

**On the existence of solutions
of nonlinear integral equations in Banach spaces
and Henstock–Kurzweil integrals**

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Abstract. We prove some existence theorems for nonlinear integral equations of the Urysohn type $x(t) = \varphi(t) + \lambda \int_0^a f(t, s, x(s)) ds$ and Volterra type $x(t) = \varphi(t) + \int_0^t f(t, s, x(s)) ds$, $t \in I_a = [0, a]$, where f and φ are functions with values in Banach spaces. Our fundamental tools are: measures of noncompactness and properties of the Henstock–Kurzweil integral.

1. Introduction. The Henstock–Kurzweil integral encompasses the Newton, Riemann and Lebesgue integrals ([8], [9], [14]). A particular feature of this integral is that integrals of highly oscillating functions such as $F'(t)$, where $F(t) = t^2 \sin t^{-2}$ on $(0, 1]$ and $F(0) = 0$, can be defined. This integral was introduced by Henstock and Kurzweil independently in 1957–58 and has since proved useful in the study of ordinary differential equations ([6], [7], [13], [17], [18]).

It is well known that Henstock’s Lemma plays an important role in the theory of the Henstock–Kurzweil integral in the real-valued case. On the other hand, in connection with the Henstock–Kurzweil integral for Banach space valued functions, S. S. Cao pointed out in [5] that Henstock’s Lemma holds for the case of finite dimensions, but it does not always hold in infinite dimensions.

In this paper we will use the HL integral which satisfies Henstock’s Lemma and which is more general than the Bochner integral.

Let E be a Banach space, E_1 a separable Banach space, and let $I_a = [0, a]$, $a \in \mathbb{R}_+$. We will prove some existence theorems for the Urysohn integral

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equation

$$(1.1) \quad x(t) = \varphi(t) + \lambda \int_0^a f(t, s, x(s)) ds, \quad \lambda > 0, t \in I_a,$$

and for the Volterra integral equation

$$(1.2) \quad x(t) = \varphi(t) + \int_0^t f(t, s, x(s)) ds, \quad t \in I_a,$$

where f and φ are functions with values in the Banach space E (or in the separable Banach space E_1) and the integral is the HL integral. Our fundamental tools are the Kuratowski and Hausdorff measures of noncompactness ([12], see also [3]), and the properties of the HL integral ([5], [17]).

The result presented in this paper contains several results concerning the Bochner integral (e.g. Januszewski [10], Krzyńska [11]) and the Henstock–Kurzweil integral for real-valued functions (Bugajewski [4]).

For any bounded subset A of E we denote by $\alpha(A)$ the *Kuratowski measure of noncompactness* of A , i.e. the infimum of all $\varepsilon > 0$ such that there exists a finite covering of A by sets of diameter smaller than ε .

For any bounded subset A of E we denote by $\alpha_1(A)$ the *Hausdorff measure of noncompactness* of A , i.e. the infimum of all $\varepsilon > 0$ such that A can be covered by a finite number of balls of radius smaller than ε .

The properties of the measures of noncompactness $\gamma = \alpha, \alpha_1$ are:

- (i) if $A \subset B$ then $\gamma(A) \leq \gamma(B)$;
- (ii) $\gamma(A) = \gamma(\bar{A})$, where \bar{A} denotes the closure of A ;
- (iii) $\gamma(A) = 0$ if and only if A is relatively compact;
- (iv) $\gamma(A \cup B) = \max\{\gamma(A), \gamma(B)\}$;
- (v) $\gamma(\lambda A) = |\lambda| \gamma(A)$ ($\lambda \in \mathbb{R}$);
- (vi) $\gamma(A + B) \leq \gamma(A) + \gamma(B)$;
- (vii) $\gamma(\text{conv } A) = \gamma(A)$;
- (viii) $\alpha_1(A) \leq \alpha(A) \leq 2\alpha_1(A)$.

We will need the following lemma:

LEMMA 1.1 ([2]). *Let $H \subset C(I_a, E)$ be a family of strongly equicontinuous functions. Then*

$$\alpha_C(H) = \sup_{t \in I_a} \alpha(H(t)) = \alpha(H(I_a)),$$

where $\alpha_C(H)$ denotes the measure of noncompactness in $C(I_a, E)$ and the function $t \mapsto \alpha(H(t))$ is continuous.

2. Henstock–Kurzweil integral in Banach spaces. In this part we define the Henstock–Kurzweil integral in a Banach space and we give some properties of this integral.

