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The chain recurrent set for maps of compacta

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Dedicated to Professor Tetsuo Furumochi on his 60th birthday

Abstract. For a self-map of a compactum we give a necessary and sufficient condition for the chain recurrent set to be precisely the set of periodic points.

1. Introduction. The purpose of this paper is to study properties of the chain recurrent set for self-maps of compacta.

In studying the dynamics, we encounter points which are not periodic but whose orbit keeps returning near where it started. As a type of such recurrence, the concept of chain recurrence was introduced by C. Conley [5] in the study of flows on manifolds. In general, the set $\mathrm{CR}(f)$ of chain recurrent points (see §2 for definition) contains the set $\mathrm{Per}(f)$ of periodic points, but equality need not hold. Block and Franke showed in [2] that for an interval self-map f, $\mathrm{CR}(f) = \mathrm{Per}(f)$ when $\mathrm{Per}(f)$ is closed. For a circle self-map f, Block and Franke [3] gave necessary and sufficient conditions for $\mathrm{CR}(f) = \mathrm{Per}(f)$. In this paper, for a self-map of a compactum, we establish a necessary and sufficient condition for the chain recurrent set to be precisely the set of periodic points. As a special case, for a graph self-map f we obtain sufficient conditions for which the equality holds. Our argument is based on a series of results [2], [3] and [4] by Block and Franke. A motivation for studying graph self-maps is that higher-dimensional dynamics can often be reduced to one-dimensional dynamics: this is the case in the study of the structure of attractors of a diffeomorphism, the quotient maps generated by maps on manifolds with an invariant foliation of codimension one and the dynamics of pseudo-Anosov homeomorphisms on a surface.

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We give the terminology and notation needed in what follows. Throughout this paper, a *compactum* means a compact metric space; by a *graph*, we mean a *connected* compact one-dimensional polyhedron, and a *tree* is a graph which contains no loops. A *map* $f : X \to X$ is a continuous function from a space X to itself; f^0 is the identity map, and for every $n \geq 0$, $f^{n+1} = f^n \circ f$. A map from a graph (respectively interval, circle, tree) to itself is called a *graph self-map* (respectively an *interval self-map*, a *circle self-map*, a *tree self-map*). We denote by $\mathrm{Fix}(f)$ and $\mathrm{Per}(f)$ the sets of fixed points and of periodic points of f, respectively. A subset A of a space X is said to be *invariant* with respect to $f : X \to X$ if $f(A) \subseteq A$, and *strongly invariant* if $f(A) = A$. For a subset K of X, $\mathrm{Bd}\,K$ and $\mathrm{Cl}\,K$ denote the boundary and closure of K in X. We define the *limit set* of a point $x \in X$ with respect to $f : X \to X$ to be the set $\omega(x, f) = \bigcap_{m\geq 0} \mathrm{Cl} \bigcup_{n\geq m}\{f^n(x)\}$.

2. Chain recurrence and elementary properties. We let $f : X \to X$ be a map from a compactum (X, d) to itself. Let $x, y \in X$. An ε-*chain* from x to y is a finite sequence of points $\{x_0, x_1, \ldots, x_n\}$ of X such that $x_0 = x$, $x_n = y$ and $d(f(x_{i-1}), x_i) < \varepsilon$ for $i = 1, \ldots, n$. We say that x *can be chained to* y if for every $\varepsilon > 0$ there exists an ε-chain from x to y, and x is *chain recurrent* if it can be chained to itself. The set of all chain recurrent points is called the *chain recurrent set* of f and denoted by $\mathrm{CR}(f)$.

The following two lemmas are basic properties of the chain recurrent set of maps.

LEMMA 2.1. $\mathrm{CR}(f) = \mathrm{CR}(f^n)$ *for any natural number* n.

LEMMA 2.2. *The chain recurrent set* $\mathrm{CR}(f)$ *is strongly invariant.*

A subset Y of X is called *positively chain invariant* if for every $y \in Y$ and $x \in X \setminus Y$, y cannot be chained to x. The following lemmas are quite useful.

LEMMA 2.3 ([2]). *Let* Y *be a positively chain invariant subset of* X. *If* $x \notin Y$ *and* $f^k(x) \in Y$ *for some natural number* k, *then* $x \notin \mathrm{CR}(f)$.

