ANNALES
POLONICI MATHEMATICI
83.3 (2004)

Applications of global bifurcation to existence theorems for Sturm–Liouville problems

by JACEK GULGOWSKI (Gdańsk)

Abstract. We prove an existence theorem for Sturm–Liouville problems

$$(*)\qquad \begin{cases} u''(t) + \varphi(t, u(t), u'(t)) = 0 & \text{for a.e. } t \in (a, b), \\ l(u) = 0, \end{cases}$$

where $\varphi : [a, b] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is a Carathéodory map.

We assume that $\varphi(t, x, y) = m_1\varphi_0(t, x, y) + o(|x| + |y|)$ as $|x| + |y| \to 0$ and $\varphi(t, x, y) = m_2\varphi_0(t, x, y) + o(|x| + |y|)$ as $|x| + |y| \to \infty$, where m_1, m_2 are positive constants and φ_0 belongs to a class of nonlinear maps. The proof bases on global bifurcation results. We define a map $f : (0, \infty) \times C^1([a, b], \mathbb{R}^k) \to C^1([a, b], \mathbb{R}^k)$ such that if $f(1, u) = 0$, then u is a solution of $(*)$. Then we show that there exists a connected set $\mathcal{C}$ of nontrivial zeroes of f such that there exist $(\lambda_1, u_1), (\lambda_2, u_2) \in \mathcal{C}$ with $\lambda_1 < 1 < \lambda_2$. In the last section we give examples of maps φ_0 leading to specific existence theorems.

1. Preliminaries. We consider the Banach space $C^1([a, b], \mathbb{R}^k)$ with the norm $\|u\|_k = \sum_{i=1}^k (\|u_i\|_0 + \|u_i'\|_0)$, $u = (u_1, \ldots, u_k)$, where $\|\cdot\|_0$ is the supremum norm in $C[a, b]$. Moreover, we set $|x| = \sum_{i=1}^k |x_i|$ for $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$.

Recall that $\psi : [a, b] \times \mathbb{R}^k \times \mathbb{R}^k \times (0, \infty) \to \mathbb{R}^k$ is a *Carathéodory map* if for almost every $t \in [a, b]$ the map $\psi(t, \cdot, \cdot, \cdot) : \mathbb{R}^k \times \mathbb{R}^k \times (0, \infty) \to \mathbb{R}^k$ is continuous; for every $(x, y, \lambda) \in \mathbb{R}^k \times \mathbb{R}^k \times (0, \infty)$ the map $\psi(\cdot, x, y, \lambda) : [a, b] \to \mathbb{R}^k$ is measurable; and for every $R > 0$ there exists $m_R \in L^1(a, b)$ such that $|\psi(t, x, y, \lambda)| \le m_R(t)$ for $|x| + |y| + |\lambda| \le R$.

We call a set $A \subset L^1((a, b), \mathbb{R}^k)$ *integrably bounded* if there exists $m_A \in L^1(a, b)$ such that $|u(t)| \le m_A(t)$ for $u \in A$ and a.e. t.

In the next section we deal with the problem of existence of solutions of the boundary value problem

$$(1.1)\qquad \begin{cases} u''(t) + \psi(t, u(t), u'(t), \lambda) = 0 & \text{for a.e. } t \in (a, b), \\ l(u) = 0, \end{cases}$$

2000 *Mathematics Subject Classification*: Primary 34B24; Secondary 34C23.
Key words and phrases: global bifurcation, Sturm–Liouville problem.

where $\psi : [a,b] \times \mathbb{R}^k \times \mathbb{R}^k \times (0,\infty) \to \mathbb{R}^k$ is a Carathéodory map and $l : C^1([a,b],\mathbb{R}^k) \to \mathbb{R}^k \times \mathbb{R}^k$ corresponds to Sturm–Liouville boundary conditions, given by

$$l(u_1,\dots,u_k) = (l_1(u_1),\dots,l_k(u_k)), \tag{1.2}$$

where

$$l_j(u_j) = (u_j(a)\sin\alpha_j - u_j'(a)\cos\alpha_j, u_j(b)\sin\beta_j + u_j'(b)\cos\beta_j),$$

and $\alpha_j,\beta_j \in [0,\pi/2]$, $\alpha_j^2+\beta_j^2 > 0$ $(j=1,\dots,k)$.

