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On uniqueness of algebroid functions with shared values in some angular domains

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Abstract. We investigate the uniqueness of transcendental algebroid functions with shared values in some angular domains instead of the whole complex plane $\mathbb{C}$. We obtain two theorems which are counterparts of results for meromorphic functions obtained by Zheng.

1. Introduction and main results. We assume that the readers are familiar with the standard notations and fundamental results of Nevanlinna theory in the unit disk $\Delta = \{z : |z| < 1\}$ and in the complex plane $\mathbb{C}$ (see [2, 16]). An $a \in \widehat{\mathbb{C}}$ is called an *IM* (ignoring multiplicities) *shared value* in $X \subseteq \mathbb{C}$ of two ν-valued algebroid functions $f(z)$ and $g(z)$ if $f(z) = a$ if and only if $g(z) = a$ in X. He [3] proved that $f(z) \equiv g(z)$ if two ν-valued algebroid functions $f(z)$ and $g(z)$ have $4\nu + 1$ distinct IM shared values in $X = \mathbb{C}$.

Zheng [22] was the first to consider the uniqueness of meromorphic functions with shared values in a proper subset of $\mathbb{C}$. After Zheng's work, many researchers have investigated the uniqueness of meromorphic functions in angular domains: Lin, Mori and Tohge [5], Lin, Mori and Yi [6], Liu and Sun [7], Mao and Liu [9]. In 2010, Liu and Sun [8] studied the uniqueness of algebroid functions in an angular domain.

In this paper, we consider the uniqueness of ν-valued algebroid functions with shared values in q angular domains. Our results extend some uniqueness theorems of [22] for meromorphic functions to algebroid functions.

Before stating the results, we give some notations and definitions. Let $w = w(z)$ be the ν-valued algebroid function defined by an irreducible equation

$$F(z, w) := A_0(z)w^{\nu} + A_1(z)w^{\nu-1} + \cdots + A_{\nu}(z) = 0, \tag{1.1}$$

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where $A_\nu(z), \ldots, A_0(z)$ are analytic functions without any common zeros. Let $\vec{A} = (A_0, \ldots, A_\nu)$ and $\vec{\infty} = (1, 0, \ldots, 0)$. For any $a \in \mathbb{C}$, denote $\vec{a} = (a^\nu, a^{\nu-1}, \ldots, 1)$. Then

$$\|\vec{A}(z)\| = (|A_0|^2 + |A_1|^2 + \cdots + |A_\nu|^2)^{1/2},$$

$$\|\vec{a}\| = \begin{cases} (|a|^{2\nu} + |a|^{2\nu-2} + \cdots + |a|^2 + 1)^{1/2}, & a \neq \infty, \\ 1, & a = \infty. \end{cases}$$

Since $F(z,w)$ is irreducible, we have $F(z,a) = \vec{A}(z) \cdot \vec{a} \not\equiv 0$, where $F(z,\infty) = A_0(z)$. Set

$$\log^+ x = \max\{0, \log x\}.$$

Define

$$m(r, \vec{a}, \vec{A}) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|\vec{A}(re^{i\theta})\| \, \|\vec{a}\|}{|F(re^{i\theta}, a)|} \, d\theta,$$

$$N(r, \vec{a}, \vec{A}) = N(r, 0, F(z,a))$$

$$= \int_0^r \frac{n(t, 0, F(z,a)) - n(0, 0, F(z,a))}{t} \, dt + n(0, 0, F(z,a)) \log r,$$

$$T(r, \vec{a}, \vec{A}) = m(r, \vec{a}, \vec{A}) + N(r, \vec{a}, \vec{A}),$$

where $n(t, 0, F(z,a))$ is the number of roots of the equation $F(z,a) = 0$ in the disk $\{z : |z| \leq t\}$, counting multiplicities. Throughout, $n(t, a, w(z))$ denotes the number of roots of $w(z) = a$ in the disk $\{z : |z| \leq t\}$, counting multiplicities. Following G. Valiron, we define the characteristic function of $w(z)$ as

$$T(r, w) = \frac{1}{2\nu\pi} \int_0^{2\pi} \log \max_{0 \leq j \leq \nu} |A_j(re^{i\theta})| \, d\theta.$$

By Yang's result [15], we get the relation between $T(r,w)$ and $T(r, \vec{a}, \vec{A})$:

$$|T(r, \vec{a}, \vec{A}) - \nu T(r, w)| = O(1).$$

The counting function of roots of $w(z) - a$ is defined as

$$N(r, a, w) = \frac{1}{\nu} N(r, 0, F(z,a)).$$

Put

$$\delta(a, w) = 1 - \limsup_{r \to \infty} \frac{N(r, a, w)}{T(r, w)} = 1 - \limsup_{r \to \infty} \frac{N(r, 0, F(z,a))}{T(r, \vec{a}, \vec{A})}.$$

The value a is called a *Nevanlinna deficient value* of w if $\delta(a, w) > 0$. The *order* and *lower order* of $w(z)$ are defined as

$$\lambda(w) := \limsup_{r \to \infty} \frac{\log T(r, w)}{\log r}, \qquad \mu(w) := \liminf_{r \to \infty} \frac{\log T(r, w)}{\log r}.$$

Give a subset $Y \subset \mathbb{C}$, let $n(r, Y, w = a)$ denote the number of roots of $w(z) - a$ in $Y \cap \{z : |z| \le r\}$, counting multiplicities. Define the counting function of a-points of $w(z)$ in Y as

$$N(r, Y, w = a) = \frac{1}{\nu} \int_0^r \frac{n(t, Y, w = a)}{t}\, dt.$$

We consider q pairs $\{\alpha_j, \beta_j\}$ of real numbers satisfying

$$-\pi \le \alpha_1 < \beta_1 \le \cdots \le \alpha_q < \beta_q \le \pi, \tag{1.2}$$

and define $\omega = \max_{1 \le i \le q}\{\pi/(\beta_i - \alpha_i)\}$.

Now we state our results.