LEMMA 2.4 ([3]). *If* Y *is an open subset of* X *with* $f(\mathrm{Cl}\,Y) \subseteq Y$, *then* $\mathrm{Cl}\,Y$ *is positively chain invariant and* $\mathrm{CR}(f) \cap \mathrm{Cl}\,Y = \mathrm{CR}(f|_{\mathrm{Cl}\,Y})$.

We also need Theorem A from [6] later. The case of interval self-maps was proved earlier by Block and Franke [2].

LEMMA 2.5 ([6]). *Let* f *be a tree self-map. Then* $\mathrm{CR}(f) = \mathrm{Per}(f)$ *if and only if* $\mathrm{Per}(f)$ *is closed.*

3. Chain recurrence and attractors. There are various definitions of an attractor; here we take the simplest one.

Definition 3.1. Let $f : X \to X$ be a map from a compactum X to itself. A non-empty closed subset A of X is called an *attractor* of f if

- for each open neighborhood U of A there exists an open neighborhood V of A such that $f^k(V) \subseteq U$ for $k \geq 0$,
- there exists an open neighborhood W of A such that $\omega(x, f) \subseteq A$ for every $x \in W$.

The following property of attractors of maps is well-known.

Proposition 3.2. *Let A be an attractor of $f : X \to X$. Then there exists an arbitrarily small open neighborhood U of A such that $f(\mathrm{Cl}\, U) \subseteq U$ and $\bigcap_{k=0}^{\infty} f^k(\mathrm{Cl}\, U) \subseteq A$.*

We establish that the chain recurrent set of the restriction to an attractor can be represented by the one of the restriction to a certain closed neighborhood. We need this result later.

Lemma 3.3. *Let $f : X \to X$ be a map from a compactum (X, d) to itself, and A an attractor of f. Then there exists an arbitrarily small open neighborhood U of A in X such that* $\mathrm{CR}(f|_{\mathrm{Cl}\, U}) = \mathrm{CR}(f|_A)$.

Proof. By Proposition 3.2, we have an (arbitrarily small) open neighborhood U of A in X such that

(1) $f(\mathrm{Cl}\, U) \subseteq U$,
(2) $\bigcap_{k=0}^{\infty} f^k(\mathrm{Cl}\, U) \subseteq A$.

The set U with such properties will satisfy our required condition. It suffices to show the inclusion $\mathrm{CR}(f|_{\mathrm{Cl}\, U}) \subseteq \mathrm{CR}(f|_A)$. We first note that

$$\mathrm{CR}(f|_{\mathrm{Cl}\, U}) \subseteq A; \tag{3}$$

this follows from (2) and from the fact that for each $k \geq 0$,

$$\mathrm{CR}(f|_{\mathrm{Cl}\, U}) = (f|_{\mathrm{Cl}\, U})^k(\mathrm{CR}(f|_{\mathrm{Cl}\, U})) \subseteq (f|_{\mathrm{Cl}\, U})^k(\mathrm{Cl}\, U) = f^k(\mathrm{Cl}\, U),$$

where the first equality is by Lemma 2.2.

Let $x \in \mathrm{CR}(f|_{\mathrm{Cl}\, U})$. We shall show that for any $\varepsilon > 0$, there exists an ε-chain from x to itself in A (note (3)).

As f is uniformly continuous, we can take a positive number $\delta < \varepsilon/3$ such that for any $y, z \in X$,

$$d(y, z) < \delta \quad \text{implies} \quad d(f(y), f(z)) < \varepsilon/3. \tag{4}$$

By (1) and (2), we have a natural number k_0 such that

$$f^{k_0}(\mathrm{Cl}\, U) \subseteq \mathbb{B}(A; \delta), \tag{5}$$

where $\mathbb{B}(A; \delta)$ means the δ-neighborhood of A. As f^{k_0} is uniformly continuous, we again take a positive number $\gamma < \delta$ such that for any $y, z \in X$,

$$d(y, z) < \gamma \quad \text{implies} \quad d(f^{k_0}(y), f^{k_0}(z)) < \delta. \tag{6}$$