DEFINITION 2.1 ([8]). Let $\delta(\cdot)$ be a positive function defined on the interval $[a, b]$. A *tagged interval* $(x, [c, d])$ consists of an interval $[c, d] \subset [a, b]$ and a point $x \in [c, d]$. The tagged interval $(x, [c, d])$ is *subordinate to δ* if $[c, d] \subset [x - \delta(x), x + \delta(x)]$. The letter P will be used to denote finite collections of nonoverlapping tagged intervals. Let

$$P = \{(s_i, [c_i, d_i]) : 1 \leq i \leq n\}, \quad n \in \mathbb{N}$$

be such a collection in $[a, b]$.

- (i) The points $\{s_i : 1 \leq i \leq n\}$ are called the *tags* of P .
- (ii) The intervals $\{[c_i, d_i] : 1 \leq i \leq n\}$ are called the intervals of P .
- (iii) If $\{(s_i, [c_i, d_i]) : 1 \leq i \leq n\}$ is subordinate to δ for each i , then we say that P is sub δ .
- (iv) If $[a, b] = \bigcup_{i=1}^n [c_i, d_i]$ then P is called a *tagged partition* of $[a, b]$.
- (v) If P is a tagged partition of $[a, b]$ and P is sub δ , then P is sub δ on $[a, b]$.
- (vi) If $f : [a, b] \rightarrow E$ then $f(P) := \sum_{i=1}^n f(s_i)(d_i - c_i)$.
- (vii) If F is a function defined on the subintervals of $[a, b]$, then

$$F(P) = \sum_{i=1}^n F([c_i, d_i]) = \sum_{i=1}^n (F(d_i) - F(c_i)).$$

DEFINITION 2.2 ([5]). The function $f : [a, b] \rightarrow E$ is *Henstock–Kurzweil integrable* on $[a, b]$ ($f \in \text{HK}([a, b], E)$) if there exists a vector z in E with the following property: for each $\varepsilon > 0$ there exists a positive function δ on $[a, b]$ such that $\|f(P) - z\| < \varepsilon$ whenever P is sub δ on $[a, b]$. The function f is *Henstock–Kurzweil integrable on $A \subset [a, b]$* if $f\chi_A$ is Henstock–Kurzweil integrable on $[a, b]$. The vector z is the *Henstock–Kurzweil integral* of f .

We remark that this definition includes the generalized Riemann integral defined by Gordon in [8].

DEFINITION 2.3 ([5]). Let f be Henstock–Kurzweil integrable on $[a, b]$. Then the function $F(t) = \int_a^t f$, which is defined on subintervals of $[a, b]$, is called the *primitive* of f .

DEFINITION 2.4 ([5]). A function $f : [a, b] \rightarrow E$ is *HL integrable* on $[a, b]$ ($f \in \text{HL}([a, b], E)$) if there exists a function $F : [a, b] \rightarrow E$ with the following property: given $\varepsilon > 0$ there exists a positive function $\delta(\cdot)$ on $[a, b]$ such that if $P = \{(s_i, [c_i, d_i]) : 1 \leq i \leq n\}$ is a tagged partition of $[a, b]$ sub δ , then

$$\sum_{i=1}^n \|f(s_i)(d_i - c_i) - F([c_i, d_i])\| < \varepsilon.$$

We note that by the triangle inequality $f \in \text{HL}([a, b], E)$ implies $f \in \text{HK}([a, b], E)$. In general, the converse is not true. For real-valued functions, the two integrals are equivalent.

We now recall Henstock's Lemma for real-valued Henstock–Kurzweil integrable functions (for the proof, see [14, Theorem 3.7]).

THEOREM 2.5 (Henstock's Lemma). *If f is Henstock–Kurzweil integrable on $[a, b]$ with primitive F , then for every $\varepsilon > 0$ there exists $\delta(\cdot) > 0$ such that for any tagged partition $P = \{(s_i, [c_i, d_i]) : 1 \leq i \leq n\}$ of $[a, b]$ sub δ , we have*

$$\sum_{i=1}^n |f(s_i)(d_i - c_i) - F([c_i, d_i])| < \varepsilon.$$

Theorem 2.5 says that in the definition of the Henstock–Kurzweil integral for real-valued functions ([8]), we may put the absolute value sign $|\cdot|$ inside the summation sign \sum . We know ([5]) that this is no longer true if we replace $|\cdot|$ with $\|\cdot\|$, i.e. Henstock's Lemma is not satisfied by Henstock–Kurzweil integrable Banach-valued functions. By the definition of HL integral, an HL integrable function with primitive F satisfies Henstock's Lemma with $|\cdot|$ replaced $\|\cdot\|$.