THEOREM 1.1. *Let $f(z)$ and $g(z)$ be ν-valued transcendental algebroid functions and let $f(z)$ be of finite lower order μ and such that for some $a \in \widehat{\mathbb{C}}$, $\delta = \delta(a, f) > 0$. For q pairs $\{\alpha_j, \beta_j\}$ of real numbers satisfying* (1.2) *and*

$$0 < \sum_{i=1}^{q} (\alpha_{i+1} - \beta_i) < \frac{4}{\sigma} \arcsin \sqrt{\delta/2}, \qquad \alpha_{q+1} = \pi + \alpha_1, \tag{1.3}$$

where $\sigma = \max\{\omega, \mu\}$, assume that $f(z)$ and $g(z)$ have $4\nu + 1$ distinct IM shared values in $X = \bigcup_{j=1}^{q}\{z : \alpha_j \le \arg z \le \beta_j\}$. If $\omega < \lambda(f)$, then $f(z) \equiv g(z)$.

If we remove the condition $\mu(f) < \infty$ in Theorem 1.1, then we have the following theorem.

THEOREM 1.2. *Let $f(z)$ and $g(z)$ be ν-valued algebroid functions such that for some $a \in \widehat{\mathbb{C}}$, $\delta = \delta(a, f) > 0$. Assume that for q rays $\arg z = \alpha_j$ $(1 \le j \le q)$ satisfying*

$$-\pi \le \alpha_1 < \cdots < \alpha_q < \pi, \qquad \alpha_{q+1} = \alpha_1 + 2\pi,$$

$f(z)$ and $g(z)$ have $4\nu + 1$ distinct IM shared values in $X = \mathbb{C} \setminus \bigcup_{j=1}^{q}\{z : \arg z = \alpha_j\}$. If

$$\max\left\{\frac{\pi}{\alpha_{j+1} - \alpha_j} : 1 \le j \le q\right\} < \lambda(f), \tag{1.4}$$

then $f(z) \equiv g(z)$.

REMARK 1.3. We mention that the total linear measure of flare angles of angular domains $X = \bigcup_{j=1}^{q}\{z : \alpha_j \le \arg z \le \beta_j\}$ in Theorem 1.1 is less than 2π under the condition (1.3). This indicates that the uniqueness of algebroid functions we study in this paper is different from the case of $\mathbb{C}$.

2. Lemmas. The following result was proved in [20] for a meromorphic function.

LEMMA 2.1. *Let $f(z)$ be an algebroid function in $\mathbb{C}$ of finite lower order $0 \le \mu < \infty$ and order $0 < \lambda \le \infty$. Then for any positive number β satisfying $\mu \le \beta \le \lambda$ and any set E of finite logarithmic measure, i.e., $\int_E t^{-1}\,dt < \infty$, there exists a sequence $\{r_n\}$ of positive numbers such that*

(1) $r_n \notin E$, $\lim_{n\to\infty} r_n/n = \infty$;
(2) $\liminf_{n\to\infty} \log T(r_n, f)/\log r_n \ge \beta$;
(3) $T(t,f) < (1+o(1))(2t/r_n)^\beta T(r_n/2, f)$, $t \in [r_n/n, nr_n]$;
(4) $T(t,f)/t^{\beta-\varepsilon_n} \le 2^{\beta+1} T(r_n, f)/r_n^{\beta-\varepsilon_n}$, $1 \le t \le nr_n$, $\varepsilon_n = [\log n]^{-2}$.

Since the characteristic function $T(r,f)$ of an algebroid function $f(z)$ is also a non-decreasing, positive and continuous function defined in $(0,\infty)$, we can derive Lemma 2.1 directly from [20]. A sequence $\{r_n\}$ satisfying (1)–(4) in Lemma 2.1 is called a sequence of *Pólya peaks* of order β outside E. For $r > 0$ and $a \in \widehat{\mathbb{C}}$, define

$$E(r,a) = \left\{\theta : \log^+ \frac{\|\vec{A}(re^{i\theta})\|\,\|\vec{a}\|}{|F(re^{i\theta}, a)|} > \frac{1}{\log r} T(r,f)\right\}.$$

The following result is a special version of the main result of Krytov [4] and Yang [15].

LEMMA 2.2 ([4, 15]). *Let $f(z)$ be a ν-valued algebroid function in $\mathbb{C}$ of finite lower order μ and order $0 < \lambda \le \infty$ and for some $a \in \widehat{\mathbb{C}}$, $\delta = \delta(a,f) > 0$. Then for any sequence $\{r_n\}$ of Pólya peaks of order $\sigma > 0$, where $\mu \le \sigma \le \lambda$, we have*

$$\liminf_{n\to\infty} \operatorname{meas} E(r_n, a) \ge \min\{2\pi, (4/\sigma)\arcsin\sqrt{\delta/2}\}, \tag{2.1}$$

where meas *denotes Lebesgue measure.*

Lemma 2.2 was proved in [4, 15] for Pólya peaks of order μ. By the same argument, one can derive it for Pólya peaks of order σ with $\mu \le \sigma \le \lambda$.

LEMMA 2.3. *Let $f(z)$ be the ν-valued algebroid function determined by* (1.1) *in the complex plane. Assume that $X \subset \mathbb{C}$ is an open simply connected domain in $\mathbb{C}$, and $u = u(z)$ is a conformal mapping from X onto the unit disk. Then $f(z(u))$ is a ν-valued algebroid function defined in the unit disk.*

Lemma 2.3 was proved by the first author [12, 11]. For completeness, we repeat the proof.

Proof. It is obvious that $f(z(u))$ is an algebroid function determined by the equation

$$F(z(u), w) := A_0(z(u))w^\nu + A_1(z(u))w^{\nu-1} + \cdots + A_\nu(z(u)) = 0.$$

The fact that $A_i(z)$ $(i = 0,\ldots,\nu)$ are entire functions implies that $A_0(z(u)),\ldots,A_\nu(z(u))$ are analytic functions. As $A_0(z),\ldots,A_\nu(z)$ have no common zeros, $A_0(z(u)),\ldots,A_\nu(z(u))$ have no common zeros either:

if $u = u_0$ were a common zero, then $z(u_0)$ would be a common zero of $A_0(z), \dots, A_\nu(z)$. Since $F(z,w)$ is irreducible, so is $F(z(u),w)$, because if $F(z(u),w) = F_1(u,w)F_2(u,w)$, then $F(z,w) = F_1(u(z),w)F_2(u(z),w)$, a contradiction. ∎