We now choose a point $\tilde{x} \in \mathrm{CR}(f|_{\mathrm{Cl}\,U})$ with $(f|_{\mathrm{Cl}\,U})^{k_0}(\tilde{x}) = x$, using Lemma 2.2. Let $\{x_0 = \tilde{x}, x_1, \ldots, x_n = \tilde{x}\}$ be a γ-chain from $\tilde{x}$ to itself in $\mathrm{Cl}\,U$. By (6), (5) and (1), the finite sequence

$$\{f^{k_0}(x_0), f^{k_0}(x_1), \ldots, f^{k_0}(x_n)\}$$

is a δ-chain from x to itself in $\mathbb{B}(A;\delta)$. Take points $z_0, z_1, \ldots, z_n \in A$ such that

$$z_0 = x = z_n \quad \text{and} \quad d(z_i, f^{k_0}(x_i)) < \delta \quad \text{for } i = 1, \ldots, n-1.$$

By (4), (6) and the above, we have for each $i = 1, \ldots, n$,

$$\begin{aligned} d(f(z_{i-1}), z_i) &\le d(f(z_{i-1}), f(f^{k_0}(x_{i-1}))) \\ &\qquad + d(f(f^{k_0}(x_{i-1})), f^{k_0}(x_i)) + d(f^{k_0}(x_i), z_i) \\ &< \varepsilon/3 + \delta + \delta < \varepsilon. \end{aligned}$$

Therefore, the finite sequence $\{z_0, z_1, \ldots, z_n\}$ is an ε-chain from x to itself in A. As $\varepsilon > 0$ is arbitrary, we conclude $x \in \mathrm{CR}(f|_A)$, and the proof is complete. ∎

4. Chain recurrence and periodicity. Here is our main result.

THEOREM 4.1. *Let $f : X \to X$ be a map from a compactum to itself. Then $\mathrm{CR}(f) = \mathrm{Per}(f)$ if and only if for every $x \in X \setminus \mathrm{Per}(f)$, there exist an open set U in X which intersects $\omega(x, f)$, and a natural number n such that $f^n(\mathrm{Cl}\,U) \subseteq U$, $\mathrm{Cl}\,U \ne X$ and $\mathrm{CR}(f^n|_{\mathrm{Cl}\,U}) = \mathrm{Per}(f^n|_{\mathrm{Cl}\,U})$.*

REMARK. This theorem is a generalization of [2] and [3], which is based on the results in [4].

Proof. We first show the sufficiency. Suppose that the condition is satisfied. Let $x \in X \setminus \mathrm{Per}(f)$. By assumption, we have a point $y \in \omega(x, f)$, an open neighborhood U of y in X and a natural number n such that

(1) $f^n(\mathrm{Cl}\,U) \subseteq U$, $\mathrm{Cl}\,U \ne X$,
(2) $\mathrm{CR}(f^n|_{\mathrm{Cl}\,U}) = \mathrm{Per}(f^n|_{\mathrm{Cl}\,U})$.

We note that $y \in \mathrm{Per}(f)$, because $y \in \omega(x, f) \cap \mathrm{Cl}\,U \subseteq \mathrm{CR}(f) \cap \mathrm{Cl}\,U = \mathrm{CR}(f^n) \cap \mathrm{Cl}\,U = \mathrm{CR}(f^n|_{\mathrm{Cl}\,U}) = \mathrm{Per}(f^n|_{\mathrm{Cl}\,U})$ (see Lemmas 2.1 and 2.4). Without loss of generality, we may assume that the period of y with respect to f divides n; that is, $f^n(y) = y$ (if necessary, take $n \times$ (the period of y) as a new "n").

We claim that for each $i \in \{0, 1, \ldots, n-1\}$, there exists an open neighborhood U_i of $f^i(y)$ in X such that

(3) $f^n(\mathrm{Cl}\,U_i) \subseteq U_i$, $\mathrm{Cl}\,U_i \ne X$,
(4) $\mathrm{CR}(f^n|_{\mathrm{Cl}\,U_i}) = \mathrm{Per}(f^n|_{\mathrm{Cl}\,U_i})$.

(In fact, this claim is stronger than what we require for our present purpose; however, we need it to understand the structure of the dynamical system in our situation; cf. Lemma 6 in [4].)