DEFINITION 2.6 ([8]). A family \mathcal{F} of functions F is said to be *uniformly absolutely continuous in the restricted sense on A* , for short uniformly $AC_*(A)$, if for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every F in \mathcal{F} and for every finite or infinite sequence of nonoverlapping intervals $\{[a_i, b_i]\}$ with $a_i, b_i \in A$ and satisfying $\sum_i |b_i - a_i| < \eta$, we have $\sum_i \omega(F, [a_i, b_i]) < \varepsilon$, where $\omega(F, [a_i, b_i])$ denotes the oscillation of F over $[a_i, b_i]$.

A family \mathcal{F} of functions F is said to be *uniformly generalized absolutely continuous in the restricted sense on $[a, b]$* , or uniformly ACG_* , if $[a, b]$ is the union of a sequence of closed sets A_i such that on each A_i the function F is uniformly $AC_*(A_i)$.

For the Henstock–Kurzweil integral, in particular for the HL integral, we have the following theorems.

THEOREM 2.7 ([5]). *Let $f : [a, b] \rightarrow E$. If $f = 0$ almost everywhere on $[a, b]$, then f is HL integrable on $[a, b]$ and $\int_a^b f(t) dt = 0$.*

THEOREM 2.8 ([5]). *Let $f : [a, b] \rightarrow E$ be HL integrable on $[a, b]$ and let $F(x) = \int_a^x f(t) dt$ for each $x \in [a, b]$. Then*

- (i) F is continuous on $[a, b]$,
- (ii) F is differentiable almost everywhere on $[a, b]$ and $F' = f$,
- (iii) f is measurable.

THEOREM 2.9 ([20, Theorem 5]). *Suppose that $f_n : [a, b] \rightarrow E$, $n = 1, 2, \dots$, is a sequence of HL integrable functions satisfying the following conditions:*

- (i) $f_n(x) \rightarrow f(x)$ almost everywhere in $[a, b]$ as $n \rightarrow \infty$;
- (ii) the set of primitives of $f_n, \{F_n(x)\}$, where $F_n(x) = \int_a^x f_n(s) ds$, is uniformly ACG_* in n ;
- (iii) the primitives F_n are equicontinuous on $[a, b]$.

Then f is HL integrable on $[a, b]$ and $\int_a^x f_n \rightarrow \int_a^x f$ uniformly on $[a, b]$ as $n \rightarrow \infty$.

We remark that this theorem for Denjoy–Bochner integrals is mentioned in [20] without proof. It is also true for HL integrals. The proof is similar to that of Theorem 7.6 in [14] (see also [19, Theorem 1.8]).

LEMMA 2.10 ([17]). *Let E_1 be a separable Banach space. Suppose that V is a countable set of HL integrable functions. Let $F = \{\int_0^t x(s) ds : x \in V, t \in I_a\}$ be equicontinuous, equibounded and uniformly ACG_* on I_a . Then $\alpha_1(\int_0^t V(s) ds) \leq \int_0^t \alpha_1(V(s)) ds, t \in I_a$, whenever $\alpha_1(V(s)) \leq \varphi(s)$ for a.e. $s \in I_a$, φ is a Lebesgue integrable function and α_1 denotes the Hausdorff measure of noncompactness.*

THEOREM 2.11. *If the function $f : I_a \rightarrow E$ is HL integrable, then*

$$\int_I f(t) dt \in |I| \overline{\text{conv}} f(I),$$

where I is an arbitrary subinterval of I_a and $|I|$ is the length of I .

The proof is similar to that of Lemma 2.1.3 in [15]. See also [1, Theorem 10.4, p. 268].

3. Main result

I. *The Urysohn integral equation.* Now we prove an existence theorem for the problem (1.1). We will apply the following fixed point theorem.

THEOREM 3.1 ([16]). *Let D be a closed convex subset of E , and let F be a continuous map from D into itself. If for some $x \in D$ the implication*

$$\bar{V} = \overline{\text{conv}}(\{x\} \cup F(V)) \Rightarrow V \text{ is relatively compact}$$

holds for every countable subset V of D , then F has a fixed point.

We recall that a function $f : I_a \times I_a \times E \rightarrow E$ is a *Carathéodory function* if for each $x \in E, f(t, s, x)$ is measurable in $(t, s) \in I_a \times I_a$ and for almost all $(t, s) \in I_a \times I_a, f(t, s, x)$ is continuous with respect to x .

A nonnegative real-valued function $(t, s, z) \mapsto h(t, s, z)$ defined on $I_a \times I_a \times E$ is a *Kamke function* if h satisfies the Carathéodory conditions, for each fixed t, s the function $z \mapsto h(t, z, s)$ is nondecreasing, and for each $q, 0 < q \leq a$, the identically zero function is the unique continuous solution of the integral equation $z(t) = \int_0^t h(t, s, z(s)) ds$ defined on $[0, q]$.

