n} \le \bar{Q},$$
$$\|\partial_x \tilde{\varphi}(t,x) - \partial_x \tilde{\varphi}(t,\bar{x})\|_{n\times n} \le \tilde{Q}\|x - \bar{x}\|,$$
$$\|\partial_x \tilde{\psi}(t,x) - \partial_x \tilde{\psi}(t,\bar{x})\|_{n\times n} \le \bar{Q}\|x - \bar{x}\|.$$

Assumption $H[\kappa]$. The functions $\kappa_i \colon E_{0.i} \to \mathbb{R}$, $1 \le i \le k$, are continuous and bounded, the derivatives $\partial_x \kappa_i = (\partial_{x_1}\kappa_i, \ldots, \partial_{x_n}\kappa_i)$ exist, and

$\partial_x \kappa_i \in C(E_{0.i}, \mathbb{R}^n)$, $1 \le i \le k$. There is $C > 0$ such that $\|\partial_x \kappa_i(t,x)\| \le C$ on $E_{0.i}$ for $1 \le i \le k$.

Let us denote by X the class of all $\kappa = (\kappa_1, \ldots, \kappa_k)$, $\kappa_i \colon E_{0.i} \to \mathbb{R}$, $1 \le i \le k$, satisfying Assumption $H[\kappa]$.

For $\chi \in X$, $\chi = (\chi_1, \ldots, \chi_k)$, we define $\|\chi_i\|_{E_{0.i}} = \sup\{|\chi_i(t,x)| : (t,x) \in E_{0.i}\}$, $i = 1, \ldots, k$, and $\|\chi\|_X = \max\{\|\chi_i\|_{E_{0.i}} : 1 \le i \le k\}$ and $\|\partial_x \chi\|_{X_{k\times n}} = \max\{\sum_{j=1}^n \|\partial_{x_j} \chi_i\|_{E_{0.i}} : 1 \le i \le k\}$. Given $\kappa \in X$, we denote by $C_\kappa(E_0 \cup E, \mathbb{R}^k)$ the set of all $z \in C(E_0 \cup E, \mathbb{R}^k)$ such that $z_i(t,x) = \kappa_i(t,x)$ on $E_{0.i}$, $1 \le i \le k$. For the above κ, we denote by $C_{\partial\kappa_i}(E_0 \cup E, \mathbb{R}^n)$, $1 \le i \le k$, the class of all $v \in C(E_0 \cup E, \mathbb{R}^n)$ such that $v(t,x) = \partial_x \kappa_i(t,x)$ on $E_{0.i}$. Let $C_{\partial\kappa}(E_0 \cup E, M_{k\times n})$ denote the set of all $w \in C(E_0 \cup E, M_{k\times n})$, $w = [w_{ij}]_{i=1,\ldots,k,\, j=1,\ldots,n}$, such that $w_{[i]} \in C_{\partial\kappa_i}(E_0 \cup E, \mathbb{R}^n)$ where $w_{[i]} = (w_{i1}, \ldots, w_{in})$, $1 \le i \le k$.

Suppose that Assumptions $H_0[f,F]$, $H[\varphi,\psi]$ are satisfied and $\kappa \in X$. Let us denote by $g_{[i]}(\,\cdot\,, t, x)$ the solution of the Cauchy problem

$$\eta'(\tau) = f_{[i]}(\tau, \eta(\tau)), \quad \eta(t) = x, \tag{2.1}$$

where $(t,x) \in E$. The function $g_{[i]}(\,\cdot\,, t, x)$ is the ith characteristic of (1.1).

For $P = (t,x,v,w) \in \Omega$ and $\tilde v = (\tilde v_1, \ldots, \tilde v_n) \in C(B, \mathbb{R}^n)$ we write $\partial_{v_\nu} F_i(P)\tilde v = (\partial_{v_\nu} F_i(P)\tilde v_1, \ldots, \partial_{v_\nu} F_i(P)\tilde v_n)$, $i, \nu = 1, \ldots, k$. In a similar way we define the expression $\partial_{w_\nu} F_i(P)\tilde w$, $1 \le i, \nu \le k$, where $\tilde w = (\tilde w_1, \ldots, \tilde w_n) \in C(D, \mathbb{R}^n)$. For the above $\tilde v$, $\tilde w$ and $q = (q_1, \ldots, q_n) \in \mathbb{R}^n$ the functions $\tilde v \circ q \colon B \to \mathbb{R}$ and $\tilde w \circ q \colon D \to \mathbb{R}$ are given by $\tilde v \circ q = \tilde v_1 q_1 + \cdots + \tilde v_n q_n$ and $\tilde w \circ q = \tilde w_1 q_1 + \cdots + \tilde w_n q_n$. Set

$$P_i[z](\tau,t,x) = (\tau, g_{[i]}(\tau,t,x), z_{\varphi(\tau, g_{[i]}(\tau,t,x))}, z_{\psi[\tau, g_{[i]}(\tau,t,x)]}), \qquad 1 \le i \le k.$$

Suppose that $z \in C_\kappa(E_0 \cup E, \mathbb{R}^k)$. Let us denote by $\mathcal{F}[z] = (\mathcal{F}_1[z], \ldots, \mathcal{F}_k[z])$ the function defined by

$$\mathcal{F}_i[z](t,x) = \kappa_i(a_i, g_{[i]}(a_i,t,x)) + \int_{a_i}^{t} F_i(P_i[z](\tau,t,x))\,d\tau \quad \text{ on } E \setminus E_{0.i},$$

$$\mathcal{F}_i[z] = \kappa_i(t,x) \quad \text{ on } E_{0.i},$$

for $i = 1, \ldots, k$. We consider the functional integral equation

$$z = \mathcal{F}[z]. \tag{2.2}$$

Assumption $H[f,F]$. The functions $f \colon E \to M_{k\times n}$ and $F \colon \Omega \to \mathbb{R}^k$ are continuous and

1) the derivatives $\partial_x f_{[\mu]} = [\partial_{x_j} f_{\mu i}]_{i,j=1}^n$ exist and $\partial_x f_{[\mu]} \in C(E, M_{n\times n})$ for $1 \le \mu \le k$,
2) the derivatives $\partial_x F = [\partial_{x_j} F_i]_{i=1,\ldots,k,\, j=1,\ldots,n}$ exist and are continuous on Ω,

3) for $P=(t,x,v,w)$ the Fréchet derivatives $\partial_v F(P)=[\partial_{w_j}F_i(P)]_{i,j=1}^k$ and $\partial_w F(P)=[\partial_{w_j}F_i(P)]_{i,j=1}^k$ exist, and $\partial_{v_j}F_i(P)\in CL(B,\mathbb{R})$ and $\partial_{w_j}F_i(P)\in CL(D,\mathbb{R})$ for $i,j=1,\ldots,k$, $P\in\Omega$,
4) there are $\alpha_0,\beta,\delta_0,\gamma\in C([0,a],\mathbb{R}_+)$ such that for $P\in\Omega$ we have
$$\|\partial_x f(t,x)\|_{k\times n}\le\alpha_0(t),\qquad \|\partial_x F(P)\|_{k\times n}\le\delta_0(t),$$
$$\|\partial_v F(P)\|_{k\times k;*}\le\beta(t),\qquad \|\partial_w F(P)\|_{k\times k;*}\le\gamma(t),$$
5) there are $L_x,L_v,L_x\in C([0,a],\mathbb{R}_+)$ such that the expressions
$$\|\partial_x F(t,x,v,w)-\partial_x F(t,\tilde x,\tilde v,\tilde w)\|_{k\times n},$$
$$\|\partial_v F(t,x,v,w)-\partial_v F(t,\tilde x,\tilde v,\tilde w)\|_{k\times k;*},$$
$$\|\partial_w F(t,x,v,w)-\partial_w F(t,\tilde x,\tilde v,\tilde w)\|_{k\times k;*},$$
are bounded by $L_x(t)\|x-\tilde x\|+L_v(t)\|v-\tilde v\|_B+L_w(t)\|w-\tilde w\|_{D[t]}$ for $(t,x),(t,\tilde x)\in E$, $v,\tilde v\in C(B,\mathbb{R}^k)$, $w,\tilde w\in C(D[t],\mathbb{R}^k)$.