When $i = 0$, then put $U_0 = U$. Let $1 \le i \le n-1$; then since

$$f^{n-i}(f^i(y)) = f^n(y) \in f^n(U) \subseteq U,$$

$f^{-(n-i)}(U)$ is an open neighborhood of $f^i(y)$ and we have

$$f^{n-i}(\mathrm{Cl}\, f^{-(n-i)}(U)) \subseteq \mathrm{Cl}\, U, \tag{5}$$

by continuity of f^{n-i}. There are two cases to consider

CASE 1: $\mathrm{Cl}\, f^{-(n-i)}(U) \ne X$.
CASE 2: $\mathrm{Cl}\, f^{-(n-i)}(U) = X$.

First, in Case 1, put $U_i = f^{-(n-i)}(U)$. Then using $f^i(\mathrm{Cl}\, U) \subseteq U_i$ with (5), we have

$$f^n(\mathrm{Cl}\, U_i) = f^i(f^{n-i}(\mathrm{Cl}\, U_i)) \subseteq f^i(\mathrm{Cl}\, U) \subseteq U_i. \tag{6}$$

Next, we shall prove

$$\mathrm{CR}(f^n|_{\mathrm{Cl}\, U_i}) = \mathrm{Per}(f^n|_{\mathrm{Cl}\, U_i}). \tag{7}$$

It suffices to show the inclusion $\mathrm{CR}(f^n|_{\mathrm{Cl}\, U_i}) \subseteq \mathrm{Per}(f^n|_{\mathrm{Cl}\, U_i})$. By (5) and uniform continuity of f^{n-i},

$$f^{n-i}(\mathrm{CR}(f^n|_{\mathrm{Cl}\, U_i})) \subseteq \mathrm{CR}(f^n|_{\mathrm{Cl}\, U}). \tag{8}$$

Let $z \in \mathrm{CR}(f^n|_{\mathrm{Cl}\, U_i})$. Using (8) and (2) yields

$$f^{n-i}(z) \in \mathrm{CR}(f^n|_{\mathrm{Cl}\, U}) = \mathrm{Per}(f^n|_{\mathrm{Cl}\, U}),$$

so there exists a natural number l such that

$$f^{n-i}(z) = (f^n)^l(f^{n-i}(z)),$$

and hence

$$f^n(z) = (f^n)^l(f^n(z)),$$

therefore $f^n(z) \in \mathrm{Per}(f^n|_{\mathrm{Cl}\, U_i})$, and so

$$f^n|_{\mathrm{Cl}\, U_i}(\mathrm{CR}(f^n|_{\mathrm{Cl}\, U_i})) \subseteq \mathrm{Per}(f^n|_{\mathrm{Cl}\, U_i}).$$

Since $\mathrm{CR}(f^n|_{\mathrm{Cl}\, U_i})$ is strongly invariant with respect to $f^n|_{\mathrm{Cl}\, U_i}$, we have $\mathrm{CR}(f^n|_{\mathrm{Cl}\, U_i}) \subseteq \mathrm{Per}(f^n|_{\mathrm{Cl}\, U_i})$, therefore we have proved the equality in (7).

It remains to consider Case 2; then by (5),

$$f^{n-i}(X) = f^{n-i}(\mathrm{Cl}\, f^{-(n-i)}(U)) \subseteq \mathrm{Cl}\, U \subsetneq X.$$

Take an open set H in X such that $\mathrm{Cl}\, U \subseteq H \subseteq \mathrm{Cl}\, H \subsetneq X$; then we easily find that

- $f^i(y) = f^{n-i}(f^{2i}(y)) \in f^{n-i}(X) \subseteq H$,
- $f^n(\mathrm{Cl}\, H) \subseteq f^n(X) \subseteq f^{n-i}(X) \subseteq H$,
- $\mathrm{CR}(f^n|_{\mathrm{Cl}\, H}) = \mathrm{Per}(f^n|_{\mathrm{Cl}\, H})$,

where the last equality follows from a similar argument to the proof of (7). In Case 2, we put $U_i = H$. We have just constructed open sets $\{U_i\}$ satisfying (3) and (4).