Lemma 2.4 ([17]). *The transformation*

$$\zeta(z) = \frac{(ze^{-i\theta_0})^{\pi/(\beta-\alpha)} - 1}{(ze^{-i\theta_0})^{\pi/(\beta-\alpha)} + 1} \qquad (\theta_0 = (\alpha+\beta)/2) \tag{2.2}$$

maps the angle $X = \{z : \alpha < \arg z < \beta\}$ $(0 \le \alpha < \beta \le 2\pi,\ 0 < \beta - \alpha \le 2\pi)$ *conformally onto the unit disk* $\{\zeta : |\zeta| < 1\}$, *and maps* $z = e^{i\theta_0}$ *to* $\zeta = 0$. *Under the transformation* (2.2), *the image of* $X_\varepsilon(r) = \{z : 1 \le |z| \le r,\ \alpha+\varepsilon \le \arg z \le \beta - \varepsilon\}$ $(0 < \varepsilon < (\beta-\alpha)/2)$ *is contained in the disk* $\{\zeta : |\zeta| \le h\}$, *where*

$$h = 1 - \frac{\varepsilon}{\beta-\alpha} r^{-\pi/(\beta-\alpha)}.$$

On the other hand, the inverse image of the disk $\{\zeta : |\zeta| \le t\}$ $(t < 1)$ *in the* z*-plane is contained in* $X \cap \{z : |z| \le \rho\}$, *where*

$$\rho = \left(\frac{2}{1-t}\right)^{(\beta-\alpha)/\pi}.$$

Moreover, for $|\zeta| \le h$, *we have*

$$\frac{\beta-\alpha}{\pi}\left(\frac{1-h}{2}\right)^{(\beta-\alpha)/\pi} \le |z'(\zeta)| \le \frac{\beta-\alpha}{\pi}\left(\frac{2}{1-h}\right)^{1+(\beta-\alpha)/\pi},$$

where $z(\zeta)$ *is the inverse of transformation* (2.2).

Lemma 2.5 ([10]). *Let* $f(z)$ *be a* ν*-valued algebroid function in the unit disk and let* $a_i \in \widehat{\mathbb{C}}$ $(i = 1, \dots, q)$ *be* q $(> 2\nu)$ *distinct complex numbers. Then*

$$(q-2\nu)T(r,f) \le \sum_{i=1}^{q} \overline{N}(r,a_i,f) + O\big(\log(1-r)^{-1} + \log T(r,f)\big)$$

except for r *in a set* $F \subset (0,1)$ *with* $\int_F dr/(1-r) < \infty$.

Lemma 2.5 is usually called the second fundamental theorem for algebroid functions in the unit disk.

By using the Poisson–Jensen formula for meromorphic functions, the first author [11, 12] established an estimate of the logarithmic module $\log^+ \frac{\|\vec{A}(z)\|\,\|\vec{a}\|}{|F(z,a)|}$. Applying the Boutroux–Cartan Theorem, we will modify that result. The proof is essentially the same and it is included here only for completeness.

LEMMA 2.6. *Let $f(\xi)$ be the ν-valued algebroid function determined by (1.1) in the unit disk. Then, for any $z = re^{i\theta} \notin (\gamma)$, $0 < r < R < 1$, we have*

$$\log^+ \frac{\|\vec{A}(z)\|\,\|\vec{a}\|}{|F(z,a)|} \le \log^+ (\nu+1)^{1/2} + \frac{4 + 2\log(2R/h)}{R-r}(\nu T(R,f) + O(1)), \tag{2.3}$$

where (γ) denotes several disks, the total sum of whose diameters does not exceed $4eh$.

Proof. For any $z = re^{i\theta}$, $0 < r < R < 1$, there exists an integer $0 \le k = k_z \le \nu$ such that

$$\max_{0\le l\le \nu} |A_l(z)| = |A_k(z)|.$$

Then

$$\begin{aligned} \log^+ \frac{\|\vec{A}(z)\|\,\|\vec{a}\|}{|F(z,a)|} &\le \log^+ \frac{(\nu+1)^{1/2}|A_k(z)|\,\|\vec{a}\|}{|F(z,a)|} \\ &\le \log^+ (\nu+1)^{1/2} + \log^+ \frac{|A_k(z)|\,\|\vec{a}\|}{|F(z,a)|} \\ &= \log^+ (\nu+1)^{1/2} + \log^+ \left|\frac{A_k(z)}{F(z,a)}\|\vec{a}\|\right|. \end{aligned} \tag{2.4}$$

Notice that $A_k(\xi)$ and $F(\xi,a)$ are entire functions, and $\|\vec{a}\|$ is a constant number, so

$$\frac{A_k(\xi)\|\vec{a}\|}{F(\xi,a)}$$

is a meromorphic function. Set $\widetilde{R} = (r+R)/2$. We apply the Poisson–Jensen formula to the meromorphic function $A_k(\xi)\|\vec{a}\|/F(\xi,a)$ to get

$$\begin{aligned} \log^+ \left|\frac{A_k(z)\|\vec{a}\|}{F(z,a)}\right| &\le \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{A_k(\widetilde{R}e^{i\phi})\|\vec{a}\|}{F(\widetilde{R}e^{i\phi},a)}\right| \frac{\widetilde{R}^2 - r^2}{\widetilde{R}^2 - 2\widetilde{R}r\cos(\theta-\phi) + r^2}\,d\phi \\ &\quad + \sum_{t=1}^{M} \log\left|\frac{\widetilde{R}^2 - \bar{b}_t z}{\widetilde{R}(z - b_t)}\right|. \end{aligned}$$

Using the inequality $\frac{\widetilde{R}^2 - r^2}{\widetilde{R}^2 - 2\widetilde{R}r\cos(\theta-\phi) + r^2} \le \frac{\widetilde{R}+r}{\widetilde{R}-r}$, we obtain

$$\begin{aligned} \log^+ \left|\frac{A_k(z)\|\vec{a}\|}{F(z,a)}\right| &\le \frac{1}{2\pi}\,\frac{\widetilde{R}+r}{\widetilde{R}-r}\int_0^{2\pi} \log^+ \left|\frac{A_k(\widetilde{R}e^{i\phi})\|\vec{a}\|}{F(\widetilde{R}e^{i\phi},a)}\right| d\phi + \sum_{t=1}^{M} \log\left|\frac{2\widetilde{R}}{z - b_t}\right| \\ &= \frac{1}{2\pi}\,\frac{\widetilde{R}+r}{\widetilde{R}-r}\int_0^{2\pi} \log^+ \frac{|A_k(\widetilde{R}e^{i\phi})|\,\|\vec{a}\|}{|F(\widetilde{R}e^{i\phi},a)|}\, d\phi + \sum_{t=1}^{M} \log\left|\frac{2\widetilde{R}}{z - b_t}\right|. \end{aligned}$$