Assumption $H[a]$. The following relations hold:

1) $\displaystyle\int_0^a\gamma(\tau)e^{\int_\tau^a\beta(s)\,ds}\,d\tau<1$,

2) $\displaystyle\bar Q\int_0^a\gamma(\tau)e^{\int_\tau^a(\alpha_0(s)+\tilde Q\beta(s))\,ds}\,d\tau<1$.

The proof of the existence of the classical solution to (2.2) is based on the following method of successive approximations. Suppose that $\kappa\in X$ and Assumptions $H[f,F]$, $H[\varphi,\psi]$ are satisfied. We consider the sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$ where $z^{(m)}\colon E_0\cup E\to\mathbb{R}^k$, $z^{(m)}=(z_1^{(m)},\ldots,z_k^{(m)})$, $u^{(m)}\colon E_0\cup E\to M_{k\times n}$, $u^{(m)}=[u_{ij}^{(m)}]_{i=1,\ldots,k,\,j=1,\ldots,n}$, $u_{[i]}^{(m)}=(u_{i1}^{(m)},\ldots,u_{in}^{(m)})$, $1\le i\le k$, defined in the following way. We put first

$$z_i^{(0)}(t,x)=\kappa_i(t,x)\quad\text{on } E_{0.i},\quad z_i^{(0)}(t,x)=\kappa_i(a_i,x)\quad\text{on } E\setminus E_{0.i},\tag{2.3}$$

$$u_{[i]}^{(0)}(t,x)=\partial_x\kappa_i(t,x)\quad\text{on } E_{0.i},\quad u_{[i]}^{(0)}(t,x)=\partial_x\kappa_i(a_i,x)\quad\text{on } E\setminus E_{0.i},\tag{2.4}$$

where $i=1,\ldots,k$. Suppose that $z^{(m)}\colon E_0\cup E\to\mathbb{R}^k$ and $u^{(m)}\colon E_0\cup E\to M_{k\times n}$ are known functions. Then $u_{[i]}^{(m+1)}$ is a solution of the equation

$$v=\mathbb{G}_{[i]}^{(m)}[v]\tag{2.5}$$

where $v=(v_1,\ldots,v_n)$ and

$$\mathbb{G}_{[i]}^{(m)}[v](t,x)=\partial_x\kappa_i(t,x)\quad\text{on } E_{0.i}\tag{2.6}$$

and

$$\mathbb{G}_{[i]}^{(m)}[v](t,x)=-\int_{a_i}^t v\big(\tau,g_{[i]}(\tau,t,x)\big)\partial_x f_{[i]}\big(\tau,g_{[i]}(\tau,t,x)\big)\,d\tau+\Gamma_{[i]}^{(m)}(t,x)\tag{2.7}$$

on $E \setminus E_{0.i}$. The functions $\Gamma^{(m)}_{[i]} \colon E \setminus E_{0.i} \to \mathbb{R}^n$, $1 \le i \le k$, are given by

$$\Gamma^{(m)}_{[i]}(t,x) = \partial_x \kappa_i(a_i, g_{[i]}(a_i,t,x)) + \int_{a_i}^{t} \partial_x F_i\big(P_i[z^{(m)}](\tau,t,x)\big)\,d\tau \tag{2.8}$$
$$+ \sum_{\nu=1}^{k} \int_{a_i}^{t} \partial_{v_\nu} F_i\big(P_i[z^{(m)}](\tau,t,x)\big)(u^{(m)}_{[\nu]})_{\varphi(\tau,g_{[i]}(\tau,t,x))} \partial_x \tilde{\varphi}(\tau, g_{[i]}(\tau,t,x))\,d\tau$$
$$+ \sum_{\nu=1}^{k} \int_{a_i}^{t} \partial_{w_\nu} F_i\big(P_i[z^{(m)}](\tau,t,x)\big)(u^{(m)}_{[\nu]})_{\psi[\tau,g_{[i]}(\tau,t,x)]} \partial_x \tilde{\psi}(\tau, g_{[i]}(\tau,t,x))\,d\tau.$$

The function $(u^{(m)}_{[\nu]})_{\varphi(\tau,y)}\,\partial_x\tilde{\varphi}(\tau,y) \colon B \to \mathbb{R}^n$, $y = g_{[i]}(\tau,t,x)$, is defined by

$$(u^{(m)}_{[\nu]})_{\varphi(\tau,y)}\,\partial_x\tilde{\varphi}(\tau,y)$$
$$= \Big(\sum_{j=1}^{n} (u^{(m)}_{\nu j})_{\varphi(\tau,y)}\,\partial_{x_1}\varphi_j(\tau,y), \ldots, \sum_{j=1}^{n} (u^{(m)}_{\nu j})_{\varphi(\tau,y)}\,\partial_{x_n}\varphi_j(\tau,y) \Big).$$

Analogously we define $(u^{(m)}_{[\nu]})_{\psi[\tau,y]}\,\partial_x\tilde{\psi}(\tau,y) \colon D[\psi_0(\tau)] \to \mathbb{R}^n$. The function $z^{(m+1)}$ is given by

$$z^{(m+1)} = \mathcal{F}[z^{(m)}]. \tag{2.9}$$

Remark 2.1. Equations (2.5) are obtained in the following way. Suppose that $z^{(m)} \colon E_0 \cup E \to \mathbb{R}^k$ and $u^{(m)} \colon E_0 \cup E \to M_{k\times n}$ are known functions. We consider system (1.1) with $z^{(m)}_{\varphi(t,x)}$, $z^{(m)}_{\psi[t,x]}$ instead of $z_{\varphi(t,x)}$, $z_{\psi[t,x]}$ respectively:

$$\partial_t z_i(t,x) + f_{[i]}(t,x) \circ \partial_x z_i(t,x) = F_i(t,x,z^{(m)}_{\varphi(t,x)}, z^{(m)}_{\psi[t,x]}),\ i = 1,\ldots,k. \tag{2.10}$$

We now introduce an additional unknown function $u = \partial_x z$ where $u = [u_{ij}]_{i=1,\ldots,k\, j=1,\ldots,n}$, $u_{[i]} = (u_{i1},\ldots,u_{in})$, $1 \le i \le k$. From (2.10) we get the differential equations for $u_{[i]}$:

$$\partial_t u_{[i]}(t,x) + u_{[i]}(t,x)\,\partial_x f_{[i]}(t,x) + f_{[i]}(t,x)[\partial_x u_{[i]}(t,x)]^T$$
$$= \partial_x F_i\big(t,x,z^{(m)}_{\varphi(t,x)}, z^{(m)}_{\psi[t,x]}\big)$$
$$+ \sum_{\nu=1}^{k} \partial_{v_\nu} F_i\big(t,x,z^{(m)}_{\varphi(t,x)}, z^{(m)}_{\psi[t,x]}\big)(\partial_x z^{(m)}_i)_{\varphi(t,x)} \partial_x \tilde{\varphi}(t,x)$$
$$+ \sum_{\nu=1}^{k} \partial_{w_\nu} F_i\big(t,x,z^{(m)}_{\varphi(t,x)}, z^{(m)}_{\psi[t,x]}\big)(\partial_x z^{(m)}_i)_{\psi[t,x]} \partial_x \tilde{\psi}(t,x), \qquad i = 1,\ldots,k,$$

and $u_{[i]}(t,x) = \partial_x \kappa_i(t,x)$ on $E_{0.i}$, $i = 1,\ldots,k$. If we assume that $\partial_x z^{(m)}_i = u^{(m)}_{[i]}$ (see Lemma 3.1) then by integrating the above system along the characteristics, we obtain (2.5).

3. Successive approximations for integral functional equations. We begin by proving important properties of the sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$.