Let $B = \{x \in C(I_a, E) : \|x\| \leq \|\varphi(\cdot)\| + \lambda b\}$, where b, λ are some positive numbers.

We define the operator F_x by

$$F_x(t) = \varphi(t) + \lambda \int_0^a f(t, s, x(s)) \, ds \quad \text{for } t \in I_a \text{ and } x \in B,$$

where the integral is the HL integral.

Moreover, let $F = \{F_x : x \in B\}$.

THEOREM 3.2. *Assume that for each continuous function $x : I_a \rightarrow E_1$, $f(\cdot, s, x(s))$ is HL integrable, f is a Carathéodory function and*

$$(3.1) \quad \alpha(f(t, s, X)) \leq h(t, s, \alpha(X)) \quad \text{for each } X \subset B, 0 \leq s \leq t \leq a,$$

where h is a Kamke function. Suppose that F is equicontinuous, equibounded and uniformly ACG_* on I_a . Then there exists a solution of the problem (1.1) on I_c for some $0 < c \leq a$ with continuous initial function φ .

Proof. By equicontinuity and equiboundedness of F there exists some number c ($0 < c \leq a$) such that $\|\int_0^c f(t, s, x(s)) \, ds\| \leq b$ for fixed $b > 0$, $t \in I_c$ and $x \in B$.

By our assumptions, the operator F_x is well defined and maps B into B and by Theorem 2.8, F_x is continuous. Observe that a fixed point of F_x is a solution of the problem (1.1). Now we prove that F_x has a fixed point using Theorem 3.1.

Suppose that $V \subset B$ is a countable set and $\bar{V} = \overline{\text{conv}}(\{x\} \cup F_x(V))$ is equicontinuous. Then $t \mapsto v(t) = \alpha(V(t))$ is continuous on I_c (by Lemma 1.1).

Let us fix $t \in I_c$. Let $\int_0^c Z(s) \, ds = \{\int_0^c x(s) \, ds : x \in Z\}$ for any $Z \subset C(I_c, E)$ and let \tilde{f}_t denote the mapping defined by $\tilde{f}_t(x(s)) = f(t, s, x(s))$ for each $x \in B$ and $s \in I_c$. Obviously, $\tilde{f}_t(V(s)) = f(t, s, V(s))$ and $F_x(V(t)) = \varphi(t) + \lambda \int_0^c \tilde{f}_t(V(s)) \, ds$.

Using (3.1), Lemma 2.10 and the properties of the measure of noncompactness α we have

$$\begin{aligned} \alpha(F_x(V(t))) &= \alpha\left(\varphi(t) + \lambda \int_0^c \tilde{f}_t(V(s)) \, ds\right) \leq 2\alpha_1\left(\lambda \int_0^c \tilde{f}_t(V(s)) \, ds\right) \\ &\leq 2\lambda \int_0^c \alpha_1(f(t, s, V(s))) \, ds \leq 2\lambda \int_0^c \alpha(f(t, s, V(s))) \, ds \\ &\leq 2\lambda \int_0^c h(t, s, \alpha(V(s))) \, ds. \end{aligned}$$

Because $V = \overline{\text{conv}}(\{x\} \cup F_x(V))$ we have

$$v(t) \leq 2\lambda \int_0^c h(t, s, v(s)) ds.$$

Hence applying now a theorem on differential inequalities we get $v(t) = \alpha(V(t)) = 0$. By Arzelà–Ascoli’s theorem, V is relatively compact. So, by Theorem 3.1, F_x has a fixed point which is a solution of the problem (1.1).

Now we present an existence theorem for the problem (1.1) in a real Banach space E . Let $r(K)$ be the spectral radius of the integral operator K defined by $Ku(t) = \int_0^a k(t, s)u(s) ds, u \in B, t \in I_a$.

THEOREM 3.3. *Assume that for each continuous function $x : I_a \rightarrow E, f(\cdot, s, x(s))$ is HL integrable, f is a Carathéodory function and there exists a function $k : I_a \times I_a \rightarrow \mathbb{R}_+$ such that $k(t, \cdot)$ is continuous and*

$$(3.2) \quad \alpha(f(t, J, X)) \leq \sup_{s \in J} k(t, s)\alpha(X) \quad \text{for each } X \subset E \text{ and } J \subset I_a.$$

Suppose that F is equicontinuous, equibounded and uniformly ACG_ on I_a . Moreover, let $\lambda r(K) < 1$. Then there exists a solution of the problem (1.1) on I_c for some $0 < c \leq a$ with continuous initial function φ .*

Proof. By equicontinuity and equiboundedness of F there exists some number c ($0 < c \leq a$) such that $\|\int_0^c f(t, s, x(s)) ds\| \leq b$ for fixed $b > 0, t \in I_c$ and $x \in B$.