In view of $|A_k(\widetilde{R}e^{i\phi})| \le \|\vec{A}(\widetilde{R}e^{i\phi})\|$, we have

$$\log^+\left|\frac{A_k(z)\|\vec{a}\|}{F(z,a)}\right| \le \frac{\widetilde{R}+r}{\widetilde{R}-r}\,m(\widetilde{R},\vec{a},\vec{A}) + \sum_{t=1}^{M}\log\left|\frac{2\widetilde{R}}{z-b_t}\right|. \tag{2.5}$$

According to the Boutroux–Cartan Theorem, for $z \notin (\gamma)$, we deduce that

$$\sum_{t=1}^{M}\log\left|\frac{2\widetilde{R}}{z-b_t}\right| = \log\frac{(2\widetilde{R})^M}{\prod_{t=1}^{M}|z-b_t|} \le M\log\left(\frac{2\widetilde{R}}{h}\right).$$

By the definition of $N(R,0,F(z,a))$, we derive that

$$n(\widetilde{R},a,f) \le \frac{N(R,0,F(z,a))}{\log R - \log\widetilde{R}} \le \frac{N(R,0,F(z,a))R}{R-\widetilde{R}} = \frac{2R}{R-r}N(R,\vec{a},\vec{A}).$$

Therefore, for $z \notin (\gamma)$,

$$\begin{aligned}(2.6)\qquad &\log^+\frac{\|\vec{A}(z)\|\,\|\vec{a}\|}{|F(z,a)|}\\ &\le \log^+(\nu+1)^{1/2} + \frac{2(R+r)}{R-r}m(R,\vec{a},\vec{A}) + \frac{2R\log(2R/h)}{R-r}N(R,\vec{a},\vec{A})\\ &\le \log^+(\nu+1)^{1/2} + \frac{4}{R-r}m(R,\vec{a},\vec{A}) + \frac{2\log(2R/h)}{R-r}N(R,\vec{a},\vec{A})\\ &\le \log^+(\nu+1)^{1/2} + \frac{4+2\log(2R/h)}{R-r}(\nu T(R,f)+O(1)). \ \blacksquare\end{aligned}$$

In order to prove Theorem 1.1, we need the following result, which is the main lemma in this paper.

LEMMA 2.7. *Let $f(z)$ be the ν-valued algebroid function of finite lower order $\mu < \infty$ in $\mathbb{C}$ determined by* (1.1), *and let $g(z)$ be a ν-valued algebroid function. Assume that $f(z)$ and $g(z)$ have $4\nu+1$ distinct IM shared values in $X = \{z : \alpha < \arg z < \beta\}$ ($0 \le \alpha < \beta \le 2\pi$, $0 < \beta - \alpha \le 2\pi$) and $f(z) \not\equiv g(z)$. If for any ε satisfying $0 < \varepsilon < (\beta-\alpha)/10$, there exists a set $\Omega \subset (\alpha,\beta)$ such that $\operatorname{meas}(\Omega \cap (\alpha+\varepsilon,\beta-\varepsilon)) = \kappa > 0$, then there exists at least one point $\phi \in \Omega\cap(\alpha+\varepsilon,\beta-\varepsilon)$ such that for all r off a set E of finite logarithmic measure, we have*

$$\log^+\frac{\|\vec{A}(re^{i\phi})\|\,\|\vec{a}\|}{|F(re^{i\phi},a)|} = O(r^{\omega}\log^2 r), \qquad \omega = \frac{\pi}{\beta-\alpha}. \tag{2.7}$$

Proof. Let $a_j \in \widehat{\mathbb{C}}$ ($j = 1,\dots,4\nu+1$) be $4\nu+1$ distinct IM shared values in X of $f(z)$ and $g(z)$. By Lemmas 2.3 and 2.4, $f(z(\zeta)), g(z(\zeta))$ are ν-valued algebroid functions in the unit disk, where $z(\zeta)$ is the inverse of the transformation in (2.2), and $a_j \in \widehat{\mathbb{C}}$ ($j = 1,\dots,4\nu+1$) are $4\nu+1$ distinct IM shared values of $f(z(\zeta))$ and $g(z(\zeta))$ in the unit disk. By applying Lemma 2.5

to $f(z(\zeta))$, we have

$$\begin{aligned}(2\nu+1)T(h,f(z(\zeta))) &\le \sum_{j=1}^{4\nu+1}\overline{N}(h,a_j,f(z(\zeta)))+R(h,f(z(\zeta)))\\ &\le N\biggl(h,\frac{1}{f(z(\zeta))-g(z(\zeta))}\biggr)+R(h,f(z(\zeta)))\\ &\le T\bigl(h,f(z(\zeta))-g(z(\zeta))\bigr)+R(h,f(z(\zeta)))\\ &\le T(h,f(z(\zeta)))+T(h,g(z(\zeta)))+R(h,f(z(\zeta))),\end{aligned}$$

where $R(h,*)=\log(1-h)^{-1}+\log T(h,*)$, $h\notin F$, and F is a set such that $\int_F dh/(1-h)<\infty$, so that

$$2\nu T(h,f(z(\zeta)))-R(h,f(z(\zeta)))\le T(h,g(z(\zeta))). \tag{2.8}$$

This implies that $R(h,f(z(\zeta)))=R(h,g(z(\zeta)))$. We also have the same formula (2.8) with $f(z(\zeta))$ and $g(z(\zeta))$ interchanged, and combining the two formulas, we obtain

$$\begin{aligned}2\nu T(h,g(z(\zeta)))-R(h,g(z(\zeta)))&\le T(h,f(z(\zeta)))\\ &\le T(h,g(z(\zeta)))+R(h,g(z(\zeta))).\end{aligned}$$