LEMMA 3.1. *If Assumptions $H[f,F]$ and $H[\varphi,\psi]$ are satisfied and $\kappa \in X$, then for $m \geq 0$ we have:*

(I_m) *the functions $z^{(m)}$ and $u^{(m)}$ are defined on $E_0 \cup E$, and $z^{(m)} \in C_\kappa(E_0 \cup E, \mathbb{R}^k)$, $u^{(m)}_{[i]} \in C_{\partial\kappa_i}(E_0 \cup E, \mathbb{R}^n)$ for $1 \leq i \leq k$,*

(II_m) *the derivatives $\partial_x z_i^{(m)}$ exist and $\partial_x z_i^{(m)}(t,x) = u^{(m)}_{[i]}(t,x)$ on $E_0 \cup E$, $i = 1,\ldots,k$.*

Proof. We will prove (I_m) and (II_m) by induction. From (2.3), (2.4) we see that (I_0) and (II_0) are satisfied. Suppose that $z^{(m)} \in C_\kappa(E_0 \cup E, \mathbb{R}^k)$, $u^{(m)}_{[i]} \in C_{\partial\kappa_i}(E_0 \cup E, \mathbb{R}^n)$, $1 \leq i \leq k$, are given and $m \geq 0$. We now prove that $u^{(m+1)}\colon E_0 \cup E \to M_{k\times n}$ exists and $u^{(m+1)}_{[i]} \in C_{\partial\kappa_i}(E_0 \cup E, \mathbb{R}^n)$ for $1 \leq i \leq k$. For $v, \tilde{v} \in C_{\partial\kappa_i}(E_0 \cup E, \mathbb{R}^n)$, $1 \leq i \leq k$, we put

$$[|v - \tilde{v}|] = \max\Big\{ \|v - \tilde{v}\|_{(t,\mathbb{R}^n)} \exp\Big[-2\int_{a_i}^{t} \alpha_0(\tau)\,d\tau\Big] : a_i \leq t \leq a \Big\}.$$

Then we have

$$\begin{aligned}\|\mathbb{G}^{(m)}_{[i]}[v](t,x) - \mathbb{G}^{(m)}_{[i]}[\tilde{v}](t,x)\| &\leq [|v - \tilde{v}|] \int_{a_i}^{t} e^{2\int_{a_i}^{\tau} \alpha_0(s)\,ds}\,d\tau \\ &\leq \tfrac{1}{2}[|v - \tilde{v}|] e^{2\int_{a_i}^{t} \alpha_0(\tau)\,d\tau} \quad \text{for } (t,x) \in E \setminus E_{0.i},\end{aligned}$$

and consequently

$$\|\mathbb{G}^{(m)}_{[i]}[v](t,x) - \mathbb{G}^{(m)}_{[i]}[\tilde{v}](t,x)\|_{(t,\mathbb{R}^n)} \leq \frac{1}{2}[|v - \tilde{v}|] e^{2\int_{a_i}^{t} \alpha_0(\tau)\,d\tau}.$$

This gives $[|\mathbb{G}^{(m)}_{[i]}[v] - \mathbb{G}^{(m)}_{[i]}[\tilde{v}]|] \leq \frac{1}{2}[|v - \tilde{v}|]$, $1 \leq i \leq k$. From the Banach fixed point theorem there is exactly one $u^{(m+1)}_{[i]}\colon E_0 \cup E \to \mathbb{R}^n$ and $u^{(m+1)}_{[i]} \in C_{\partial\kappa_i}(E_0 \cup E, \mathbb{R}^n)$ for $1 \leq i \leq k$.

Suppose that $z^{(m+1)}$ is given by (2.9). Now we prove (II_{m+1}). Write

$$\Lambda_i(t,x,y) = z_i^{(m+1)}(t,y) - z_i^{(m+1)}(t,x) - u^{(m+1)}_{[i]}(t,x) \circ (y - x), \qquad 1 \leq i \leq k.$$

We prove that there exists $K > 0$ such that

$$(3.1) \qquad |\Lambda_i(t,x,y)| \leq K\|x - y\|^2, \qquad (t,x),(t,y) \in E \setminus E_{0.i},\ 1 \leq i \leq k.$$

It follows from (2.6)–(2.9) that for $1 \leq i \leq k$ we have

$$\Lambda_i(t,x,y) = \mathcal{F}_i[z^{(m)}](t,y) - \mathcal{F}_i[z^{(m)}](t,x) - \mathbb{G}^{(m)}_{[i]}[u^{(m+1)}](t,x) \circ (y - x).$$

Write $Q_i^{(m)}(s,\tau,t,x,y)=(1-s)P_i[z^{(m)}](\tau,t,x)+sP_i[z^{(m)}](\tau,t,y)$, $0\le s\le 1$, $g_i(\tau,t,x,y)=g_{[i]}(\tau,t,y)-g_{[i]}(\tau,t,x)$ and

$$\Lambda_{\kappa.i}(t,x,y)=\kappa_i(a_i,g_{[i]}(a_i,t,y))-\kappa_i(a_i,g_{[i]}(a_i,t,x))$$
$$-\partial_x\kappa_i(a_i,g_{[i]}(a_i,t,x))\circ g_i(a_i,t,x,y),$$

$$\Lambda_{F.i}(t,x,y)=\sum_{\nu=1}^{k}\int_{a_i}^{t}\int_0^1\big[\partial_{v_\nu}F_i(Q_i^{(m)}(s,\tau,t,x,y))-\partial_{v_\nu}F_i(P_i[z^{(m)}](\tau,t,x))\big]$$
$$\cdot\big[(z_\nu^{(m)})_{\varphi(\tau,g_{[i]}(\tau,t,y))}-(z_\nu^{(m)})_{\varphi(\tau,g_{[i]}(\tau,t,x))}\big]\,ds\,d\tau$$
$$+\sum_{\nu=1}^{k}\int_{a_i}^{t}\int_0^1\big[\partial_{v_\nu}F_i(Q_i^{(m)}(s,\tau,t,x,y))-\partial_{v_\nu}F_i(P_i[z^{(m)}](\tau,t,x))\big]$$
$$\cdot\big[(z_\nu^{(m)})_{\varphi(\tau,g_{[i]}(\tau,t,y))}-(z_\nu^{(m)})_{\varphi(\tau,g_{[i]}(\tau,t,x))}\big]\,ds\,d\tau$$
$$+\int_{a_i}^{t}\int_0^1\big[\partial_xF_i(Q_i^{(m)}(s,\tau,t,x,y))-\partial_xF_i(P_i[z^{(m)}](\tau,t,x))\big]$$
$$\circ\, g_i(\tau,t,x,y)\,ds\,d\tau$$

$$\Lambda_{v.i}(t,x,y)=\sum_{\nu=1}^{k}\int_{a_i}^{t}\partial_{v_\nu}F_i(P_i[z^{(m)}(\tau,t,x)])\big\{(z_\nu^{(m)})_{\varphi(\tau,g_{[i]}(\tau,t,y))}$$
$$-(z_\nu^{(m)})_{\varphi(\tau,g_{[i]}(\tau,t,x))}-(u_{[\nu]}^{(m)})_{\varphi(\tau,g_{[i]}(\tau,t,x))}\partial_x\tilde{\varphi}(\tau,g_{[i]}(\tau,t,x))\circ g_i(\tau,t,x,y)\big\}\,d\tau,$$

$$\Lambda_{w.i}(t,x,y)=\sum_{\nu=1}^{k}\int_{a_i}^{t}\partial_{w_\nu}F_i(P_i[z^{(m)}(\tau,t,x)])\big\{(z_\nu^{(m)})_{\psi[\tau,g_{[i]}(\tau,t,y)]}$$
$$-(z_\nu^{(m)})_{\psi[\tau,g_{[i]}(\tau,t,x)]}-(u_{[\nu]}^{(m)})_{\psi[\tau,g_{[i]}(\tau,t,x)]}\partial_x\tilde{\psi}(\tau,g_{[i]}(\tau,t,x))\circ g_i(\tau,t,x,y)\big\}\,d\tau,$$

and

$$\Lambda_{*.i}(t,x,y)=\partial_x\kappa_i(a_i,g_{[i]}(a_i,t,x))\circ[g_i(a_i,t,x,y)-(y-x)] \tag{3.2}$$
$$+\int_{a_i}^{t}\partial_xF_i(P_i[z^{(m)}](\tau,t,x))\circ[g_i(\tau,t,x,y)-(y-x)]\,d\tau$$
$$+\int_{a_i}^{t}u_{[i]}^{(m+1)}(\tau,g_{[i]}(\tau,t,x))\partial_xf_{[i]}(\tau,g_{[i]}(\tau,t,x))\,d\tau\circ(y-x)$$
$$+\sum_{\nu=1}^{n}\int_{a_i}^{t}\partial_{v_\nu}F_i(P_i[z^{(m)}](\tau,t,x))\cdot(u_{[\nu]}^{(m)})_{\varphi(\tau,g_{[i]}(\tau,t,x))}$$
$$\cdot\,\partial_x\tilde{\varphi}(\tau,g_{[i]}(\tau,t,x))\circ[g_i(\tau,t,x,y)-(y-x)]\,d\tau$$
$$+\sum_{\nu=1}^{n}\int_{a_i}^{t}\partial_{w_\nu}F_i(P_i[z^{(m)}](\tau,t,x))\cdot(u_{[\nu]}^{(m)})_{\psi[\tau,g_{[i]}(\tau,t,x)]}$$
$$\cdot\,\partial_x\tilde{\psi}(\tau,g_{[i]}(\tau,t,x))\circ[g_i(\tau,t,x,y)-(y-x)]\,d\tau.$$