We are now in a position to show $x \in X \setminus \mathrm{CR}(f)$. If $x \in \mathrm{Cl}\, U_i$ for some i, then by (4), Lemma 2.4 and Lemma 2.1, we have

$$x \notin \mathrm{Per}(f^n|_{\mathrm{Cl}\, U_i}) = \mathrm{CR}(f^n|_{\mathrm{Cl}\, U_i}) = \mathrm{CR}(f^n) \cap \mathrm{Cl}\, U_i = \mathrm{CR}(f) \cap \mathrm{Cl}\, U_i,$$

therefore $x \notin \mathrm{CR}(f)$.

Next, we assume $x \notin \bigcup_{i=0}^{n-1} \mathrm{Cl}\, U_i$. As $y \in \omega(x, f)$, we can take numbers $l_1 < l_2 < \cdots$ and $s \in \{0, 1, \ldots, n-1\}$ with $\lim_{i\to\infty} f^{l_i n+s}(x) = y$; so $\lim_{i\to\infty} f^{(l_i+1)n}(x) = f^{n-s}(y)$, and hence there exists a natural number k such that $(f^n)^k(x) \in U_{n-s}$. This with Lemma 2.3 yields $x \notin \mathrm{CR}(f^n) = \mathrm{CR}(f)$.

We now show the necessity. This is analogous to the proof of Theorem A in [3], so we give an outline only. We assume $\mathrm{CR}(f) = \mathrm{Per}(f)$, and let $x \in X \setminus \mathrm{Per}(f)$ and $y \in \omega(x, f)$. For $\varepsilon > 0$, let $\mathrm{R}_\varepsilon(y)$ denote the set of $z \in X$ such that y can be ε-chained to z. Then $\mathrm{R}_\varepsilon(y)$ is open in X and $f(\mathrm{Cl}\, \mathrm{R}_\varepsilon(y)) \subseteq \mathrm{R}_\varepsilon(y)$.

Since $x \notin \mathrm{Per}(f) = \mathrm{CR}(f)$ and $y \in \omega(x, f)$, there exists a positive number ε_0 such that for every $\varepsilon > 0$, $\varepsilon_0 \geq \varepsilon$ implies $x \notin \mathrm{R}_\varepsilon(y)$. Then we see that $\mathrm{Cl}\, \mathrm{R}_{\varepsilon_0/2}(y) \subsetneq X$, as $\mathbb{B}(x; \varepsilon_0/2) \cap \mathrm{R}_{\varepsilon_0/2}(y) = \emptyset$.

Put $U = \mathrm{R}_{\varepsilon_0/2}(y)$; then U is an open neighborhood of y satisfying $f(\mathrm{Cl}\, U) \subseteq U$ and $\mathrm{Cl}\, U \neq X$. By our assumption and Lemma 2.4, we obtain $\mathrm{CR}(f|_{\mathrm{Cl}\, U}) = \mathrm{CR}(f) \cap \mathrm{Cl}\, U = \mathrm{Per}(f) \cap \mathrm{Cl}\, U = \mathrm{Per}(f|_{\mathrm{Cl}\, U})$. The proof of Theorem 4.1 is finished. ∎

We reformulate the result above by using the concept of attractor.

THEOREM 4.2. *Let $f : X \to X$ be a map from a compactum to itself. Then $\mathrm{CR}(f) = \mathrm{Per}(f)$ if and only if for every $x \in X \setminus \mathrm{Per}(f)$, there exist an element $y \in \omega(x, f)$, a natural number n and a proper attractor A of f^n such that $\omega(y, f^n) \subseteq A$ and $\mathrm{CR}(f^n|_A) = \mathrm{Per}(f^n|_A)$.*

Proof. We first show the sufficiency. Suppose that the condition is satisfied. To show that $\mathrm{CR}(f) = \mathrm{Per}(f)$, we prove that the condition of Theorem 4.1 is satisfied; and let $x \in X \setminus \mathrm{Per}(f)$. By assumption, we have a $y \in \omega(x, f)$, an $n \in \mathbb{N}$, and a proper attractor A of f^n such that

(1) $\omega(y, f^n) \subseteq A$,
(2) $\mathrm{CR}(f^n|_A) = \mathrm{Per}(f^n|_A)$.