Hence

$$T(h,f(z(\zeta)))=O\biggl(\log\frac{1}{1-h}\biggr),\qquad h\notin F, \tag{2.9}$$

where F is the set described in Lemma 2.5 satisfying $\int_F dh/(1-h)<\infty$. Set

$$E=\biggl\{r:h=1-\frac{\varepsilon}{4(\beta-\alpha)}r^{-\omega},\ h\in F\biggr\}, \tag{2.10}$$

where $\varepsilon>0$ is small enough. Put

$$\begin{aligned}\zeta&=\zeta(re^{i\phi})\quad(\alpha+\varepsilon\le\phi\le\beta-\varepsilon),\\ h&=1-\frac{\varepsilon}{(\beta-\alpha)}r^{-\omega},\\ h'&=1-\frac{\varepsilon}{4(\beta-\alpha)}r^{-\omega}\notin F,\end{aligned} \tag{2.11}$$

where $\zeta=\zeta(z)$ is the mapping described in Lemma 2.4. Combining (2.10) with (2.11), we can see that if $h'\notin F$, then $r\notin E$ and E is a set of finite logarithmic measure, because

$$\int_E\frac{dr}{r}=\frac{1}{\omega}\int_F\frac{dh}{1-h}<\infty.$$

Next we apply (2.3) to estimate the logarithmic module:

$$\log^+\frac{\|\vec{A}(re^{i\phi})\|\,\|\vec{a}\|}{|F(re^{i\phi},a)|}=\log^+\frac{\|\vec{A}(z(\zeta))\|\,\|\vec{a}\|}{|F(z(\zeta),a)|}.$$

We now adapt a line of reasoning used by Zhang and Wu [13, 17, 18, 19].

According to Lemma 2.4, X is mapped onto the unit disk $\{|\zeta|<1\}$ and $z=e^{i(\alpha+\beta)/2}$ is mapped to $\zeta=0$. The image of $X_\varepsilon(r)=\{z:1\le|z|\le r,$ $\alpha+\varepsilon\le\arg z\le\beta-\varepsilon\}$ in the ζ-plane must be contained in the disk $\{|\zeta|<h\}$, where

$$h=1-\frac{\varepsilon}{\beta-\alpha}r^{-\pi/(\beta-\alpha)}.$$

On the other hand, the inverse image of the disk $|\zeta|\le(h+1)/2$ in the z-plane is contained in $X\cap\{|z|\le r_1\}$, where

$$r_1=\left(\frac{4(\beta-\alpha)}{\varepsilon}\right)^{(\beta-\alpha)/\pi}r.$$

In addition, for $|\zeta|\le(h+1)/2$,

$$\frac{\beta-\alpha}{\pi}\left(\frac{\varepsilon}{4(\beta-\alpha)}\right)^{(\beta-\alpha)/\pi}\frac{1}{r}\le|z'(\zeta)|\le\frac{\beta-\alpha}{\pi}\left(\frac{4(\beta-\alpha)}{\varepsilon}\right)^{1+(\beta-\alpha)/\pi}r^{1+\pi/(\beta-\alpha)}. \tag{2.12}$$

The transformation

$$\xi=\xi(\zeta)=\frac{2}{1+h}\zeta \tag{2.13}$$

maps the disks $|\zeta|\le(1+h)/2$ and $|\zeta|\le h$ to the unit disk $|\xi|\le1$ and the disk $|\xi|\le\tau$, respectively, where

$$\tau=2h/(1+h). \tag{2.14}$$

Let $\zeta(\xi)$ be the inverse of the mapping in (2.13). Then according to (2.12), we have $1/2\le|\zeta'(\xi)|\le1$. If $z(\xi)=z(\zeta(\xi))$, we have, for $|\xi|\le1$,

$$\frac{\beta-\alpha}{2\pi}\left(\frac{\varepsilon}{4(\beta-\alpha)}\right)^{(\beta-\alpha)/\pi}\frac{1}{r}\le|z'(\xi)|\le\frac{\beta-\alpha}{\pi}\left(\frac{4(\beta-\alpha)}{\varepsilon}\right)^{1+(\beta-\alpha)/\pi}r^{1+\pi/(\beta-\alpha)}. \tag{2.15}$$

Evidently the image $\Gamma_\xi(\alpha,\beta,r)$ in the ξ-plane of the circular arc $\Gamma(\alpha,\beta,r)$ in the z-plane under the mappings (2.2) and (2.13) is orthogonal to the circle $|\xi|=2/(1+h)$, while the image arc in the ξ-plane of the circular arc $\Gamma(\alpha+\varepsilon,\beta-\varepsilon,r)$ in the z-plane is contained in the disk $\{|\xi|\le\tau\}$. Putting

$$\xi_0=\frac{2}{1+h}\zeta(re^{i(\alpha+\beta)/2}),$$

we obtain $\xi_0\in\Gamma_\xi(\alpha+\varepsilon,\beta-\varepsilon,r)$. The linear transformation

$$x=x(\xi)=\frac{\xi-\xi_0}{1-\bar{\xi}_0\xi} \tag{2.16}$$

maps $\Gamma_\xi(\alpha,\beta,r)$ to a straight line segment $\Gamma_x(\alpha,\beta,r)$ passing through the origin $x = 0$. On the other hand, the inverse of (2.16) is expressed as $\xi = \xi(x)$. When $|x| < 1$,

$$\frac{1-\xi_0}{2} \le |\xi'(x)| \le \frac{2}{1-\xi_0}.$$

Letting $\varepsilon > 0$ be small enough, we have

$$1-\xi_0 \ge r^{-\pi/(\beta-\alpha)}.$$

Writing $z(x) = z(\zeta(\xi(x)))$, and combining the above with (2.15), when $|x| < 1$, we have

$$\frac{\beta-\alpha}{2\pi}\left(\frac{\varepsilon}{4(\beta-\alpha)}\right)^{(\beta-\alpha)/\pi}\left(\frac{1}{r}\right)^{1+\pi/(\beta-\alpha)} \le |z'(x)| \le \frac{2(\beta-\alpha)}{\pi}\left(\frac{4(\beta-\alpha)}{\varepsilon}\right)^{1+(\beta-\alpha)/\pi} r^{1+2\pi/(\beta-\alpha)}. \tag{2.17}$$

