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Differential inclusions in the Almgren sense on unbounded domains

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Abstract. We prove the existence of solutions of differential inclusions on a half-line. Our results are based on an approximation method combined with a diagonalization method.

1. Introduction. The theory of multiple-valued functions in the sense of Almgren [2] has several applications in the framework of geometric measure theory. It gives a very useful tool to approximate some abstract objects arising from geometric measure theory. For example, Almgren [2] used multiple-valued functions to approximate mass-minimizing rectifiable currents, hence successfully obtaining their partial interior regularity. Solomon [11] succeeded in proving the closure theorem without using the structure theorem. His proof relies on various facts about multiple-valued functions. There are also other objects similar to these functions, such as the union of Sobolev functions graphs introduced by Ambrosio, Gobbino and Pallara (see [4]). In complex function theory one often speaks of the multiple-valued function $\sqrt{z}$. It can be considered as a function $\mathbb{C} \to \mathcal{A}_2(\mathbb{C})$. Almgren [3] introduced $\mathcal{A}_Q(\mathbb{R}^n)$-valued functions to tackle the problem of estimating the size of the singular set of mass-minimizing integral currents (see [2] for a summary). Almgren's multiple-valued functions are a fundamental tool for understanding geometric variational problems in codimension higher than 1.

The success of Almgren's regularity theory raises the need of further studying multiple-valued functions. For more information concerning multi-

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ple-valued functions, see [5, 6, 7–10]. For some local existence results for differential inclusions in the sense of Almgren, we cite Goblet [7].

Agarwal and O'Regan [1] considered some classes of boundary value problems on a half-line, in which they used the diagonalization process combined with fixed point theory.

We use the iteration method combined with the diagonalization process for the existence of a continuously differentiable solution for a class of differential inclusions with nonconvex right-hand side in the sense of Almgren.

2. Preliminaries. In this section, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout the paper.

DEFINITION 2.1. We denote by $[[p_i]]$ the Dirac mass at $p_i \in \mathbb{R}^n$, and we define the space of *Q-points* as

$$\mathcal{A}_Q(\mathbb{R}^n) := \Big\{ \sum_{i=1}^{Q} [[p_i]] : p_i \in \mathbb{R}^n \text{ for every } i = 1, \ldots, Q \Big\}.$$

DEFINITION 2.2. For any $T_1, T_2 \in \mathcal{A}_Q(\mathbb{R}^n)$, with $T_1 = \sum [[p_i]]$ and $T_2 = \sum_i [[s_i]]$, we define

$$d_{\mathcal{A}}(T_1, T_2) := \min_{\sigma \in \mathcal{P}_Q} \sqrt{\sum_{i=1}^{Q} |p_i - s_{\sigma(i)}|^2},$$

$$d_{\mathcal{A}}(T_1, T_2) := \min_{\sigma \in \mathcal{P}_Q} \sum_{i=1}^{Q} |p_i - s_{\sigma(i)}|,$$

or

$$d_{\mathcal{A}}(T_1, T_2) := \min_{\sigma \in \mathcal{P}_Q} \{\max |p_i - s_{\sigma(i)}| : i = 1, \ldots, Q\},$$

where $\mathcal{P}_Q$ denotes the group of permutations of $\{1, \ldots, Q\}$.

A *multiple-valued function in the sense of Almgren* is an $\mathcal{A}_Q(\mathbb{R}^n)$-valued function.

DEFINITION 2.3. Let $\Omega \subset \mathbb{R}^m$ and $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ be an $\mathcal{A}_Q(\mathbb{R}^n)$-valued function. If there exist single-valued maps $g_i : \Omega \to \mathbb{R}^m$, $i = 1, \ldots, Q$, such that

$$f(x) = \sum_{i=1}^{Q} [[g_i(x)]] \quad \text{for each } x \in \mathbb{R}^m,$$

then we say that the vector $(g_1, \ldots, g_Q)$ is a *selection* for f.