By Proposition 3.2, there exists an open neighborhood U of A in X such that

- $f^n(\mathrm{Cl}\, U) \subseteq U$, $\mathrm{Cl}\, U \neq X$,
- $\bigcap_{k=0}^{\infty} (f^n)^k(\mathrm{Cl}\, U) \subseteq A$.

Then by (1) and $\omega(y, f^n) \subseteq \omega(x, f)$, U intersects $\omega(x, f)$; and from (the proof of) Lemma 3.3 and (2), we conclude

$$\mathrm{CR}(f^n|_{\mathrm{Cl}\,U}) = \mathrm{CR}(f^n|_A) = \mathrm{Per}(f^n|_A) \subseteq \mathrm{Per}(f^n|_{\mathrm{Cl}\,U});$$

the converse inclusion is trivial.

We now show the necessity. Assume that $\mathrm{CR}(f) = \mathrm{Per}(f)$. Let $x \in X \setminus \mathrm{Per}(f)$. By Theorem 4.1, we have a $y \in \omega(x, f)$, an open neighborhood U of y in X, and an $n \in \mathbb{N}$ such that

(3) $f^n(\mathrm{Cl}\,U) \subseteq U$, $\mathrm{Cl}\,U \neq X$,
(4) $\mathrm{CR}(f^n|_{\mathrm{Cl}\,U}) = \mathrm{Per}(f^n|_{\mathrm{Cl}\,U})$.

Put $A = \bigcap_{k=0}^{\infty} (f^n)^k(\mathrm{Cl}\,U)$. Then A is a proper attractor of f^n satisfying $\omega(y, f^n) \subseteq A$. To see $\mathrm{CR}(f^n|_A) = \mathrm{Per}(f^n|_A)$, let $z \in \mathrm{CR}(f^n|_A)$. Using $\mathrm{CR}(f^n|_A) \subseteq \mathrm{CR}(f^n|_{\mathrm{Cl}\,U})$ and (4) implies $z \in \mathrm{Per}(f^n|_{\mathrm{Cl}\,U})$; and since z is an element of an f^n-invariant set A, we have $z \in \mathrm{Per}(f^n|_A)$. The converse inclusion is trivial. ∎

We extend the results for interval (or circle) self-maps in [2] and [3] to graph self-maps.

THEOREM 4.3. *Let $f : G \to G$ be a graph self-map. If the set* $\mathrm{Per}(f)$ *is closed, and for every $x \in G \setminus \mathrm{Per}(f)$, there exist an element $y \in \omega(x, f)$ and an open neighborhood U of y in G for which the closure* $\mathrm{Cl}\,U$ *is a tree and $f^n(\mathrm{Cl}\,U) \subseteq U$ for some natural number n, then* $\mathrm{CR}(f) = \mathrm{Per}(f)$.

Proof. If G is a tree, this follows directly from Lemma 2.5 (Theorem A in [6]). We prove the statement in the case when G is not a tree. Let $x \in G \setminus \mathrm{Per}(f)$. Then by assumption, there exist a $y \in \omega(x, f)$ and an open neighborhood U of y in G for which $\mathrm{Cl}\,U$ is a tree and $f^n(\mathrm{Cl}\,U) \subseteq U$ for some $n \in \mathbb{N}$. By applying Lemma 2.5 to the tree map $f^n|_{\mathrm{Cl}\,U} : \mathrm{Cl}\,U \to \mathrm{Cl}\,U$, we have $\mathrm{CR}(f^n|_{\mathrm{Cl}\,U}) = \mathrm{Per}(f^n|_{\mathrm{Cl}\,U})$, and note $\mathrm{Cl}\,U \neq G$; therefore, by Theorem 4.1, $\mathrm{CR}(f) = \mathrm{Per}(f)$. ∎

5. Examples. The converse of Theorem 4.3 is not generally true.

EXAMPLE 1. Let G be the graph defined by the union of the unit circle and the interval $[1, 2] \times \{0\}$ in $\mathbb{R}^2$, drawn below in Figure 1. The map f :