Let us recall some properties of the problem

$$\begin{cases} u''(t)+h(t)=0 & \text{for a.e. } t\in[a,b], \\ l(u)=0, \end{cases} \tag{1.3}$$

where $h \in L^1((a,b),\mathbb{R}^k)$.

We call $u \in C^1([a,b],\mathbb{R}^k)$ a solution of (1.3) if $u' : [a,b] \to \mathbb{R}^k$ is absolutely continuous and satisfies (1.3). It is known (cf. [H]) that there exists a continuous linear map $T : L^1((a,b),\mathbb{R}^k) \to C^1([a,b],\mathbb{R}^k)$ such that $Th = u$ iff u is a solution of (1.3). Let us now recall some properties of the map T.

(1.4) (cf. [H]) If $u \in C^1([a,b],\mathbb{R}^k)$ and $h \in L^1((a,b),\mathbb{R}^k)$, then $\langle u, Th\rangle_k = \langle Tu, h\rangle_k$, where $\langle w,v\rangle_k = \int_a^b \sum_{i=1}^k w_i(t)v_i(t)\,dt$.

(1.5) (cf. [P]) If $A \subset L^1((a,b),\mathbb{R}^k)$ is integrably bounded, then $T(A) \subset C^1([a,b],\mathbb{R}^k)$ is relatively compact.

Moreover, if $\Psi : (0,\infty)\times C^1([a,b],\mathbb{R}^k) \to L^1((a,b),\mathbb{R}^k)$ is the Nemytskiĭ map associated with the Carathéodory map $\psi : [a,b]\times\mathbb{R}^k\times\mathbb{R}^k\times(0,\infty)\to\mathbb{R}^k$, given by $\Psi(\lambda,u)(t) = \psi(t,u(t),u'(t),\lambda)$, then the map $T\circ\Psi : (0,\infty)\times C^1([a,b],\mathbb{R}^k) \to C^1([a,b],\mathbb{R}^k)$ is completely continuous.

Let $f : (0,\infty)\times C^1([a,b],\mathbb{R}^k) \to C^1([a,b],\mathbb{R}^k)$ be given by $f(\lambda,u) = u - T\Psi(\lambda,u)$. Assume that $\psi(\cdot,0,0,\cdot)=0$. Then $f(\lambda,0)=0$ for any $\lambda\in(0,\infty)$. Let $\mathcal{R}_f$ denote the closure of the set of nontrivial zeroes of f, i.e.

$$\mathcal{R}_f = \overline{\{(\lambda,u)\in(0,\infty)\times C^1([a,b],\mathbb{R}^k) \mid f(\lambda,u)=0,\ u\neq 0\}}.$$

We will use the global bifurcation theorem 1 given below. We recall that $(\lambda_0,0)\in(0,\infty)\times C^1([a,b],\mathbb{R}^k)$ is a *bifurcation point* of $f:(0,\infty)\times C^1([a,b],\mathbb{R}^k)\to C^1([a,b],\mathbb{R}^k)$ if for any open set $U\subset(0,\infty)\times C^1([a,b],\mathbb{R}^k)$ such that $(\lambda_0,0)\in U$, there exists $(\lambda,u)\in U$ with $u\neq 0$ and $f(\lambda,u)=0$. The set of all bifurcation points of f will be denoted by $\mathcal{B}_f$.

If $[a,b]\subset(0,\infty)$ and $\mathcal{B}_f\subset[a,b]\times\{0\}$, then we may define the *bifurcation index* of f in the interval $[a,b]$ as

$$s[f,a,b] = \lim_{\lambda\to b^+} d_f(\lambda) - \lim_{\lambda\to a^-} d_f(\lambda),$$

where $d_f(\lambda) = \deg(f(\lambda,\cdot),B(0,r),0)$ for $(\lambda,0)\notin\mathcal{B}_f$ and $r>0$ small enough.

The theorem given below is a direct consequence of the theorem given in [LS] (see also [CH]).