We define a set $E_z(r)$ in the z-plane as

$$E_z(r) = \{re^{i\varphi} : \varphi \in \Omega \cap (\alpha+\varepsilon, \beta-\varepsilon)\}.$$

Let $E_\xi(r)$ and $E_x(r)$ be the images of $E_z(r)$ in the ξ-plane and the x-plane, respectively. Evidently, $E_\xi(r) \subset \Gamma_\xi(\alpha+\varepsilon,\beta-\varepsilon,r)$, $E_x(r) \subset \Gamma_x(\alpha,\beta,r)$, and it follows that

$$\kappa r \le \int_{E_z(r)} |dz| = \int_{E_x(r)} |z'(x)|\,|dx| \le \frac{2(\beta-\alpha)}{\pi}\left(\frac{4(\beta-\alpha)}{\varepsilon}\right)^{1+(\beta-\alpha)/\pi} r^{1+2\pi/(\beta-\alpha)} \operatorname{meas} E_x(r)$$

and

$$\operatorname{meas} E_x(r) \ge \frac{\kappa\pi}{2(\beta-\alpha)}\left(\frac{\varepsilon}{4(\beta-\alpha)}\right)^{1+(\beta-\alpha)/\pi} r^{-2\pi/(\beta-\alpha)}. \tag{2.18}$$

Let $G(\xi) = f(z(\zeta(\xi)))$. Then $G(\xi)$ is a ν-valued algebroid function in $\{\xi : |\xi| \le 1\}$. Let $(\gamma)_\xi$ be the non-Euclidean exceptional disks with regard to the $n(1,a,G)$ points and the number H, and let $(\gamma)_x$ be their images in the x-plane. We take

$$H = \frac{1}{8e}\,\frac{\kappa\pi}{2(\beta-\alpha)}\left(\frac{\varepsilon}{4(\beta-\alpha)}\right)^{1+(\beta-\alpha)/\pi} r^{-2\pi/(\beta-\alpha)}.$$

The sum of the non-Euclidean radii is less than or equal to $2eH$. Since a non-Euclidean disk is also a Euclidean disk, with its non-Euclidean radius larger than or equal to its Euclidean radius, and since $\Gamma_x(\alpha,\beta,r)$ is a segment, it follows from (2.18) that there is a point $x_1 \in E_x(r) \setminus (\gamma)_x$. If ξ_1 is the image of x_1, then $\xi_1 \in E_\xi(r) \setminus (\gamma)_\xi$. Let $(\gamma)_\zeta$ be the inverse image of $(\gamma)_\xi$ in the

ζ-plane, and let ζ_1 be the inverse image of ξ_1. Then $\zeta_1 \in E_\zeta(r)\backslash(\gamma)_\zeta$, where $E_\zeta(r)$ is the image of $E_z(r)$ and $(\gamma)_\zeta$ are the non-Euclidean exceptional circles with regard to the $n\big((1+h)/2, a, f(z(\zeta))\big)$ points and the number H. Let $z_1 = z(\zeta_1)$ be the inverse image of ζ_1 in the z-plane. Then $z_1 \in E_z(r)$.

For $R = (3+h)/4$, $r = h$ and $\zeta_1 \in E_\zeta(r) \setminus (\gamma)_\zeta$, Lemma 2.6 gives

$$\tag{2.19} \begin{aligned} \log^+ \frac{\|\vec{A}(z_1)\|\,\|\vec{a}\|}{|F(z_1,a)|} &= \log^+ \frac{\|\vec{A}(z(\zeta_1))\|\,\|\vec{a}\|}{|F(z(\zeta_1),a)|} \\ &\le \log^+(\nu+1)^{1/2} + \frac{4+2\log(2/H)}{R-r}(\nu T(R, f(z(\zeta))) + O(1)). \end{aligned}$$

For $r \notin E$, we infer from (2.11) that $h' = (h+3)/4 \notin F$. Then combining (2.9) with (2.19), we have

$$\tag{2.20} \begin{aligned} &\log^+ \frac{\|\vec{A}(z_1)\|\,\|\vec{a}\|}{|F(z_1,a)|} \\ &\qquad \le \log^+(\nu+1)^{1/2} + \frac{4\nu}{1-h} O\bigg(\log\frac{4}{1-h}\bigg)\bigg[4 + 2\log\frac{2}{H}\bigg]. \end{aligned}$$

Noticing that $h = 1 - \frac{\varepsilon}{\beta-\alpha} r^{-\omega}$ and applying the expression on H, we have

$$\log^+ \frac{\|\vec{A}(z_1)\|\,\|\vec{a}\|}{|F(z_1,a)|} \le \log^+(\nu+1)^{1/2} + O(r^\omega \log^2 r).$$

Thus (2.7) follows from the existence of ϕ, which is deduced from the fact that $z_1 \in E_z(r)$. ∎

The following lemma is crucial to Theorem 1.2, as it is a generalization of Edrei [1]. The proof has been given by the first author [11]; for completeness we repeat it here.

LEMMA 2.8. *Let $f(z)$ be the ν-valued algebroid function determined by* (1.1) *with $\delta = \delta(a,f) > 0$ for some $a \in \widehat{\mathbb{C}}$. Then, given $\varepsilon > 0$, we have*

$$\operatorname{meas} E(r,a) > \frac{1}{T^\varepsilon(r,f)[\log r]^{1+\varepsilon}}, \qquad r \notin F,$$

where

$$E(r,a) = \bigg\{\theta \in [-\pi,\pi) : \log^+ \frac{\|\vec{A}(re^{i\theta})\|\,\|\vec{a}\|}{|F(re^{i\theta},a)|} > \frac{\delta}{4} T(r,f)\bigg\},$$

and F is a set of positive real numbers of finite logarithmic measure depending on ε.