By applying the Hadamard mean value theorem to the expressions $F_i(P_i[z^{(m)}](\tau,t,y)) - F_i(P_i[z^{(m)}](\tau,t,x))$ we get $\Lambda_i(t,x,y) = \Lambda_{\kappa.i}(t,x,y) + \Lambda_{F.i}(t,x,y) + \Lambda_{v.i}(t,x,y) + \Lambda_{w.i}(t,x,y) + \Lambda_{*.i}(t,x,y)$ for $1 \le i \le k$. Since $g_{[i]}(\,\cdot,t,x)$ satisfy (2.1) we have

$$g_i(\tau,t,x,y) = y - x + \int_\tau^t \big[f_{[i]}(\xi, g_{[i]}(\xi,t,x)) - f_{[i]}(\xi, g_{[i]}(\xi,t,y))\big]\, d\xi.$$

Substituting the above relations in (3.2) and changing the order of integrals, where necessary, we get

$$\begin{aligned}\Lambda_{*.i}(t,x,y) = &\int_{a_i}^t u^{(m+1)}_{[i]}(\tau, g_{[i]}(\tau,t,x))\partial_x f_{[i]}(\tau, g_{[i]}(\tau,t,x)) \circ g_i(\tau,t,x,y)\, d\tau \\ &+ \int_{a_i}^t \big[f_{[i]}(\tau, g_{[i]}(\tau,t,x)) - f_{[i]}(\tau, g_{[i]}(\tau,t,y))\big] \circ U_i(\tau,t,x)\, d\tau\end{aligned}$$

where

$$\begin{aligned}&U_i(\tau,t,x)\\ &\quad= \partial_x \kappa_i(a_i, g_{[i]}(a_i,t,x)) + \int_{a_i}^\tau \partial_x F_i(P_i[z^{(m)}](\xi,t,x))\, d\xi \\ &\qquad- \int_{a_i}^\tau u^{(m+1)}_{[i]}(\xi, g_{[i]}(\xi,t,x))\partial_x f_{[i]}(\xi, g_{[i]}(\xi,t,x))\, d\xi \\ &\qquad+ \sum_{\nu=1}^k \int_{a_i}^\tau \partial_{v_\nu} F_i(P_i[z^{(m)}](\xi,t,x))(u^{(m)}_{[i]})_{\varphi(\xi,g_{[i]}(\xi,t,x))}\partial_x \tilde{\varphi}(\xi, g_{[i]}(\xi,t,x))\, d\xi \\ &\qquad+ \sum_{\nu=1}^k \int_{a_i}^\tau \partial_{w_\nu} F_i(P_i[z^{(m)}](\xi,t,x))(u^{(m)}_{[i]})_{\psi[\xi,g_{[i]}(\xi,t,x)]}\partial_x \tilde{\psi}(\tau, g_{[i]}(\tau,t,x))\, d\xi.\end{aligned}$$

We see at once that the characteristics satisfy the relations $g_{[i]}(\tau,\xi,g_{[i]}(\xi,t,x)) = g_{[i]}(\tau,t,x)$ for $(t,x)\in E$ and $\tau,\xi\in[a_i,t]$. We thus get $u^{(m+1)}_{[i]}(\tau, g(\tau,t,x)) = U_i(\tau,t,x)$ for $(t,x) \in E$ and $\tau \in [a_i,t]$, and consequently $\Lambda_i(t,x,y) = \Lambda_{\kappa.i}(t,x,y) + \Lambda_{F.i}(t,x,y) + \Lambda_{v.i}(t,x,y) + \Lambda_{w.i}(t,x,y) + \Lambda_{f.i}(t,x,y)$ where

$$\begin{aligned}\Lambda_{f.i}(t,x,y) = \int_{a_i}^t u^{(m+1)}_{[i]}(\tau, g_{[i]}(\tau,t,x)) \circ \big[&f_{[i]}(\tau, g_{[i]}(\tau,t,x)) - f_{[i]}(\tau, g_{[i]}(\tau,t,y)) \\ &- (g_{[i]}(\tau,t,x) - g_{[i]}(\tau,t,y))\partial_x f_{[i]}(\tau, g(\tau,t,x))\big]\, d\tau.\end{aligned}$$

For $(t,y) \in E$ we see that there is $K > 0$ such that

$$\|\Lambda_i(t,x,y)\| \le K\|x-y\|^2, \qquad (t,x) \in E,\ 1 \le i \le k.$$

Consequently, $\partial_x z_i^{(m+1)}(t,x)$ exists and $\partial_x z_i^{(m+1)} = u_{[i]}^{(m+1)}$, $1 \le i \le k$. This completes the proof. ■

4. Integral inequalities. In [10]–[9] initial value problems for partial functional differential equations are replaced by integral functional equations, and sequences of successive approximations for problems thus obtained are investigated by using theorems on linear integral inequalities of Volterra type.

In the present paper we consider functional differential equations which do not satisfy the Volterra condition. It follows that we need a new comparison result for integral inequalities. More precisely, we consider integral inequalities generated by the equation

$$y(t) = \eta + \int_0^t C_0(\tau)\,d\tau + \int_0^t B(\tau)y(\tau)\,d\tau + y(a)\int_0^t C(\tau)\,d\tau \tag{4.1}$$

where $B, C, C_0\colon [0,a] \to \mathbb{R}_+$ and $\eta \in \mathbb{R}_+$.

Lemma 4.1. *Suppose that $A, B, C, C_0 \in C([0,a],\mathbb{R}_+)$, and $\eta \in \mathbb{R}_+$.*

(I) *There exists exactly one solution of the integral equation* (4.1) *if*

$$\int_0^a C(s)e^{\int_s^a B(\tau)\,d\tau}\,ds < 1. \tag{4.2}$$

(II) *If*

$$\int_0^a C(s)e^{\int_s^a (A(\tau)+B(\tau))\,d\tau}\,ds < 1, \tag{4.3}$$

and $\tilde{y}\colon [0,a] \to \mathbb{R}_+$ is a solution of the integral equation

$$y(t) = \eta + \int_0^t C_0(\tau)\,d\tau + \int_0^t [A(\tau)+B(\tau)]y(\tau)\,d\tau + y(a)\int_0^t C(\tau)\,d\tau, \tag{4.4}$$

and $\tilde{\omega} \in C([0,a],\mathbb{R}_+)$ where $\tilde{\omega}$ is a solution of the integral inequality

$$\begin{aligned} y(t) \le \eta + \int_0^t C_0(\tau)\,d\tau + \int_0^t A(\tau)y(\tau)\,d\tau \\ + \int_0^t B(\tau)\tilde{y}(\tau)\,d\tau + \tilde{y}(a)\int_0^t C(\tau)\,d\tau, \end{aligned} \tag{4.5}$$

then

$$\tilde{\omega}(t) \le \tilde{y}(t) \quad \text{for } t \in [0,a]. \tag{4.6}$$

Proof. (I) Set

$$\bar{y}(t) = \eta e^{\int_0^t B(\tau)\,d\tau} + \int_0^t C_0(s)e^{\int_s^t B(\tau)\,d\tau}\,ds + C_\star \int_0^t C(s)e^{\int_s^t B(\tau)\,d\tau}\,ds, \qquad t \in (0,a],$$

where

$$C_\star = \Big[\eta e^{\int_0^a B(\tau)\,d\tau} + \int_0^a C_0(s)e^{\int_s^a B(\tau)\,d\tau}\,ds\Big]\Big[1 - \int_0^a C(s)e^{\int_s^a B(\tau)\,d\tau}\,ds\Big]^{-1}.$$

It follows that $\bar{y}$ is the unique solution of the Cauchy problem corresponding to (4.1). This completes the proof of the first part of the lemma.