THEOREM 1. *Let* $F : (0,\infty) \times C^1([a,b],\mathbb{R}^k) \to C^1([a,b],\mathbb{R}^k)$ *be a completely continuous map such that* $F(\cdot,0) = 0$, *and let* $f : (0,\infty)\times C^1([a,b],\mathbb{R}^k) \to C^1([a,b],\mathbb{R}^k)$ *be given by* $f(\lambda,u) = u - F(\lambda,u)$. *If* $[a,b] \subset (0,\infty)$, $\mathcal{B}_f \subset [a,b] \times \{0\}$ *and* $s[f,a,b] \neq 0$, *then there exists a noncompact component* $\mathcal{C} \subset \mathcal{R}_f \cup ([a,b] \times \{0\})$ *such that* $\mathcal{C} \cap \mathcal{B}_f \neq \emptyset$.

2. Existence theorem. In this section we will be assuming that $\varphi_0 : [a,b] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$, $\varphi_0 = (\varphi_{0,1},\dots,\varphi_{0,k})$, is a Carathéodory map, satisfying the following conditions (A1)–(A3):

(A1) $\varphi_0(t,mx,my) = m\varphi_0(t,x,y)$ for all $(x,y) \in \mathbb{R}^{2k}$, $m \geq 0$ and almost every $t \in [a,b]$.

(A2) The set Λ of $\lambda \in (0,\infty)$ for which there exists $u \in C^1([a,b],\mathbb{R}^k)$, $u \neq 0$, such that (λ,u) is a solution of

$$\begin{cases} u''(t) + \lambda\varphi_0(t,u(t),u'(t)) = 0, \\ l(u) = 0, \end{cases} \tag{2.1}$$

is nonempty and bounded.

(A3) There exist a positive constant $\alpha > 0$ and a nonzero solution $(\mu_0,u_0) \in (0,\infty) \times C^1([a,b],\mathbb{R}^k)$, $u_0 = (u_{0,1},\dots,u_{0,k})$, of (2.1) such that

$$\sum_{i=1}^{k} \varphi_{0,i}(t,x,y)u_{0,i}(t) \geq \alpha \sum_{i=1}^{k} |x_i|\,|u_{0,i}(t)|$$

for all $(x,y) \in \mathbb{R}^{2k}$ and almost every $t \in [a,b]$.

Observe that $0 \notin \overline{\Lambda}$ because of the boundary conditions. Moreover, there exists $r > 0$ such that $\Lambda \subset (r,\infty)$. To prove this assume, contrary to our claim, that there exists a sequence $\{(\lambda_n,u_n)\} \subset (0,\infty) \times C^1([a,b],\mathbb{R}^k)$ of solutions of (2.1) such that $\lambda_n \to 0$ and $\|u_n\|_k = 1$. Then the sequence $T\Phi_0(u_n)$ contains a convergent subsequence and the corresponding subsequence of $\{u_n\} = \{\lambda_n T\Phi_0(u_n)\}$ converges to 0, which contradicts our assumption. A similar reasoning shows that Λ is a closed subset of $\mathbb{R}$.

THEOREM 2. *Assume* $0 < m_1 < \min\Lambda \leq \max\Lambda < m_2$ *and Carathéodory map* $\varphi_0 : [a,b] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ *satisfies* (A1)–(A3). *Assume moreover that* $\varphi : [a,b] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ *is a Carathéodory map satisfying*

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{(x,y)\in\mathbb{R}^{2k}}\ \forall_{t\in[a,b]} \quad |x|+|y| \leq \delta \ \Rightarrow\ |\varphi(t,x,y) - m_i\varphi_0(t,x,y)| \leq \varepsilon(|x|+|y|), \tag{2.2}$$

$$\forall_{\varepsilon>0}\ \exists_{R>0}\ \forall_{(x,y)\in\mathbb{R}^{2k}}\ \forall_{t\in[a,b]} \quad |x|+|y| \geq R \ \Rightarrow\ |\varphi(t,x,y) - m_j\varphi_0(t,x,y)| \leq \varepsilon(|x|+|y|), \tag{2.3}$$

where $(i,j)=(1,2)$ *or* $(2,1)$. *Then there exists a nonzero solution of the boundary value problem*

$$\begin{cases} u''(t)+\varphi(t,u(t),u'(t))=0 & \text{for a.e. } t\in(a,b),\\ l(u)=0, \end{cases} \tag{2.4}$$

where $l : C^1([a,b],\mathbb{R}^k)\to\mathbb{R}^{2k}$ *is given by* (1.2).