THEOREM 2.1 ([3, 6]). *Let $f : [0, b] \to \mathcal{A}_Q(\mathbb{R}^n)$ be a continuous multiple-valued function. Then there are continuous functions $f_1, \dots, f_Q : [0, b] \to \mathbb{R}^n$ such that*

$$f = \sum_{i=1}^{Q} f_i.$$

REMARK 2.1. If for each $i \in \{1, \dots, Q\}$, g_i is continuous, then f has a continuous selection.

LEMMA 2.2 ([7]). *Let $f : \mathbb{R} \to \mathcal{A}_Q(\mathbb{R}^n)$ be a continuous multiple-valued function and $g : \mathbb{R} \to \mathbb{R}^n$ be a continuous function. If $h : [0, b] \times \mathbb{R} \to \mathcal{A}_{Q-1}(\mathbb{R}^n)$ satisfies*

$$f = [[g]] + h,$$

then h is a continuous function.

REMARK 2.2. An $\mathcal{A}_Q(\mathbb{R}^n)$-valued function is essentially a rule assigning Q unordered and not necessarily distinct elements of $\mathbb{R}^n$ to each element of its domain.

LEMMA 2.3 ([7]). *Let $\{f_i\} : [0, b] \to \mathcal{A}_Q(\mathbb{R}^n)$ be a sequence of multiple-valued functions pointwise converging to f, and let $\{g_i\} : [0, b] \to \mathbb{R}^n$ be a sequence of functions pointwise converging to g such that g_i is a selection of f_i for each $i \in \mathbb{N}$. Then g is a selection of f.*

THEOREM 2.4 ([3]). *Suppose $f_1, \dots, f_Q : [0, b] \to \mathbb{R}^n$ are continuous functions and $f = \sum_{i=1}^{Q} [[f_i]] : [0, b] \to \mathcal{A}_Q(\mathbb{R}^n)$. Then there exists a constant $C_{n,Q} > 0$, depending only on n and Q, such that*

$$\omega_{f_i} \leq C_{n,Q}\, \omega_f \quad \textit{for each } i = 1, \dots, Q,$$

where ω_f is the modulus of continuity of f, i.e.,

$$\omega_f(\delta) = \sup\{d_{\mathcal{A}}(f(s_1), f(s_2)) : s_1, s_2 \in [0, b] \textit{ and } |s_1 - s_2| \leq \delta\},$$

and

$$\omega_{f_i}(\delta) = \sup\{|f_i(s_1) - f(s_2)| : s_1, s_2 \in [0, b] \textit{ and } |s_1 - s_2| \leq \delta\}.$$

3. Existence result. We consider the following problem:

$$\text{(3.1)} \qquad \begin{cases} y'(t) \in \{f_1(t, y(t)), \dots, f_Q(t, y(t))\}, & t \in [0, \infty), \\ y(0) = a, \end{cases}$$

where $f_i : [0, \infty) \times \mathbb{R}^N \to \mathbb{R}^N, 1 \leq i \leq Q$, are single-valued functions.

THEOREM 3.1. *Let $f_i : [0, b] \times \mathbb{R}^N \to \mathbb{R}^N$, $i = 1, \dots, Q$, be single-valued functions with which we associate the continuous multiple-valued function* *in the sense of Almgren*

$$f = \sum_{i=1}^{Q} [[f_i]] : [0, \infty) \times \mathbb{R} \to \mathcal{A}_Q(\mathbb{R}^N).$$

Assume that there exists $M_1, M_2 > 0$ *such that*

$$d_{\mathcal{A}}(f(t,x), Q(0)) \leq M_1 + M_2|x| \quad \text{for all } x \in \mathbb{R}^N,\ t \in [0,\infty). \tag{3.2}$$

Then problem (3.1) *has at least one solution in* $C([0,\infty), \mathbb{R}^N)$.

Proof. The proof involves several steps.