(II) It follows from (4.3) that there exists exactly one solution $\tilde{y}\colon [0,a] \to \mathbb{R}_+$ of (4.4). Write $y_\epsilon(t) = \tilde{\omega}(t) - \tilde{\omega}_\epsilon(t)$, $t \in [0,a]$, where $\epsilon > 0$ and $\tilde{\omega}_\epsilon$ is the solution of the integral equation $\omega(t) = \epsilon + \int_0^t A(\tau)\omega(\tau)\,d\tau$. We will show that $y_\epsilon(t) < \tilde{y}(t)$ for $t \in [0,a]$. It is clear that $y_\epsilon(0) < \tilde{y}(0)$. Suppose that there is $\tilde{t} \in (0,a]$ such that $y_\epsilon(t) < \tilde{y}(t)$ for $t \in [0,\tilde{t})$ and

$$y_\epsilon(\tilde{t}) = \tilde{y}(\tilde{t}). \tag{4.7}$$

Then

$$y_\epsilon(\tilde{t}) - \tilde{y}(\tilde{t}) = \bar{y}(\tilde{t}) - \omega_\epsilon(\tilde{t}) - \tilde{y}(\tilde{t}) \le \int_0^{\tilde{t}} A(\tau)[\bar{y}(\tau) - \tilde{y}(\tau)]\,d\tau - \omega_\epsilon(\tilde{t}) \le -\epsilon,$$

which contradicts (4.7). Therefore $y_\epsilon(t) < \tilde{y}(t)$ for $t \in [0,a]$. Letting ϵ tend to zero, we obtain (4.6). ∎

Now we prove a lemma on integral inequalities of Fredholm type.

Lemma 4.2. *Suppose that $A, B, C \in C([0,a],\mathbb{R}_+)$, A is nondecreasing and condition (4.2) holds, and the function $\tilde{\omega} \in C([0,a],\mathbb{R}_+)$ is the solution of the integral inequality*

$$y(t) \le A(t) + \int_0^t B(\tau)y(\tau)\,d\tau + y(a)\int_0^t C(\tau)\,d\tau. \tag{4.8}$$

Then

$$\tilde{\omega}(t) \le A(t)e^{\int_0^t B(\tau)\,d\tau} + A(a)\Lambda_\star \int_0^t C(s)e^{\int_s^t B(\tau)\,d\tau}\,ds \tag{4.9}$$

where

$$\Lambda_\star = e^{\int_0^a B(\tau)\,d\tau}\Big[1 - \int_0^a C(\tau)e^{\int_\tau^a B(s)\,ds}\,d\tau\Big]^{-1}. \tag{4.10}$$

Proof. Write $\Psi(t) = \int_0^t B(\tau)\tilde{\omega}(\tau)\,d\tau + \tilde{\omega}(a)\int_0^t C(\tau)\,d\tau$. Then $\Psi'(t) \le B(t)[\Psi(t) + A(t)] + C(t)\tilde{\omega}(a)$ for $t \in (0,a]$ and

$$\frac{d}{dt}[\Psi(t)e^{-\int_0^t B(\tau)\,d\tau}] \le e^{-\int_0^t B(\tau)\,d\tau}[A(t)B(t) + C(t)\tilde{\omega}(a)].$$

This gives $\Psi(t) \le A(t)[e^{\int_0^t B(\tau)\,d\tau} - 1] + \tilde{\omega}(a)\int_0^t C(\tau)e^{\int_\tau^t B(s)\,ds}\,d\tau$ and

$$\tilde{\omega}(t) \le A(t)e^{\int_0^t B(\tau)\,d\tau} + \tilde{\omega}(a)\int_0^t C(\tau)e^{\int_\tau^t B(s)\,ds}\,d\tau. \tag{4.11}$$

It follows from (4.2) that $\tilde{\omega}(a) \le \Lambda_\star A(a)$. The last inequality and (4.11) imply (4.9). ∎

5. Existence of solutions. First we will give an estimate for the sequence $\{u^{(m)}\}$ for $m \ge 0$.

LEMMA 5.1. *Suppose that Assumptions $H[f,F]$, $H[\kappa]$, $H[\varphi,\psi]$ and $H[a]$ are satisfied. Then*

$$\|u^{(m)}\|_{(t,M_{k\times n})} \le \Lambda(t), \qquad t \in [0,a],\ m \ge 0, \tag{5.1}$$

where Λ is the solution to the integral equation

$$y(t) = C + \int_0^t \delta_0(\tau)\,d\tau + \int_0^t \alpha_0(\tau)y(\tau)\,d\tau + \tilde{Q}\int_0^t \beta(\tau)y(\tau)\,d\tau + y(a)\bar{Q}\int_0^t \gamma(\tau)\,d\tau.$$

Proof. For $m = 0$ the inequality comes directly from Assumption $H[\kappa]$. Suppose that $\|u^{(m)}_{[i]}\|_{(t,\mathbb{R}^n)} \le \Lambda(t)$ for $t \in [0,a]$, $1 \le i \le k$. From Assumption $H[f,F]$ we have

$$\|u^{(m+1)}_{[i]}\|_{(t,\mathbb{R}^n)} \le \int_0^t \alpha_0(\tau)\|u^{(m+1)}_{[i]}\|_{(\tau,\mathbb{R}^n)}\,d\tau + C + \int_0^t \delta_0(\tau)\,d\tau + \tilde{Q}\int_0^t \beta(\tau)\Lambda(\tau)\,d\tau + \Lambda(a)\bar{Q}\int_0^t \gamma(\tau)\,d\tau.$$

It follows from Lemma 4.1 that $\|u^{(m+1)}_{[i]}\|_{(t,\mathbb{R}^n)} \le \Lambda(t)$ for $t \in [0,a]$, $1 \le i \le k$. The proof of (5.1) is completed by induction. ∎

Now we formulate a theorem on the existence of solutions of (1.1), (1.2).

THEOREM 5.2. *If Assumptions $H[f,F]$, $H[\varphi,\psi]$, $H[a]$, $H[z^{(0)}]$ are satisfied and $\kappa \in X$ then there is a classical solution $\bar{z}\colon E_0 \cup E \to \mathbb{R}^k$ of* (1.1), (1.2). *If* $\tilde{\kappa} \in X$ *and* $\tilde{z}$ *is a classical solution of equation* (1.1) *with the initial condition* $z_i(t,x) = \tilde{\kappa}_i(t,x)$ *on* $E_{0.i}$, $1 \le i \le k$, *then*

$$\|\bar{z} - \tilde{z}\|_{(t,\mathbb{R}^k)} \le e^{\int_0^t \beta(\tau)\,d\tau}\Big(1 + \tilde{\Lambda}\int_0^t \gamma(\tau)\,d\tau\Big)\|\kappa - \tilde{\kappa}\|_X \tag{5.2}$$

and

$$\|u - \tilde{u}\|_{(t,M_{k\times n})} \le e^{\int_0^t (\alpha_0(\tau)+\tilde{Q}\beta(\tau))\,d\tau}\Big(1 + \bar{Q}\bar{\Lambda}\int_0^t \gamma(\tau)\,d\tau\Big) \cdot \Big(\|\partial_x\kappa - \partial_x\tilde{\kappa}\|_{X_{k\times n}} + \|\kappa - \tilde{\kappa}\|_X \int_0^a \bar{\zeta}(\tau)\,d\tau\Big) \tag{5.3}$$

for $t \in [0,a]$ *and* $\bar{\zeta}(t) = \big(1 + \Lambda(t)\tilde{Q} + \Lambda(a)\bar{Q}\big)\big(L_v(t)\tilde{\zeta}(t) + L_w(t)\tilde{\zeta}(a)\big)$, *and* $\tilde{\Lambda}$, $\bar{\Lambda}$ *are given in Lemma* 4.2.

Proof. The proof will be divided into four steps.