Proof. Without loss of generality we may assume that the constant α in (A3) satisfies $\alpha\in(0,1)$. Fix $\nu>\max\Lambda/m_1\alpha$. Let $q_1,q_2:(0,\infty)\to[0,\infty)$ be a partition of unity associated with the covering $U_1=(0,2\nu)$, $U_2=(\nu,\infty)$ of $(0,\infty)$. Let $\psi:[a,b]\times\mathbb{R}^k\times\mathbb{R}^k\times(0,\infty)\to\mathbb{R}^k$ be the Carathéodory map given by

$$\psi(t,x,y,\lambda)=\lambda q_1(\lambda)\varphi(t,x,y)+\lambda q_2(\lambda)m_j\varphi_0(t,x,y).$$

Let $\Psi:(0,\infty)\times C^1([a,b],\mathbb{R}^k)\to L^1((a,b),\mathbb{R}^k)$ be the Nemytskiĭ map associated with ψ. Define $f:(0,\infty)\times C^1([a,b],\mathbb{R}^k)\to C^1([a,b],\mathbb{R}^k)$ by $f(\lambda,u)=u-T\Psi(\lambda,u)$. We can see that if $f(1,u)=0$, then u is a solution of (2.4).

First we prove that $\mathcal{B}_f\subset\{\lambda/m_i \mid \lambda\in\Lambda\}$. To show this take a sequence $\{(\lambda_n,u_n)\}\subset(0,\infty)\times C^1([a,b],\mathbb{R}^k)$ such that $\lambda_n\to\lambda_0\in[0,\infty)$, $u_n\neq0$ and $u_n\to0$. Set $v_n=u_n/\|u_n\|_k$. Then

$$v_n=\lambda_n q_1(\lambda_n)T\,\frac{\Phi(u_n)-m_i\Phi_0(u_n)}{\|u_n\|_k}+\lambda_n(m_iq_1(\lambda_n)+m_jq_2(\lambda_n))T\Phi_0(v_n).$$

By (2.2) the first term on the right hand side converges to 0. Since $\{\Phi_0(v_n)\}$ is integrably bounded, $\{T\Phi_0(v_n)\}$ contains a convergent subsequence. So the corresponding subsequence of $\{v_n\}$ converges to some $v_0\in C^1([a,b],\mathbb{R}^k)$. Then $v_0=\lambda_0(m_iq_1(\lambda_0)+m_jq_2(\lambda_0))T\Phi_0(v_0)$.

Therefore $\lambda_0(m_iq_1(\lambda_0)+m_jq_2(\lambda_0))\in\Lambda$. Because $m_iq_1(\lambda_0)+m_jq_2(\lambda_0)\in[m_1,m_2]$, we must have $\lambda_0\le\max\Lambda/m_1<\nu$, and $q_2(\lambda_0)=0$. Hence $m_i\lambda_0\in\Lambda$, and $\mathcal{B}_f\subset\{\lambda/m_i\mid\lambda\in\Lambda\}$.

Now we show that $s[f,\min\Lambda/m_i,\max\Lambda/m_i]=-1$. First observe that by (2.2) and the homotopy property of the topological degree, $s[f,\min\Lambda/m_i,\max\Lambda/m_i]=s[f_0,\min\Lambda/m_i,\max\Lambda/m_i]$, where $f_0:(0,\infty)\times C^1([a,b],\mathbb{R}^k)\to C^1([a,b],\mathbb{R}^k)$ is given by

$$f_0(\lambda,u)=u-\lambda(m_iq_1(\lambda)+m_jq_2(\lambda))T\Phi_0(u).$$

Let $\lambda\in(0,\min\Lambda/m_i)\cup(\max\Lambda/m_i,\infty)$ and $r>0$. The map $f_0(\lambda,\cdot):\overline{B(0,r)}\to C^1([a,b],\mathbb{R}^k)$ is homotopic to $f_1(\lambda,\cdot):\overline{B(0,r)}\to C^1([a,b],\mathbb{R}^k)$ given by $f_1(\lambda,u)=u-\lambda m_iT\Phi_0(u)$. Indeed, for $\lambda\le\nu$ the maps are just equal. Let now $\lambda\ge\nu$. Then the required homotopy $h:[0,1]\times\overline{B(0,r)}\to C^1([a,b],\mathbb{R}^k)$ is given by

$$h(t,u)=u-\lambda(t(m_iq_1(\lambda)+m_jq_2(\lambda))+(1-t)m_i)T\Phi_0(u).$$

Observe that for $h(t,u)=0$ and $u\neq 0$,

$$\lambda(t(m_iq_1(\lambda)+m_jq_2(\lambda))+(1-t)m_i)\in \Lambda.$$

Because $m_iq_1(\lambda)+m_jq_2(\lambda)\geq m_1$, we must have

$$\max \Lambda \geq \lambda(tm_1+(1-t)m_1)=\lambda m_1,$$

which contradicts $\lambda\geq\nu$. So we conclude that

$$s[f_0,\min\Lambda/m_i,\max\Lambda/m_i]=s[f_1,\min\Lambda/m_i,\max\Lambda/m_i].$$