STEP 1. We begin by constructing two sequences $\{y_m\}_{m=0}^{\infty}$ and $\{g_m\}_{m=0}^{\infty}$ by first defining

$$y_0(t) = a \quad \text{for all } t \in [0, n],$$

$$y_1(t) = \begin{cases} y_0(t) & \text{if } t \in [0, n/2], \\ a + \int_0^{t-n/2} g_{2,1}(s)\, ds & \text{if } t \in (n/2, n], \end{cases}$$

where $g_{2,1} : [0, n/2] \to \mathbb{R}^N$ is a continuous selection of $f(\cdot, y_1(\cdot)) : [0, n/2] \to \mathcal{A}_Q(\mathbb{R}^N)$. From Theorem 2.1 we can find a continuous selection $g_1 : [0, n] \to \mathbb{R}^N$ for $f(\cdot, y_1(\cdot)) : [0, n] \to \mathcal{A}_Q(\mathbb{R}^N)$ such that $g_1(\cdot) = g_{2,1}(\cdot)$ on $[0, n/2]$. Next, we define

$$y_2(t) = \begin{cases} y_0(t) & \text{if } t \in [0, n/3], \\ a + \int_0^{t-n/3} g_{3,1}(s)\, ds & \text{if } t \in [n/3, 2n/3], \\ a + \int_0^{t-n/3} g_{3,2}(s)\, ds & \text{if } t \in [2n/3, n], \end{cases}$$

where $g_{3,1} : [0, n/3] \to \mathbb{R}^N$ is a continuous selection of $f(\cdot, y_2(\cdot)) : [0, n/3] \to \mathcal{A}_Q(\mathbb{R}^N)$ and $g_{3,2} : [0, 2n/3] \to \mathbb{R}^N$ is a continuous selection of $f(\cdot, y_2(\cdot)) : [0, 2n/3] \to \mathcal{A}_Q(\mathbb{R}^N)$ such that $g_{3,1}(\cdot) = g_{3,2}(\cdot)$ on $[0, n/3]$. Again by Theorem 2.1 we can choose a continuous selection of $f(\cdot, y_3(\cdot)) : [0, n] \to \mathcal{A}_Q(\mathbb{R}^N)$ such that $g_2(\cdot) = g_{3,2}(\cdot)$ on $[0, 2n/3]$. Finally, for $m > 2$, we define inductively,

$$y_m(t) = \begin{cases} y_0(t) & \text{if } t \in [0, n/m], \\ a + \int_0^{t-n/m} g_m(s)\, ds & \text{if } t \in (n/m, n], \end{cases}$$

where $g_m : [0, n] \to \mathbb{R}^N$ is a continuous selection of $f(\cdot, y_m(\cdot)) : [0, n] \to \mathcal{A}_Q(\mathbb{R}^N)$.

Now, we show that $\{y_m : m \in \mathbb{N} \cup \{0\}\}$ is relatively compact. First, we exhibit that $\{y_m\}_{m=0}^{\infty}$ is bounded. Since

$$|y_m(t)| \leq |a| + \int_0^t |g_m(s)|\, ds \leq |a| + \int_0^t (M_1 + M_2|y_m(s)|)\, ds,$$

it follows from Gronwall's lemma that there exists $M > 0$ such that

$$\|y_m\|_\infty \le M \qquad \text{for each } m \in \mathbb{N} \cup \{0\}.$$

Next, we show that $\{y_m\}_{m=0}^\infty$ is equicontinuous. Let $t_1, t_2 \in [0, n/m]$. Then

$$|y_m(t_1) - y_m(t_2)| = 0 \qquad \text{if } t_1, t_2 \in [0, n/m];$$

for $0 < t_1 \le b/m < t_2 < n$, we have

$$|y_m(t_1) - y_m(t_2)| \le \int_0^{t_2 - b/m} |g_m(s)|\, ds \le M|t_2 - b/m| \le M|t_2 - t_1|;$$

and

$$|y_m(t_1) - y_m(t_2)| \le \int_{t_1 - n/m}^{t_2 - b/m} |g_m(s)|\, ds \le M|t_1 - t_2|, \qquad t_1, t_2 \in (b/m, n].$$