Proof. Let $\{b_t\}$ be the sequence of roots of $f(z) = a$, and also the sequence of roots of $F(z,a) = 0$ (we assume that $\{|b_t|\}$ is non-decreasing and that the multiplicities of roots have been taken into account by suitable repetitions of elements). Let

$$I_t := (|b_t| - 1/t^2, |b_t| + 1/t^2).$$

If $r > 0$ and $r \notin \bigcup_{t=1}^{\infty} I_t$, then combining (2.4) with (2.5), we have $(r < R)$

$$\log^+ \frac{\|\vec{A}(re^{i\theta})\| \, \|\vec{a}\|}{|F(re^{i\theta}, a)|} \le \log^+(\nu+1)^{1/2} + \frac{R+r}{R-r} m(R, \vec{a}, \vec{A}) + n(R)[\log 2R + 2\log n(R)], \tag{2.21}$$

where $n(R)$ denotes the number of roots of $F(z,a) = 0$ in $|z| \le R$. Setting $R' - R = R - r$, we deduce

$$N(R', 0, F(z,a)) - N(R, 0, F(z,a)) = \int_R^{R'} \frac{n(t)}{t} \, dt > \frac{n(R)}{R'} \frac{R'-r}{2}.$$

Substituting the above into (2.21), we derive

$$\begin{aligned}
\log^+ \frac{\|\vec{A}(re^{i\theta})\| \, \|\vec{a}\|}{|F(re^{i\theta}, a)|} &\le \log^+(\nu+1)^{1/2} + \frac{4\nu R'}{R'-r} T(R', f) \\
&\quad + \frac{2R' N(R', 0, F(z,a))}{R'-r} \left[\log 2R' + 2 \log\left(\frac{2R'}{R'-r} N(R', 0, F(z,a)) \right) \right] \\
&\le \log^+(\nu+1)^{1/2} \\
&\quad + \frac{4\nu R'}{R'-r} T(R', f) \left[\left(1 + \frac{3}{2}\log 2\right) + \frac{1}{2}\log R' + \frac{R'}{R'-r} + \log T(R', f) \right].
\end{aligned}$$

If r is large enough, say larger than some r_0, we obtain

$$\log^+ \frac{\|\vec{A}(re^{i\theta})\| \, \|\vec{a}\|}{|F(re^{i\theta}, a)|} \le A \left(\frac{R'}{R'-r} \right)^2 (\log R') T(R', f) \log T(R', f), \tag{2.22}$$

where A is a positive constant depending on ν, for $r > r_0$, $r \notin \bigcup I_t$. For $r > r_0$, the function

$$V(r) = [T(r, f) \log T(r, f)] \log r$$

is positive, continuous and non-decreasing to infinity. Hence, for any $\eta > 0$, according to Borel's lemma, we have

$$V\left(r \left(1 + \frac{1}{\log V(r)} \right) \right) < V^{1+\eta}(r),$$

except possibly for values of $r > r_0$ which belong to an exceptional set $\mathcal{E}(\eta)$ of finite logarithmic measure. Taking, in (2.22),

$$R' = r\left(1 + \frac{1}{\log V(r)} \right),$$

we obtain

$$\log^+ \frac{\|\vec{A}(re^{i\theta})\|\,\|\vec{a}\|}{|F(re^{i\theta},a)|} \le V^{1+2\eta}(r) \tag{2.23}$$

provided r is sufficiently large and $r \notin \mathcal{E}(\eta) \cup \bigcup_t I_t$. It is easy to see that the set $\mathcal{E}(\eta) \cup \bigcup_t I_t$ is of finite logarithmic measure. For $r \notin \mathcal{E}(\eta) \cup \bigcup_t I_t$, consider the set

$$E = \Big\{\theta \in [-\pi,\pi) : \log^+ \frac{\|\vec{A}(re^{i\theta})\|\,\|\vec{a}\|}{|F(re^{i\theta},a)|} > \frac{1}{2} m(r,\vec{a},\vec{A})\Big\}.$$

Then

$$2\pi m(r,\vec{a},\vec{A}) \le \int_E \log^+ \frac{\|\vec{A}(re^{i\theta})\|\,\|\vec{a}\|}{|F(re^{i\theta},a)|}\,d\theta + 2\pi \frac{m(r,\vec{a},\vec{A})}{2},$$

$$\pi m(r,\vec{a},\vec{A}) < V^{1+2\eta}(r)\operatorname{meas} E.$$

If r is sufficiently large, we have

$$m(r,\vec{a},\vec{A}) > \frac{\delta}{2} T(r,\vec{a},\vec{A}) = \frac{\nu\delta}{2} T(r,f) + O(1) > \frac{\delta}{2} T(r,f),$$

$$E \subset E(r,a).$$

Therefore,

$$\operatorname{meas} E(r,a) > \frac{\delta\pi/2T(r,f)}{T^{1+2\eta}(r,f)[\log T(r,f)]^{1+2\eta} \log^{1+2\eta} r}.$$

Taking $\eta = \varepsilon/3$, we obtain the result. ∎

3. Proof of Theorem 1.1. The idea of the proof comes from [21, 22]. Suppose the theorem is not true, i.e. $\lambda(f) > \omega$. We consider the following two cases.

I. $\lambda(f) > \sigma \ge \mu(f)$. By (2.9), we can choose $\varepsilon > 0$ such that

$$\sum_{i=1}^{q} (\alpha_{i+1} - \beta_i + 2\varepsilon) + 2\varepsilon < \frac{4}{\sigma + 2\varepsilon} \arcsin\sqrt{\delta/2}, \tag{3.1}$$

where $\alpha_{q+1} = 2\pi + \alpha_1$, and $\lambda(f) > \sigma + 2\varepsilon > \mu$. Lemma 2.1 gives the existence of a sequence $\{r_n\} \notin E$ of Pólya peaks of order $\sigma + 2\varepsilon$ for $f(z)$, where E is the set of Lemma 2.7. Then by Lemma 2.2, for sufficiently large n, we have

$$\operatorname{meas} E(r_n,a) > \frac{4}{\sigma + 2\varepsilon} \arcsin\sqrt{\delta/2} - \varepsilon, \tag{3.2}$$

since $\sigma + 2\varepsilon > 1/2$. We can assume (3.2) holds for all n. Put

$$K := \operatorname{meas}\Big(E(r_n,a) \cap \bigcup_{i=1}^{q} (\alpha_i + \varepsilon, \beta_i - \varepsilon)\Big).$$