Now fix $\lambda\in(0,\min\Lambda/m_i)$. Because for $t\in[0,1]$ the maps $f_1(t\lambda,\cdot):\overline{B(0,r)}\to C^1([a,b],\mathbb{R}^k)$ do not have nontrivial zeroes, $f_1(\lambda,\cdot)$ is homotopic to the identity map, so $\deg(f(\lambda,\cdot),B(0,r),0)=1$.

Assume now that $\lambda>\max\Lambda/m_i$. As above, $f(\lambda_1,\cdot)\sim f(\lambda_2,\cdot)$ for all $\lambda_1,\lambda_2\in(\max\Lambda/m_i,\infty)$, so we may assume that $\lambda>\max\Lambda/\alpha m_i$. Now the map $f_1(\lambda,\cdot)$ may be joined by a homotopy to $f_2:\overline{B(0,r)}\to C^1([a,b],\mathbb{R}^k)$ given by $f_2(u)=u-\lambda m_iT\Phi_0(u)-u_0$, where u_0 is given in (A3); the homotopy $h_2:[0,1]\times\overline{B(0,r)}\to C^1([a,b],\mathbb{R}^k)$ is given by $h_2(t,u)=u-\lambda m_iT\Phi_0(u)-tu_0$.

We now show that $h_2(t,u)\neq 0$ for $t\in(0,1]$ and $u\in\overline{B(0,r)}$. Assume, contrary to our claim, that $h_2(t,u)=0$. Then

$$u-\lambda m_iT\Phi_0(u)=tu_0,$$
$$\langle u,u_0\rangle_k-\lambda m_i\langle T\Phi_0(u),u_0\rangle_k=t\langle u_0,u_0\rangle_k.$$

So

$$\begin{aligned}
0&<\langle u,u_0\rangle_k-\lambda m_i\langle T\Phi_0(u),u_0\rangle_k=\langle u,u_0\rangle_k-\frac{\lambda m_i}{\mu_0}\langle\Phi_0(u),u_0\rangle_k\\
&\leq\int_a^b\sum_{i=1}^k|u_i(t)|\,|u_{0,i}(t)|\,dt-\frac{\lambda m_i}{\mu_0}\int_a^b\sum_{i=1}^k\varphi_{0,i}(t,u(t),u'(t))u_{0,i}(t)\,dt\\
&\leq\int_a^b\sum_{i=1}^k|u_i(t)|\,|u_{0,i}(t)|\,dt-\frac{\alpha\lambda m_i}{\mu_0}\int_a^b\sum_{i=1}^k|u_i(t)|\,|u_{0,i}(t)|\,dt\\
&=\left(1-\frac{\alpha\lambda m_i}{\mu_0}\right)\int_a^b\sum_{i=1}^k|u_i(t)|\,|u_{0,i}(t)|\,dt.
\end{aligned}$$

Hence $\lambda<\max\Lambda/\alpha m_i$, a contradiction. So $f_1(\lambda,\cdot)\sim f_2$ and $f_2(u)\neq 0$ for $u\in\overline{B(0,r)}$. Hence $\deg(f_2,B(0,r),0)=0$ and $s[f,\min\Lambda/m_i,\max\Lambda/m_i]=-1$.

By Theorem 1 there exists a noncompact component

$$\mathcal{C}\subset\mathcal{R}_f\cup\left(\left[\frac{\min\Lambda}{m_i},\frac{\max\Lambda}{m_i}\right]\times\{0\}\right)$$

containing $[\min \Lambda/m_i, \max \Lambda/m_i] \times \{0\}$. Now we show that there exists a sequence $\{(\lambda_n, u_n)\} \subset \mathcal{C}$ such that $\|u_n\|_k \to \infty$. Assume, contrary to our claim, that $\{u_n\}$ is bounded. Then, because $\mathcal{C}$ is not compact, either $\lambda_n \to 0$ or $\lambda_n \to \infty$.