(I) We first prove that the sequence $\{z^{(m)}\}$ is uniformly convergent on $E_0 \cup E$. From Assumptions $H[\kappa]$ and $H[f,F]$ we deduce that there is $\gamma_0 \in C([0,a],\mathbb{R}_+)$ such that $\|z^{(0)} - \mathcal{F}[z^{(0)}]\|_{(t,\mathbb{R}^k)} \le \int_0^t \gamma_0(\tau)\,d\tau$. Define the sequence $\{\omega^{(k)}\}$ as follows: $\omega^{(0)}$ is a solution of the Cauchy problem

$$\omega'(t) = \beta(t)\omega(t) + \omega(a)\gamma(t) + \gamma_0(t), \qquad \omega(0) = 0.$$

For given $\omega^{(m)}$ we have

$$\omega^{(m+1)}(t) = \int_0^t \beta(\tau)\omega^{(m)}(\tau)\,d\tau + \omega^{(m)}(a)\int_0^t \gamma(\tau)\,d\tau.$$

Then for $m \ge 0$ we have $0 \le \omega^{(m+1)}(t) \le \omega^{(m)}(t)$ for $t \in [0,a]$ and $\lim_{m\to\infty}\omega^{(m)}(t) = 0$ uniformly on $[0,a]$.

Now we will prove that $\{z^{(m)}\}$ is a Cauchy sequence in $C_\psi(E_0 \cup E, \mathbb{R}^k)$. More precisely, we will show by induction on m that for every $m,p \ge 0$ we have $\|z^{(m+p)} - z^{(m)}\|_{(t,\mathbb{R}^k)} \le \omega^{(m)}(t)$ for $t \in [0,a]$.

Suppose that $m = 0$. Then

$$\|z^{(1)} - z^{(0)}\|_{(t,\mathbb{R}^k)} \le \|\mathcal{F}[z^{(0)}] - z^{(0)}\|_{(t,\mathbb{R}^k)} \le \omega^{(0)}(t), \qquad t \in [0,a].$$

Now take $p > 0$ and assume that $\|z^{(p)} - z^{(0)}\|_{(t,\mathbb{R}^k)} \le \omega^{(0)}(t)$ for $t \in [0,a]$. Then

$$\begin{aligned}\|z^{(p+1)} - z^{(0)}\|_{(t,\mathbb{R}^k)} &\le \|\mathcal{F}[z^{(p)}] - \mathcal{F}[z^{(0)}]\|_{(t,\mathbb{R}^k)} + \int_0^t \gamma_0(\tau)\,d\tau \\ &\le \int_0^t \beta(\tau)\omega^{(0)}(\tau)\,d\tau + \omega^{(0)}(a)\int_0^t \gamma(\tau)\,d\tau + \int_0^t \gamma_0(\tau)\,d\tau = \omega^{(0)}(t).\end{aligned}$$

Suppose that for a fixed $m > 0$ we have $\|z^{(m+p)} - z^{(m)}\|_{(t,\mathbb{R}^k)} \le \omega^{(m)}(t)$ for all $p \ge 0$ and $t \in [0,a]$. Then

$$\begin{aligned}\|z^{(m+1+p)} &- z^{(m+1)}\|_{(t,\mathbb{R}^k)} \\ &\le \int_0^t \beta(\tau)\|z^{(m+p)} - z^{(m)}\|_{(\tau,\mathbb{R}^k)}\,d\tau + \|z^{(m+p)} - z^{(m)}\|_{(a,\mathbb{R}^k)} \int_0^t \gamma(\tau)\,d\tau \\ &\le \int_0^t \beta(\tau)\omega^{(m)}(\tau)\,d\tau + \omega^{(m)}(a)\int_0^t \gamma(\tau)\,d\tau = \omega^{(m+1)}(t).\end{aligned}$$

By induction we find that $\{z^{(m)}\}$ is a Cauchy sequence in $C_\psi(E_0 \cup E, \mathbb{R}^k)$. Therefore there is $\bar{z} \in C_\psi(E_0 \cup E, \mathbb{R}^k)$ such that $\lim_{m\to\infty} z^{(m)}(t,x) = \bar{z}(t,x)$ uniformly on $E_0 \cup E$.

(II) Now we will prove that $\{u^{(m)}\}$ is a Cauchy sequence in $C_{\partial\psi}(E_0 \cup E, \mathbb{R}^k)$. Set $\zeta^{(m)}(t) = [1 + \Lambda(a)(\tilde{Q} + \bar{Q})]\,[L_v(t)\omega^{(m)}(t) + L_w(t)\omega^{(m)}(a)]$, $m \ge 0$, $\bar{\zeta}(t) = \Lambda(a)[\alpha_0(t) + \tilde{Q}\beta(t) + \bar{Q}\gamma(t)] + \delta_0(t)$, $\tilde{\zeta}(t) = \max\{\zeta^{(0)}(t), \bar{\zeta}(t)\}$ where $t \in [0,a]$. Consider the sequence $\{\bar{\omega}^{(m)}\}$ defined in the following way. The function $\bar{\omega}^{(0)}$ is the solution of the integral equation

$$y(t) = \int_0^t [\alpha_0(\tau) + \tilde{Q}\beta(\tau)]y(\tau)\,d\tau + \bar{Q}y(a)\int_0^t \gamma(\tau)\,d\tau + \int_0^t \tilde{\zeta}(\tau)\,d\tau.$$

For a given $\bar{\omega}^{(m)}$ we can compute $\bar{\omega}^{(m+1)}$ as the solution of the integral equation

$$\begin{aligned}y(t) = \int_0^t \alpha_0(\tau)y(\tau)\,d\tau &+ \tilde{Q}\int_0^t \beta(\tau)\bar{\omega}^{(m)}(\tau)\,d\tau \\ &+ \bar{Q}\bar{\omega}^{(m)}(a)\int_0^t \gamma(\tau)\,d\tau + \int_0^t \zeta^{(m)}(\tau)\,d\tau.\end{aligned}$$

It follows from Assumption $H[a]$ and Lemma 4.1 that $0 \le \bar{\omega}^{(m+1)}(t) \le \bar{\omega}^{(m)}(t)$ for $t \in [0,a]$, $m \ge 0$, and $\lim_{m\to\infty} \bar{\omega}^{(m)}(t) = 0$ uniformly on $[0,a]$.

We claim that

$$\|u^{(m+p)} - u^{(m)}\|_{(t,M_{k\times n})} \le \bar{\omega}^{(m)}(t), \qquad t \in [0,a], \tag{5.4}$$

where $m, p \ge 0$. We prove (5.4) by induction on m.

We first prove that

$$\|u^{(p)} - u^{(0)}\|_{(t,M_{k\times n})} \le \int_0^t \beta_0(\tau)\,d\tau \le \bar{\omega}^{(0)}(t), \qquad t \in [0,a], \tag{5.5}$$

where $p \ge 0$. It is clear that (5.5) is satisfied for $p = 0$. Assuming that (5.5) holds for p, we will prove it for $p+1$. It follows from Assumptions $H[f,F]$,

$H[a]$ that the function $\bar{y}(t) = \|u^{(p+1)} - u^{(0)}\|_{(t,M_{k\times n})}$, $t \in [0,a]$, is a solution of the integral inequality

$$y(t) \le \int_0^t \alpha_0(\tau) y(\tau)\, d\tau + \tilde{Q} \int_0^t \beta(\tau)\bar{\omega}^{(0)}(\tau)\, d\tau + \bar{Q}\bar{\omega}^{(0)}(a) \int_0^t \gamma(\tau)\, d\tau + \int_0^t \bar{\zeta}(\tau)\, d\tau.$$

Then $\bar{y}(t) \le \bar{\omega}^{(0)}(t)$ for $t \in [0,a]$ and by induction the proof of (5.5) is complete.