Consequently, $\{y_m\}_{m=0}^\infty$ is bounded and equicontinuous. By the Arzelà–Ascoli theorem, there exists a subsequence of $\{y_m\}_{m=0}^\infty$ converging to some y in $C([0,b], \mathbb{R}^N)$. Let $K = [0,b] \times B(0,M)$, and

$$\omega|_{f|_K}(\delta) = \sup\{d_{\mathcal{A}}(f(t_1,x_1), f(t_2,x_2)) : |(t_1,x_1) - (t_2,x_2)| \le \delta, \text{ where } (t_1,x_1),\ (t_2,x_2) \in K\}$$

be the modulus of continuity of f restricted to K. Hence for each $m \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
\omega|_{f(\cdot, y_m(\cdot))}(\delta_2) &= \sup\{d_{\mathcal{A}}(f(t_1, y_m(t_1)), f(t_2, y_m(t_2))) : |t_1 - t_2| \le \delta, \\
&\qquad\qquad \text{and } t_1, t_2 \in [0,n]\} \\
&\le \sup\{d_{\mathcal{A}}(f(t_1,x_1), f(t_2,x_2)) : |t_1 - t_2| \le \delta_2, |x_1 - x_2| \le \psi(M)\delta, \\
&\qquad\qquad \text{and } (t_1,x_1), (t_2,x_2) \in K\} \\
&\le \omega|_{f_K}(\delta\sqrt{1+M^2}).
\end{aligned}$$

It is clear that $f(\cdot, y_m(\cdot)) - [[g_m(\cdot)]] : [0,n] \to \mathcal{A}_{Q-1}(\mathbb{R}^N)$ is a continuous multiple-valued function. Then there exist $h_1^m, \ldots, h_{Q-1}^m : [0,n] \to \mathbb{R}^N$ continuous functions such that

$$f(\cdot, y_m(\cdot)) = [[g_m(\cdot)]] + \sum_{i=1}^{Q-1} [[h_i^m(\cdot)]].$$

Then

$$\|g_m\|_\infty \le L_1 \qquad \text{for each } m \in \mathbb{N} \cup \{0\}$$

and

$$\omega|_{g_m} \le \omega|_{f_K}(\delta_2) \qquad \text{for every } m \in \mathbb{N} \cup \{0\}.$$

Consequently, $\{g_m\}_{m=0}^{\infty}$ is bounded and equicontinuous. From the Arzelà–Ascoli theorem, we conclude that $\{g_m\}_{m=0}^{\infty}$ is compact in $C([0,n],\mathbb{R}^N)$. Hence there exists a subsequence, denoted $\{g_m\}_{m=0}^{\infty}$, converging uniformly to g. Hence

$$\|y_m - z\|_\infty \le n\|g_m - g_n\|_\infty \to 0 \quad \text{as } m \to \infty,$$

where

$$z(t) = a + \int_0^t g(s)\,ds =: y^n(t), \qquad t \in [0,n].$$

By Lemma 2.3 we conclude that g_n is a continuous selection of $f(\cdot, y_n(\cdot))$ on $[0,n]$. Then

$$y^n(t) = a + \int_0^t g_n(s)\,ds, \qquad t \in [0,n],$$

is a solution of problem (3.1) on $[0,n]$.

Step 2. By the same methods used in Step 1, we construct two new sequences $\{y_m\}_{m=0}^{\infty}$ and $\{g_m\}_{m=0}^{\infty}$ by

$$y_0(t) = y^n(n) \quad \text{for all } t \in [n, n+1],$$

$$y_1(t) = \begin{cases} y_0(t) & \text{if } t \in [n,(n+1)/2],\\ y^n(n) + \int_0^{t-(n+1)/2} g_{2,1}(s)\,ds & \text{if } t \in ((n+1)/2, n+1], \end{cases}$$