First consider the case of $\lambda_n \to 0$. Since $\{u_n\}$ is bounded, $\{T\Psi(\lambda_n, u_n)\}$ has a convergent subsequence. Because $u_n = \lambda_n T\Phi(u_n)$ for large $n \in \mathbb{N}$, the corresponding subsequence of $\{u_n\}$ converges to 0. But this contradicts our earlier observation that for $u_n \to 0$, the sequence $\{\lambda_n\}$ cannot converge to 0.

Now let $\lambda_n \to \infty$. We may assume that $\Psi(\lambda_n, u_n) = \lambda_n m_j T\Phi_0(u_n)$, so $u_n = \lambda_n m_j T\Phi_0(u_n)$. By (A2), $\lambda_n \in \{\lambda/m_j \mid \lambda \in \Lambda\}$, which contradicts $\lambda_n \to \infty$.

So there exists a sequence $\{(\lambda_n, u_n)\} \subset \mathcal{C}$ such that $\|u_n\|_k \to \infty$. We now show that $\lambda_n \to \lambda_0 \in \{\lambda/m_j \mid \lambda \in \Lambda\}$. Assume that $\lambda_n \to \lambda_0 \in [0, \infty)$. Then

$$u_n = T\Psi(\lambda_n, u_n),$$
$$u_n = \lambda_n q_1(\lambda_n) T(\Phi(u_n) - m_j \Phi_0(u_n)) + \lambda_n (q_1(\lambda_n) + q_2(\lambda_n)) m_j T\Phi_0(u_n),$$

and if we set $v_n = u_n/\|u_n\|_k$, then

$$v_n = \lambda_n q_1(\lambda_n) T \frac{\Phi(u_n) - m_j \Phi_0(u_n)}{\|u_n\|_k} + \lambda_n m_j T\Phi_0(v_n).$$

Observe that the set $\{\Phi(u_n) - m_j \Phi_0(u_n)\}$ is integrably bounded. Hence $\frac{1}{\|u_n\|_k} T(\Phi(u_n) - m_j \Phi_0(u_n)) \to 0$ in $C^1([a,b], \mathbb{R}^k)$. Because $\{\Phi_0(v_n)\}$ is integrably bounded as well, $\{T\Phi_0(v_n)\}$ has a convergent subsequence. So we may assume that $v_n \to v_0$ and then

$$v_0 = \lambda_0 m_j T\Phi_0(v_0)$$

for $v_0 \neq 0$. This implies $\lambda_0 \in \{\lambda/m_j \mid \lambda \in \Lambda\}$.

Because $\{\lambda/m_1 \mid \lambda \in \Lambda\} \subset (1, \infty)$ and $\{\lambda/m_2 \mid \lambda \in \Lambda\} \subset (0,1)$, there must exist pairs $(\lambda_1, u_1), (\lambda_2, u_2) \in \mathcal{C}$ such that $\lambda_1 < 1 < \lambda_2$. From the connectedness of $\mathcal{C}$ we conclude that there exists $(1, u) \in \mathcal{C}$. Because $1 < \nu$, the function u is a solution of (2.4). $\square$

3. Examples. In this section we give examples of Carathéodory maps $\varphi_0 : [a,b] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ satisfying (A1)–(A3), so leading to different versions of Theorem 2. First, we recall the basic spectral properties of the scalar linear Sturm–Liouville problem (cf. [H])

$$\begin{cases} v''(t) + \lambda v(t) = 0 & \text{for } t \in [a,b], \\ l_s(v) = 0, \end{cases} \tag{3.1}$$

where $v \in C^1[a,b]$, $\lambda \in \mathbb{R}$ and $l_s : C^1[a,b] \to \mathbb{R}^1 \times \mathbb{R}^1$ is given by (cf. (1.2))

$$l_s(v) = (v(a) \sin \alpha_s - v'(a) \cos \alpha_s, v(b) \sin \beta_s + v'(b) \cos \beta_s).$$

Problem (3.1) has a minimal eigenvalue $\mu_s \in \mathbb{R}$. Let v_0 denote an eigenvector associated with μ_s. Then $\mu_s \in (0,\infty)$ and v_0 does not change sign in (a,b). Additionally $|v_0|$ is the only nonzero and nonnegative solution of (3.1).