Suppose that estimate (5.4) is satisfied for a fixed $m \ge 0$ and for all $p \ge 0$. We prove that

$$\|u^{(m+p+1)} - u^{(m+1)}\|_{(t,M_{k\times n})} \le \bar{\omega}^{(m+1)}(t), \qquad t \in [0,a],\ p \ge 0. \tag{5.6}$$

Set $\tilde{y}(t) = \|u^{(m+p+1)} - u^{(m+1)}\|_{(t,M_{k\times n})}$ for $t \in [0,a]$. It follows from Assumption $H[f,F]$ that $\tilde{y}$ is a solution of the integral inequality

$$y(t) \le \int_0^t \alpha_0(\tau) y(\tau)\, d\tau + \tilde{Q} \int_0^t \beta(\tau)\bar{\omega}^{(m)}(\tau)\, d\tau + \bar{Q}\bar{\omega}^{(m)}(a) \int_0^t \gamma(\tau)\, d\tau + \int_0^t \zeta^{(m)}(\tau)\, d\tau.$$

This gives $\tilde{y}(t) \le \bar{\omega}^{(m+1)}(t)$ for $t \in [0,a]$ and consequently estimates (5.6) are satisfied.

It follows by induction on m that the proof of (5.4) is complete. This implies that $\{u^{(m)}\}$ is a Cauchy sequence in $C_{\partial\kappa}(E_0 \cup E, M_{k\times n})$. Therefore there is $\bar{u} \in C_{\partial\kappa}(E_0 \cup E, M_{k\times n})$ such that $\lim_{m\to\infty} u^{(m)}(t,x) = \bar{u}(t,x)$ uniformly on $E_0 \cup E$.

It follows from Lemma 3.1 that $\partial_x \bar{z}_i$, $1 \le i \le k$, exist on E and $\partial_x \bar{z}_i = \bar{u}_{[i]}$. Furthermore, we have

$$\bar{z}(t,x) = \mathcal{F}[\bar{z}](t,x), \qquad (t,x) \in E. \tag{5.7}$$

For a given $(t,x) \in E$ let us put $y = g_{[i]}(0,t,x)$, $1 \le i \le k$. It follows that $g_{[i]}(\tau,t,x) = g_{[i]}(\tau,0,y)$. We conclude from (5.7) that

$$\bar{z}_i(t, g_{[i]}(t,0,y)) = \psi_i(0,y) + \int_0^t F_i\big(\tau, g_{[i]}(\tau,0,y), \bar{z}_{\varphi(\tau,g_{[i]}(\tau,0,y))}, \bar{z}_{\psi[\tau,g_{[i]}(\tau,0,y)]}\big)\, d\tau. \tag{5.8}$$

The relations $y = g_{[i]}(0,t,x)$ and $x = g_{[i]}(t,0,y)$ are equivalent. By differentiating (5.8) with respect to t and by putting again $x = g_{[i]}(t,0,y)$ we infer that $\bar{z}$ satisfies (1.1) on E.

(III) Now we will prove inequality (5.2). First we will investigate $\|\bar{z} - \tilde{z}\|_{(t,\mathbb{R}^k)}$. From Assumption $H[f,F]$ we have

$$\|\bar{z} - \tilde{z}\|_{(t,\mathbb{R}^k)} \le \|\kappa - \tilde{\kappa}\|_X + \int_0^t \beta(\tau)\|\bar{z} - \tilde{z}\|_{(\tau,\mathbb{R}^k)}\,d\tau + \|\bar{z} - \tilde{z}\|_{(a,\mathbb{R}^k)} \int_0^t \gamma(\tau)\,d\tau.$$

From Lemma 4.2 we have (5.2).

(IV) Write $\partial_x \bar{z} = u$ and $\partial_x \tilde{z} = \tilde{u}$. From Assumptions $H[f,F]$, $H[a]$ we deduce the integral inequality

$$\begin{aligned}\|u - \tilde{u}\|_{(t,M_{k\times n})} \le &\int_0^t (\alpha_0(\tau) + \tilde{Q}\beta(\tau))\|u - \tilde{u}\|_{(\tau,M_{k\times n})}\,d\tau \\ &+ \bar{Q}\|u - \tilde{u}\|_{(a,M_{k\times n})} \int_0^t \gamma(\tau)\,d\tau + \|\partial_x\kappa - \partial_x\tilde{\kappa}\|_{X_{k\times n}} \\ &+ \|\kappa - \tilde{\kappa}\|_X \int_0^t \bar{\zeta}(\tau)\,d\tau.\end{aligned}$$

Using Lemma 4.2 we obtain (5.3). ∎

6. Differentiability of solutions with respect to initial functions. Suppose that Assumptions $H[f,F]$, $H[\varphi,\psi]$, $H[a]$ are satisfied and $\kappa \in X$. Let us denote by $\Xi[\kappa]$ the solution of the Cauchy problem (1.1), (1.2). It follows from Theorem 5.2 that $\Xi\colon X \to C_\kappa(E_0 \cup E, \mathbb{R}^k)$. The next theorem states that for each $\kappa \in X$ the Fréchet derivative $\partial\Xi[\kappa]$ of the operator Ξ exists at $\kappa \in X$. Moreover, if $\kappa, \chi \in X$ and $z_* = \partial\Xi[\kappa]\chi$ then z_* is a solution of an integral functional equation generated by (1.1).

We will denote by $z(\,\cdot\,;\kappa)$ the solution of (1.1) with initial condition $z_i(t,x) = \kappa_i(t,x)$ on $E_{0.i}$, $1 \le i \le k$.

THEOREM 6.1. *If Assumptions $H[f,F]$, $H[\varphi,\psi]$ and $H[a]$ are satisfied then for each $\kappa \in X$ the Fréchet derivative $\partial\Xi[\kappa]$ exists. Moreover, if $\kappa, \chi \in X$ and $z_* = \partial\Xi[\kappa]\chi$ then z_* is a solution of the equation*

$$z = \Lambda[z] \tag{6.1}$$

where $\Lambda[z] = (\Lambda_1[z], \ldots, \Lambda_k[z])$ and

$$\Lambda_i[z](t,x) = \chi_i(t,x) \quad \text{on } E_{0.i},\ 1 \le i \le k,$$

and

$$\begin{aligned}\Lambda_i[z](t,x) &= \chi_i(0, g_{[i]}(0,t,x)) \\ &\quad + \sum_{\nu=1}^{k}\int_{a_i}^{t} \partial_{v_\nu} F_i(P_i[z](\tau,t,x,\kappa))(z_\nu)_{\varphi(\tau,g_{[i]}(\tau,t,x))}\,d\tau \\ &\quad + \sum_{\nu=1}^{k}\int_{a_i}^{t} \partial_{w_\nu} F_i(P_i[z](\tau,t,x,\kappa))(z_\nu)_{\psi[\tau,g_{[i]}(\tau,t,x)]}\,d\tau \quad \text{on } E\setminus E_{0.i},\end{aligned}$$

$$P_i[z](\tau,t,x,\kappa) = (\tau, g_{[i]}(\tau,t,x), (z(\,\cdot\,;\kappa))_{\varphi(\tau,g_{[i]}(\tau,t,x))}, (z(\,\cdot\,;\kappa))_{\psi[\tau,g_{[i]}(\tau,t,x)]}).$$

Proof. Write

$$\begin{aligned}&Q_i[\kappa,\chi](\xi,\tau,t,x) = \big(\tau, g_{[i]}(\tau,t,x), \\ &(1-\xi)z(\,\cdot\,;\kappa)_{\varphi(\tau,g_{[i]}(\tau,t,x))} + \xi z(\,\cdot\,;\kappa+s\chi)_{\varphi(\tau,g_{[i]}(\tau,t,x))}, z(\,\cdot\,;\kappa+s\chi)_{\psi[\tau,g_{[i]}(\tau,t,x)]}\big), \\ &\tilde{Q}_i[\kappa,\chi](\xi,\tau,t,x) = \big(\tau, g_{[i]}(\tau,t,x), z(\,\cdot\,;\kappa)_{\varphi(\tau,g_{[i]}(\tau,t,x))}, \\ &\qquad\qquad (1-\xi)z(\,\cdot\,;\kappa)_{\psi[\tau,g_{[i]}(\tau,t,x)]} + \xi z(\,\cdot\,;\kappa+s\chi)_{\psi[\tau,g_{[i]}(\tau,t,x)]}\big),\end{aligned}$$