where $g_{2,1} : [n,(n+1)/2] \to \mathbb{R}^N$ is a continuous selection of $f(\cdot, y_1(\cdot)) : [n,(n+1)/2] \to \mathcal{A}_Q(\mathbb{R}^N)$. From Theorem 2.1 we can find a continuous selection $g_1 : [n,n+1] \to \mathbb{R}^N$ of $f(\cdot, y_1(\cdot)) : [n,n+1] \times \mathbb{R}^N \to \mathcal{A}_Q(\mathbb{R}^N)$ such that $g_1(\cdot) = g_{2,1}(\cdot)$ on $[n,(n+1)/2]$. We define

$$y_2(t) = \begin{cases} y_0(t) & \text{if } t \in [n,(n+1)/3],\\ y^n(n) + \int_n^{t-(n+1)/3} g_{3,1}(s)\,ds & \text{if } t \in [(n+1)/3, 2(n+1)/3],\\ y^n(n) + \int_n^{t-(n+1)/3} g_{3,2}(s)\,ds & \text{if } t \in [2(n+1)/3, n+1], \end{cases}$$

where $g_{3,1} : [n,(n+1)/3] \to \mathbb{R}^N$ is a continuous selection of $f(\cdot, y_2(\cdot)) : [n,(n+1)/3] \to \mathcal{A}_Q(\mathbb{R}^N)$ and $g_{3,2} : [n, 2(n+1)/3] \to \mathbb{R}^N$ is a continuous selection of $f(\cdot, y_2(\cdot)) : [n, 2(n+1)/3] \to \mathcal{A}_Q(\mathbb{R}^N)$ such that $g_{3,1}(\cdot) = g_{3,2}(\cdot)$ on $[n,(n+1)/3]$. By Theorem 2.1 we can choose a continuous selection of $f(\cdot, y_3(\cdot)) : [n,n+1] \to \mathcal{A}_Q(\mathbb{R}^N)$ such that $g_2(\cdot) = g_{3,2}(\cdot)$ on $[n, 2(n+1)/3]$. Finally, we define inductively

$$y_m(t) = \begin{cases} y_0(t) & \text{if } t \in [n,(n+1)/m],\\ y^n(n) + \int_0^{t-(n+1)/m} g_m(s)\,ds & \text{if } t \in ((n+1)/m, n+1], \end{cases}$$

where $g_m : [n, n+1] \to \mathbb{R}^N$ is a continuous selection of $f(\cdot, y_m(\cdot)) : [n, n+1] \to \mathcal{A}_Q(\mathbb{R}^N)$. From Step 1, we can show that there exist $y^{n+1} \in C([n, n+1], \mathbb{R}^N)$ and $g_{n+1} \in C([n, n+1], \mathbb{R}^N)$, $g_{n+1}(n) = g_n(n)$, a continuous selection of $f(\cdot, y^{n+1}(\cdot)) : [n, n+1] \to \mathcal{A}_Q(\mathbb{R}^N)$ such that

$$y^{n+1}(t) = y^n(n) + \int_n^t g_{n+1}(s)\,ds, \qquad t \in [n, n+1],$$

which is a solution of problem (3.1) on $[n, n+1]$, with the initial condition $y(n) = y^n(n)$.

For the last part of the proof, we now employ a diagonalization process. For $k \in \mathbb{N}$, let

$$u_k(t) = \begin{cases} \widetilde{y}_k(t), & t \in [0, n_k], \\ \widetilde{y}_k(n_k), & t \in [n_k, \infty), \end{cases}$$

where $\{n_k\}_{k \in \mathbb{N}}$ is a sequence of numbers satisfying

$$0 < n_1 < \cdots < n_k < \cdots \uparrow \infty,$$

and

$$\widetilde{y}_2(t) = \begin{cases} y_1(t), & t \in [0, n_1], \\ y_2(t), & t \in [n_1, n_2], \end{cases}$$

where

$$y_1(t) = \begin{cases} y^1(t), & t \in [0, 1], \\ y^2(t), & t \in [1, 2], \\ \quad\vdots \\ y^{n_1}(t), & t \in [n_1 - 1, n_1], \end{cases}$$

$$y_2(t) = \begin{cases} y^{n_1+1}(t), & t \in [n_1, n_1 + 1], \\ y^{n_1+2}(t), & t \in [n_1 + 1, n_1 + 2], \\ \quad\vdots \\ y^{n_2}(t), & t \in [n_2 - 1, n_2]. \end{cases}$$