Let $\varphi_0 : \mathbb{R}^k \to \mathbb{R}^k$ be given by $\varphi_0(x_1,\dots,x_k) = (\xi_1|x_1|,\dots,\xi_k|x_k|)$, where $\xi_1,\dots,\xi_k \in [0,\infty)$ for $\xi_1^2+\dots+\xi_k^2 > 0$, and let $\Lambda = \{\mu_s/\xi_s \mid \xi_s > 0,\ s = 1,\dots,k\}$.

THEOREM 3. *Let φ_0 be as above. Assume moreover $0 < m_1 < \min\Lambda \le \max\Lambda < m_2$ and the Carathéodory map $\varphi : [a,b]\times\mathbb{R}^k\times\mathbb{R}^k \to \mathbb{R}^k$ satisfies*

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{(x,y)\in\mathbb{R}^{2k}}\ \forall_{t\in[a,b]}$$
$$|x|+|y|\le\delta \ \Rightarrow\ |\varphi(t,x,y)-m_i\varphi_0(t,x,y)| \le \varepsilon(|x|+|y|),$$
$$\forall_{\varepsilon>0}\ \exists_{R>0}\ \forall_{(x,y)\in\mathbb{R}^{2k}}\ \forall_{t\in[a,b]}$$
$$|x|+|y|\ge R \ \Rightarrow\ |\varphi(t,x,y)-m_j\varphi_0(t,x,y)| \le \varepsilon(|x|+|y|),$$

where $(i,j) = (1,2)$ or $(2,1)$. Then there exists a nonzero solution of

$$\begin{cases} u''(t)+\varphi(t,u(t),u'(t)) = 0 & \textit{for a.e. } t\in(a,b),\\ l(u) = 0, \end{cases}$$

where $l : C^1([a,b],\mathbb{R}^k) \to \mathbb{R}^k\times\mathbb{R}^k$ is given by (1.2).

Proof. By Theorem 2 it is enough to check that φ_0 satisfies (A1)–(A3). Condition (A1) is obvious.

We show that if (λ,u) is a solution of (1.2) such that $u\ne 0$, then $\lambda\in\Lambda$. If $u\ne0$ then there exists $s\in\{1,\dots,k\}$ such that $u_s\ne 0$ and

$$\begin{cases} u_s''(t)+\lambda\xi_s|u_s(t)| = 0 & \text{for a.e. } t\in(a,b),\\ l_s(u_s) = 0. \end{cases}$$

From the maximum principle (cf. [PW]) we conclude that $u_s \ge 0$, so $\lambda\xi_s = \mu_s$. This implies $\xi_s \ne 0$ and $\lambda\in\Lambda$.

Because the set Λ is finite and nonempty, condition (A2) is satisfied as well.

Let $s\in\{1,\dots,k\}$ be such that $\xi_s>0$ and (μ_s,v_0) is a solution of (3.1) such that $v_0(t)>0$ for $t\in(a,b)$. Let $u_0 = (0,\dots,v_0,\dots,0)$, where the sth coordinate is the only nonzero one. Observe that

$$\sum_{l=1}^k \varphi_{0,l}(t,x)u_{0,l}(t) = \xi_s|x_s|v_0(t) = \xi_s\sum_{l=1}^k |x_l|\,|u_{0,l}(t)|.$$

Hence condition (A3) is satisfied as well. ∎

Now consider the scalar ($k=1$) Picard problem. Fix $m\in\mathbb{N}$ and let $\varphi_m : [0,\pi]\times\mathbb{R}^1\to\mathbb{R}^1$ be given by

$$\varphi_m(t,x) = \begin{cases} |x| & \text{if } \sin(mt)\ge 0,\\ -|x| & \text{if } \sin(mt)<0. \end{cases} \tag{3.2}$$

Lemma 1. *There exists a constant $r > 0$ such that if $(\lambda, u) \in (0,\infty) \times C^1[a,b]$ is a solution of*

$$\begin{cases} u''(t) + \lambda\varphi_m(t,u(t)) = 0 & \text{for a.e. } t \in [0,\pi], \\ u(0) = u(\pi) = 0, \end{cases} \tag{3.3}$$

such that $u \neq 0$, then $\lambda \in [r, m^2]$.