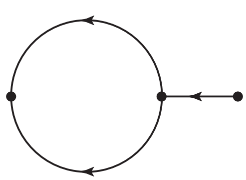

Fig. 1

$G \to G$ is given by

$$f((\cos\theta, \sin\theta)) = \big(\cos\big(\theta + \tfrac{1}{2}\sin\theta\big), \sin\big(\theta + \tfrac{1}{2}\sin\theta\big)\big)$$

on the unit circle ($0 \le \theta \le 2\pi$), and

$$f((x,0)) = ((x-1)^2 + 1, 0)$$

on the interval ($1 \le x \le 2$). Then $\mathrm{CR}(f) = \mathrm{Per}(f) = \{(-1,0),(1,0),(2,0)\}$; however, $\{(1,0)\}$ does not have an open neighborhood U in G for which the closure $\mathrm{Cl}\,U$ is a tree and $f^n(\mathrm{Cl}\,U) \subseteq U$ for some n.

In the proof of Theorem 4.3, we used the theorem of Li and Ye for tree self-maps (Lemma 2.5). This conclusion does not always hold for a dendrite (or disk) self-map. We also note that Block [1, Example D] constructed a graph self-map f such that $\#\,\mathrm{Per}(f) < \infty$ and $\mathrm{Per}(f) \neq \mathrm{CR}(f)$ to illustrate another property.

Example 2. Let D be the dendrite (that is, a connected and locally connected compactum which contains no simple closed curve) which is the union of infinitely many segments with end points z and a_n, $n \in \mathbb{Z}$, drawn below in Figure 2. The map $f : D \to D$ is given by $g(z) = z$ and f maps the segment $[z, a_n]$ homeomorphically onto the segment $[z, a_{n+1}]$ for $n \in \mathbb{Z}$. Then $\mathrm{Per}(f) = \{z\}$ and $\mathrm{CR}(f) = D$.

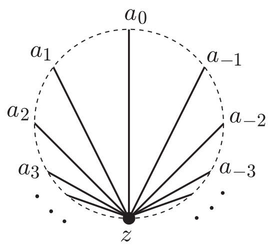

Fig. 2

Example 3. Let D^2 be the unit disk and ω an irrational number in $(0,1)$. The map $f : D^2 \to D^2$ is given by

$$f((r\cos\theta, r\sin\theta)) = (\sqrt{r}\cos(\theta + 2\pi\omega), \sqrt{r}\sin(\theta + 2\pi\omega)),$$

where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Then $\mathrm{Per}(f) = \{(0,0)\}$ and $\mathrm{CR}(f) = \mathrm{Bd}\,D^2 \cup \{(0,0)\}$.

The notion of chain recurrence can be defined on a non-compact metric space; however, Theorem 4.1 does not always hold for a self-map of a non-compact space.

EXAMPLE 4. Let X be a subspace $\{(\cos\theta, \sin\theta) \mid 0 \leq \theta \leq 2\pi,\ \theta \neq \pi\}$ of $\mathbb{R}^2$, drawn below in Figure 3. The map $f : X \to X$ is given by

$$f((\cos\theta, \sin\theta)) = \begin{cases} \big(\cos\big(\theta + \frac{1}{2}\sin\theta\big), \sin\big(\theta + \frac{1}{2}\sin\theta\big)\big), & 0 \leq \theta < \pi. \\ \big(\cos(\theta + \frac{1}{2}\sin(\theta - \pi)), \sin\big(\theta + \frac{1}{2}\sin(\theta - \pi)\big)\big), & \pi < \theta \leq 2\pi. \end{cases}$$

Then $\mathrm{Per}(f) = \{(1,0)\}$ and $\mathrm{CR}(f) = X$. Define the neighborhood U of $\{(1,0)\}$ by $\{(\cos\theta, \sin\theta) \mid 0 \leq \theta < \pi, 3\pi/2 < \theta \leq 2\pi\}$. Then $f(\mathrm{Cl}_X U) \subseteq U$, $\mathrm{Cl}_X U \neq X$ and $\mathrm{CR}(f|_{\mathrm{Cl}_X U}) = \{(1,0)\} = \mathrm{Per}(f|_{\mathrm{Cl}_X U})$.

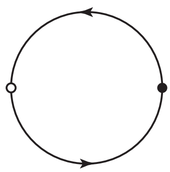

Fig. 3

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