where $0\le\xi\le 1$, $1\le i\le k$ and

$$\begin{aligned}\Delta_{s.i}(t,x) &= \frac{1}{s}[z_i(t,x;\kappa+s\chi) - z_i(t,x;\kappa)] \quad \text{on } E,\ 1\le i\le k, \\ \Delta_{s.i}(t,x) &= \chi_i(t,x) \quad \text{on } E_{0.i},\end{aligned}$$

where $1\le i\le k$, $s\in\mathbb{R}$, $s\ne 0$. For a function $z\in C_\kappa(E_0\cup E,\mathbb{R}^k)$ we put

$$\Lambda_{s.i}[z](t,x) = \chi_i(t,x) \quad \text{on } E_{0.i},$$

$$\begin{aligned}\Lambda_{s.i}[z](t,x) &= \chi_i(0, g_{[i]}(0,t,x)) \\ &\quad + \sum_{\nu=1}^{k}\int_{a_i}^{t}\int_0^1 \partial_{v_\nu} F_i(Q_i[\kappa,\chi](\xi,\tau,t,x))(z_\nu)_{\varphi(\tau,g_{[i]}(\tau,t,x))}\,d\xi\,d\tau \\ &\quad + \sum_{\nu=1}^{k}\int_{a_i}^{t}\int_0^1 \partial_{w_\nu} F_i(\tilde{Q}_i[\kappa,\chi](\xi,\tau,t,x))(z_\nu)_{\psi[\tau,g_{[i]}(\tau,t,x)]}\,d\xi\,d\tau \quad \text{on } E\setminus E_{0.i},\end{aligned}$$

where $1\le i\le k$. We conclude from Assumption $H[\,f,F\,]$ that the function Δ_s satisfies the integral functional equation $z=\Lambda_s[z]$. It is easily seen that there exists exactly one solution $z_*\in C_\chi(E_0\cup E,\mathbb{R}^k)$, $z_*=(z_{*.1},\dots,z_{*.k})$, of (6.1). We thus get

$$(z_{*.i} - \Delta_{s.i})(t,x) = \sum_{\nu=1}^{n} \int_{a_i}^{t} \int_0^1 \big[\partial_{v_\nu} F_i(P_i[z](\tau,t,x,\kappa)) \\ - \partial_{v_\nu} F_i(Q_i[\kappa,\chi](\xi,\tau,t,x))\big](z_{*.\nu})_{\varphi(\tau,g_{[i]}(\tau,t,x))}\,d\xi\,d\tau \\ + \sum_{\nu=1}^{n} \int_{a_i}^{t} \int_0^1 \partial_{v_\nu} F_i(Q_i[\kappa,\chi](\xi,\tau,t,x))(z_{*.\nu} - \Delta_{s.\nu})_{\varphi(\tau,g_{[i]}(\tau,t,x))}\,d\xi\,d\tau \\ + \sum_{\nu=1}^{n} \int_{a_i}^{t} \int_0^1 \big[\partial_{w_\nu} F_i(P_i[z](\tau,t,s,\kappa)) \\ - \partial_{w_\nu} F_i(\tilde{Q}_i[\kappa,\chi](\xi,\tau,t,x))\big](z_{*.\nu})_{\psi[\tau,g_{[i]}(\tau,t,x)]}\,d\xi\,d\tau \\ + \sum_{\nu=1}^{n} \int_{a_i}^{t} \int_0^1 \partial_{w_\nu} F_i(\tilde{Q}_i[\kappa,\chi](\xi,\tau,t,x))(z_{*.\nu} - \Delta_{s.\nu})_{\psi[\tau,g_{[i]}(\tau,t,x)]}\,d\xi\,d\tau,\ 1 \le i \le k,$$

and $(z_{*.i} - \Delta_{s.i})(t,x) = 0$ on $E_{0.i}$, $1 \le i \le k$. It follows from the above relations and from Assumption $H[f,F]$ that there is $\tilde{L}_0 \in C([0,a],\mathbb{R}_+)$ such that

$$\|z_* - \Delta_s\|_{(t,\mathbb{R}^k)} \\ \le \int_0^t \tilde{L}_0(\tau)\big(\|z(\,\cdot\,;\kappa+s\chi) - z(\,\cdot\,;\kappa)\|_{(\tau,\mathbb{R}^k)} + \|z(\,\cdot\,;\kappa+s\chi) - z(\,\cdot\,;\kappa)\|_{(a,\mathbb{R}^k)}\big)\,d\tau \\ + \int_0^t \beta(\tau)\|z_* - \Delta_s\|_{(\tau,\mathbb{R}^k)}\,d\tau + \|z_* - \Delta_s\|_{(a,\mathbb{R}^k)} \int_0^t \gamma(\tau)\,d\tau, \qquad t \in [0,a].$$

We conclude from Theorem 5.2 that $\|z(\,\cdot;\kappa+s\chi) - z(\,\cdot;\kappa)\|_{(t,\mathbb{R}^k)} \le \tilde{\zeta}(t)|s|\,\|\chi\|_X$. Then

$$\|z_* - \Delta_s\|_{(t,\mathbb{R}^k)} \le |s|\,\|\chi\|_X \int_0^t \tilde{L}_0(\tilde{\zeta}(\tau) + \tilde{\zeta}(a))\,d\tau + \int_0^t \beta(\tau)\|z_* - \Delta_s\|_{(\tau,\mathbb{R}^k)}\,d\tau \\ + \|z_* - \Delta_s\|_{(a,\mathbb{R}^k)} \int_0^t \gamma(\tau)\,d\tau$$

for $t \in [0,a]$. Hence from Lemma 4.2 there exists $L \in C([0,a],\mathbb{R}_+)$ such that $\|z_* - \Delta_s\|_{(t,\mathbb{R}^k)} \le L(t)|s|\,\|\chi\|_X$. From the above inequality we conclude that $\lim_{s\to 0} \Delta_s$ exists and $\lim_{s\to 0} \Delta_s(t,x) = z_*(t,x)$ uniformly on E. This proves the theorem. ∎

Remark 6.2. It is easy to see that the integral functional equation (6.1) is generated by the linear differential functional equation

$$
\begin{aligned}
\partial_t z_i(t,x) &+ \sum_{i=1}^{n} f_{[i]}(t,x)\partial_{x_i} z_i(t,x) \\
&= \sum_{\nu=1}^{k} \partial_{v_\nu} F_i\big(t,x,z(\,\cdot\,;\kappa)_{\varphi(t,x)}, z(\,\cdot\,;\kappa)_{\psi[t,x]}\big)(z_\nu)_{\varphi(t,x)} \\
&\quad + \sum_{\nu=1}^{k} \partial_{w_\nu} F_i\big(t,x,z(\,\cdot\,;\kappa)_{\psi[t,x]}, z(\,\cdot\,;\kappa)_{\varphi[t,x]}\big)(z_\nu)_{\psi[t,x]},
\end{aligned}
$$

and the initial condition $z_i(t,x) = \chi_i(t,x)$ on $E_{0.i}$, $1 \le i \le k$.

REMARK 6.3. Let us consider the system of functional differential equations

$$
(6.2) \qquad \partial_t z_i(t,x) + f_{[i]}(t,x) \circ \partial_x z_i(t,x) = F_i(t,x,z_{(t,x)}, z_{[t,x]}), \quad i = 1,\dots,k,
$$

which is a particular case of (1.1). The functional differential problem that consists of (6.2) and (1.2) is a *generalized Cauchy problem*.

There are the following motivations for investigation of (1.1), (1.2) instead of (6.2), (1.2). Differential systems with deviated variables are obtained from (6.2) in the following way. Suppose that $G\colon E \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$, $G = (G_1,\dots,G_k)$, is a given function. Write

$$
(6.3) \qquad F(t,x,v,w) = G\big(t,x,v(\varphi(t,x)-(t,x)), w(\psi(t,x)-(t,x))\big) \quad \text{on } \Omega.
$$

Then (6.2) is equivalent to (1.5).

Note that for the function $(F_1,\dots,F_k)$ given by (6.3) Assumption $H[f,F]$ is not satisfied. More precisely the derivatives $\partial_x F = [\partial_{x_j} F_i]_{i=1,\dots,k,\, j=1,\dots,n}$ do not exist on Ω. It is clear that under natural assumptions on G the function F given by (1.4) satisfies Assumption $H[f,F]$.

With the above motivation we have considered problem (1.1), (1.2).

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