From (3.1) and (3.2), it follows that

$$\begin{aligned} K &\geq \operatorname{meas} E(r_n, a) - \operatorname{meas}\Big([-\pi,\pi) \setminus \bigcup_{i=1}^{q} (\alpha_i + \varepsilon, \beta_i - \varepsilon)\Big) \\ &= \operatorname{meas} E(r_n, a) - \operatorname{meas}\Big(\bigcup_{i=1}^{q} (\beta_i - \varepsilon, \alpha_{i+1} + \varepsilon)\Big) \\ &= \operatorname{meas} E(r_n, a) - \sum_{i=1}^{q} (\alpha_{i+1} - \beta_i + 2\varepsilon) > \varepsilon > 0. \end{aligned}$$

It is easy to see that there exists i_0 such that for infinitely many n, we have

$$\operatorname{meas}\big(E(r_n, a) \cap (\alpha_{i_0} + \varepsilon, \beta_{i_0} - \varepsilon)\big) > K/q > \varepsilon/q. \tag{3.3}$$

We can assume (3.3) holds for all n. Set $E_n = E(r_n, a) \cap (\alpha_{i_0} + \varepsilon, \beta_{i_0} - \varepsilon)$. By the definition of $E(r_n, a)$, it follows that

$$\log^+ \frac{\|\vec{A}(r_n e^{i\theta})\| \, \|\vec{a}\|}{|F(r_n e^{i\theta}, a)|} > \frac{T(r_n, f)}{\log r_n}, \qquad \forall \theta \in E_n. \tag{3.4}$$

On the other hand, by (2.7), there exists a $\theta \in E_n$ such that

$$\log^+ \frac{\|\vec{A}(r_n e^{i\theta})\| \, \|\vec{a}\|}{|F(r_n e^{i\theta}, a)|} = O(r_n^{\omega_{i_0}} \log^2 r_n), \qquad \omega_{i_0} = \frac{\pi}{\beta_{i_0} - \alpha_{i_0}}. \tag{3.5}$$

Combining (3.4) with (3.5), we have

$$T(r_n, f) \leq O(r_n^{\omega_{i_0}} \log^3 r_n)$$

Thus from (2) of Lemma 2.1 for $\sigma + 2\varepsilon$, we obtain

$$\sigma + 2\varepsilon \leq \limsup_{n \to \infty} \frac{\log T(r_n, f)}{\log r_n} \leq \omega_{i_0} \leq \sigma + \varepsilon.$$

This is impossible.

II. $\lambda(f) = \mu(f)$. Then $\sigma = \mu = \lambda(f)$. By the same argument as in I, with $\sigma + 2\varepsilon$ replaced by $\sigma = \mu$ everywhere, we can derive

$$\max\{\omega, \mu\} = \sigma \leq \omega < \lambda(f).$$

This is impossible.

Therefore, Theorem 1.1 follows.

4. Proof of Theorem 1.2. Applying Lemma 2.8 to $f(z)$ implies the existence of a sequence $\{r_n\}$ of positive numbers such that $r_n \to \infty$ and $r_n \notin E \cup F$ and

$$\operatorname{meas} E(r_n, a) \geq \frac{1}{T^{\varepsilon}(r_n, f)[\log r_n]^{1+\varepsilon}}, \tag{4.1}$$

where E is the set of Lemma 2.7, and F the set of Lemma 2.8. Put

$$\varepsilon_n = \frac{1}{2q+1}\,\frac{1}{T^{\varepsilon}(r_n, f)[\log r_n]^{1+\varepsilon}}.$$

Then it follows from (4.1) that

$$\begin{aligned}\operatorname{meas}\Big(E(r_n, a) \cap \bigcup_{i=1}^{q}(\alpha_i + \varepsilon_n, \alpha_{i+1} - \varepsilon_n)\Big)& \\ \geq \operatorname{meas} E(r_n, a) - \operatorname{meas}\Big(\bigcup_{i=1}^{q}(\alpha_i + \varepsilon_n, \alpha_{i+1} - \varepsilon_n)\Big)& \\ \geq (2q+1)\varepsilon_n - 2q\varepsilon_n = \varepsilon_n > 0.&\end{aligned}$$

Thus there exists a j such that for infinitely many n, we have

$$\operatorname{meas} E_n > \varepsilon_n / q, \tag{4.2}$$

where $E_n = E(r_n, a) \cap (\alpha_j + \varepsilon_n, \alpha_{j+1} - \varepsilon_n)$. We can assume that (4.2) holds for all n. Thus it follows from the definition of $E(r, a)$ that

$$\log^{+} \frac{\|\vec{A}(r_n e^{i\theta})\|\,\|\vec{a}\|}{|F(r_n e^{i\theta}, a)|} > \frac{\delta}{4} T(r_n, f), \qquad \forall \theta \in E_n. \tag{4.3}$$

On the other hand, as in Lemma 2.7, for each j, there exists a point $\phi \in E_n$ such that

$$\log^{+} \frac{\|\vec{A}(r_n e^{i\phi})\|\,\|\vec{a}\|}{|F(r_n e^{i\phi}, a)|} = O(r_n^{\omega_j} \log^2 r_n). \tag{4.4}$$

Combining (4.3) with (4.4) gives

$$\frac{\delta}{4} T(r_n, f) < O(r_n^{\omega_j} \log^2 r_n).$$

Thus $\mu(f) \leq \omega_j < \infty$, and Theorem 1.2 follows from Theorem 1.1.

5. Conclusion. Corresponding to the uniqueness theorems established for meromorphic functions with shared values in an angular domain [23, 14], we can establish their counterparts for algebroid functions with shared values in angular domains. For example, using the methods of Zheng [23, 14] and of this paper, we can prove the following

THEOREM 5.1. *Let $f(z)$, $g(z)$ and a be as in Theorem* 1.1. *For q pairs $\{\alpha_j, \beta_j\}$ of real numbers satisfying* (1.2) *and* (1.3), *assume that $f(z)$ and $g(z)$ have 4ν distinct IM shared values $a_i \neq a$ $(i = 1, \ldots, 2\nu)$ in $X = \bigcup_{j=1}^{q}\{z : \alpha_j \leq \arg z \leq \beta_j\}$. If $\omega < \lambda(f)$, then $f(z) \equiv g(z)$.*

We can also establish a result similar to Theorem 5.1 corresponding to Theorem 1.2. Finally, we point out that it would be of interest to investigate the uniqueness of algebroid functions with shared values in an unbounded proper subset of the complex plane.

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