Set $S = \{u_{n_k}\}_{k=1}^{\infty}$. It is clear that there exists $M_* > 0$ such that, for every solution y of problem (3.1), we have

$$\|y\|_* = \sup\{e^{-M_2 t}|y(t)| : t \in [0, \infty)\} \le M_*.$$

Notice that

$$|u_{n_k}(t)| \le e^{n_k M_2} M_* \quad \text{for each } t \in [0, n_k],\ k \in \mathbb{N},$$

and

$$u_{n_k}(t) = a + \int_0^t g_{n_k}(t)\,dt \quad \text{for every } t \in [0, n_k].$$

Then, for each $t, \tau \in [0, n_1]$ and $k \in \mathbb{N}$, we have

$$|u_{n_k}(t) - u_{n_k}(\tau)| = \Big| \int_0^t g_{n_1}(s)\, ds - \int_0^\tau g_{n_1}(s)\, ds \Big| \le \int_\tau^t |g_{n_1}(s)|\, ds \le e^{n_1 M_2} M_* |t - \tau|.$$

The Arzelà–Ascoli theorem guarantees that there is a subsequence N_1 of $\mathbb{N}$ and a function $z_1 \in C([0, n_1], \mathbb{R}^N)$ such that $u_{n_k} \to z_1$ in $C([0, n_1], \mathbb{R}^N)$ as $k \to \infty$ through N_1. Let $N_1^* = N_1 \backslash \{1\}$.

Notice that

$$|u_{n_k}(t)| \le M \quad \text{for every } t \in [0, n_2],\ k \in \mathbb{N}.$$

Also for $k \in \mathbb{N}$, and $t, \tau \in [0, n_2]$, we have

$$|u_{n_k}(t) - u_{n_k}(\tau)| = \Big| \int_0^t g_{n_2}(s)\, ds - \int_0^\tau g_{n_2}(s)\, ds \Big| \le \int_\tau^t |g_{n_2}(s)|\, ds \le M_* e^{n_2 M_2} |t - \tau|.$$

Again the Arzelà–Ascoli theorem guarantees that there is a subsequence N_2 of N_1^* and a function $z_2 \in C([0, n_2], \mathbb{R}^N)$ such that $u_{n_k} \to z_2$ in $C([0, n_2], \mathbb{R}^N)$ as $k \to \infty$ through N_2. Note $z_1 = z_2$ on $[0, n_1]$ since $N_2 \subset N_1^*$. Let $N_2^* = N_2 \backslash \{2\}$.

Proceed inductively to obtain, for each $m \in \{2, 3, \ldots\}$, a subsequence N_m of N_{m-1}^* and a function $z_m \in C([0, n_m], \mathbb{R}^N)$ with $u_{n_k} \to z_m$ in $C([0, n_m], \mathbb{R}^N)$ as $k \to \infty$ through N_m. Let $N_m^* = N_m \backslash \{m\}$. Define a function as follows: for $t \in [0, \infty)$ and $n \in \mathbb{N}$ with $t \le n_m$, define $y(t) = z_m(t)$. Then $y \in C^1([0, \infty), \mathbb{R}^N)$, $y(0) = a$ and $|y(t)| \le M$ for each $t \in [0, \infty)$. Fix $t \in [0, \infty)$ and let $m \in \mathbb{N}$ with $t \le n_m$. Then for each $n \in N_m^*$,

$$u_{n_k}(t) = a + \int_0^t g_{n_k}(s)\, ds.$$

Let $n_k \to \infty$ through N_m^* to obtain

$$z_m(t) = a + \int_0^t g_m(s)\, ds,$$

where g_m is a continuous selection for $f(\cdot, z_m(\cdot))$. Thus

$$y(t) = a + \int_0^t g(s)\, ds, \qquad t \in [0, \infty),$$

where g is a continuous selection for $f(\cdot, y(\cdot))$. ∎

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