Proof. First observe that for $\lambda = 0$ problem (3.3) has no solution $u \neq 0$. Also, there is no sequence $\{(\lambda_u, u_n)\} \subset (0,\infty) \times C^1([a,b],\mathbb{R}^k)$ such that $\lambda_n \to 0$ and $u_n \neq 0$ (see the remark after (A3)).

From steps (A)–(D) of Lemma 3.1 of [G] we conclude that all zeroes of u are isolated (in the set of zeroes of u), and at each of them u changes sign. Assume now, contrary to our claim, that $\lambda > m^2$. By (A) of the above mentioned Lemma 3.1 of [G] we can see that if $u(t)\sin mt < 0$, then

$$u(t) = Ae^{\sqrt{\lambda}t} + Be^{-\sqrt{\lambda}t}, \tag{$\mathcal{E}$}$$

and if $u(t)\sin mt > 0$, then

$$u(t) = A\sin(\sqrt{\lambda}t) + B\cos(\sqrt{\lambda}t) \tag{$\mathcal{T}$}$$

for some constants $A, B \in \mathbb{R}$. We see that if $\lambda > m^2$, then half the period of $(\mathcal{T})$ is less than π/m. So if there exists $t_0 \in (l\pi/m, (l+1)\pi/m)$ such that $u(t_0)\sin mt_0 > 0$, then the interval $(l\pi/m, (l+1)\pi/m)$ contains a zero of u. Thus for any $l \in \{1,\ldots,m\}$ there exists a left hand neighbourhood of $l\pi/m$ such that u restricted to this neighbourhood is given by $(\mathcal{E})$. So there must exist $t_0 \in ((m-1)\pi/m, \pi)$ such that u is given by $(\mathcal{E})$ in (t_0,π) and $u(t_0) = u(\pi) = 0$. This implies that $u = 0$ for $t \in (t_0,\pi)$, which contradicts the fact that all zeroes of u are isolated (in the set of zeroes of u). ∎

Theorem 4. *Let $\varphi_m : [0,\pi] \times \mathbb{R}^1 \to \mathbb{R}^1$ be given by* (3.2),

$$r_0 = \inf\{\lambda \in (0,\infty) \mid \exists_{u \in C^1([a,b],\mathbb{R}^k)}\ u \neq 0 \text{ and } (\lambda,u) \text{ is a solution of } (3.3)\}$$

and $0 < m_1 < r_0 \le m^2 < m_2$. Assume moreover the Carathéodory map $\varphi : [a,b] \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ satisfies

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{(x,y)\in\mathbb{R}^2}\ \forall_{t\in[0,\pi]}$$
$$|x| + |y| \le \delta \ \Rightarrow\ |\varphi(t,x,y) - m_i\varphi_m(t,x)| \le \varepsilon(|x|+|y|),$$
$$\forall_{\varepsilon>0}\ \exists_{R>0}\ \forall_{(x,y)\in\mathbb{R}^2}\ \forall_{t\in[0,\pi]}$$
$$|x| + |y| \ge R \ \Rightarrow\ |\varphi(t,x,y) - m_j\varphi_m(t,x)| \le \varepsilon(|x|+|y|),$$

for $(i,j) = (1,2)$ or $(2,1)$. Then there exists a nonzero solution of the problem

$$\begin{cases} u''(t) + \varphi(t,u(t),u'(t)) = 0 & \text{for a.e. } t \in (a,b), \\ u(0) = u(\pi) = 0. \end{cases}$$

Proof. First observe that $r_0 > 0$ by Lemma 1, and condition (A2) is satisfied.

Condition (A1) is obvious. Because $(m^2, \sin mt)$ is a solution of (3.3), if $u_0(t) = \sin mt$, then

$$\varphi_0(t,x)u_0(t) = |x|\,|\sin mt|,$$

which proves (A3). ∎

Remark (cf. [G]). For $m = 2$ we have $r_0 = \lambda^*$, where $\lambda^* \in (1,4)$ is the only solution of $\tan(\sqrt{\lambda^*}\,\pi/2) = -\tanh(\sqrt{\lambda^*}\,\pi/2)$.

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Institute of Mathematics
University of Gdańsk
Wita Stwosza 57
80-952 Gdańsk, Poland
E-mail: dzak@math.univ.gda.pl

Reçu par la Rédaction le 14.4.2003 (1439)