By our assumptions, the operator F_x is well defined and maps B into B , and by Theorem 2.8, F_x is continuous. Observe that a fixed point of F_x is a solution of the problem (1.1). Now we prove that F_x has a fixed point using Theorem 3.1. Suppose that $V \subset B$ is a countable set and $\bar{V} = \overline{\text{conv}}(\{x\} \cup F_x(V))$ is equicontinuous. Then $t \mapsto v(t) = \alpha(V(t))$ is continuous on I_c (by Lemma 1.1).

We divide the interval $I_c: 0 = t_0 < t_1 < \dots < t_m = c$, where $t_i = ic/m, i = 0, 1, \dots, m$. Let $V([t_i, t_{i+1}]) = \{u(s) : u \in V, t_i \leq s \leq t_{i+1}, i = 0, 1, \dots, m - 1\}$. By Lemma 1.1 and the continuity of v there exists $s_i \in T_i = [t_i, t_{i+1}]$ such that

$$(3.3) \quad \alpha(V([t_i, t_{i+1}])) = \sup\{\alpha(V(s)) : t_i \leq s \leq t_{i+1}\} =: v(s_i).$$

On the other hand, by the definition of the operator F_x and Theorem 2.11 we have

$$\begin{aligned} F_u(t) &= \varphi(t) + \lambda \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} f(t, s, u(s)) ds \\ &\in \varphi(t) + \lambda \sum_{i=0}^{m-1} (t_{i+1} - t_i) \overline{\text{conv}} f(t, T_i, V([t_i, t_{i+1}])) \end{aligned}$$

for each $u \in V$. Therefore

$$F_u(V(t)) \subset \varphi(t) + \lambda \sum_{i=0}^{m-1} (t_{i+1} - t_i) \overline{\text{conv}} f(t, T_i, V([t_i, t_{i+1}])).$$

Using (3.2), (3.3), and the properties of the measure of noncompactness α we obtain

$$\begin{aligned} \alpha(F_u(V(t))) &\leq \lambda \sum_{i=0}^{m-1} (t_{i+1} - t_i) \sup_{s \in T_i} k(t, s) \alpha(V[t_i, t_{i+1}]) \\ &= \lambda \sum_{i=0}^{m-1} (t_{i+1} - t_i) k(t, p_i) v(s_i), \end{aligned}$$

where $s_i, p_i \in T_i$, hence

$$\begin{aligned} \alpha(F_u(V(t))) &\leq \lambda \sum_{i=0}^{m-1} (t_{i+1} - t_i) k(t, p_i) v(p_i) \\ &\quad + \lambda \sum_{i=0}^{m-1} (t_{i+1} - t_i) [k(t, p_i)(v(s_i) - v(p_i))] \\ &= \lambda \sum_{i=0}^{m-1} (t_{i+1} - t_i) k(t, p_i) v(p_i) \\ &\quad + \lambda \frac{c}{m} \sum_{i=0}^{m-1} k(t, p_i) (v(s_i) - v(p_i)). \end{aligned}$$

By continuity of v we have $v(s_i) - v(p_i) < \varepsilon$ and $\varepsilon \rightarrow 0$ as $m \rightarrow \infty$. So

$$\alpha(F_u(V(t))) < \lambda \int_0^c k(t, s) v(s) ds + \lambda c \sup_{p \in I_c} k(t, p) \varepsilon.$$

Therefore

$$(3.4) \quad \alpha(F_u(V(t))) \leq \lambda \int_0^c k(t, s) v(s) ds \quad \text{for } t \in I_c.$$

Since $V = \overline{\text{conv}}(\{u\} \cup F_u(V))$, by the property of the measure of noncompactness we have $\alpha(V(t)) \leq \alpha(F_u(V(t)))$ and so in view of (3.4) it follows that $v(t) \leq \lambda \int_0^c k(t, s) v(s) ds$ for $t \in I_c$. Because this inequality holds for every $t \in I_c$ and $\lambda r(K) < 1$, by applying a theorem on integral inequalities, we conclude that $\alpha(V(t)) = 0$ for $t \in I_c$. Hence Arzelà–Ascoli’s theorem proves that the set V is relatively compact. Consequently, by Theorem 3.1, F_x has a fixed point which is a solution of the problem (1.1).

II. *The Volterra integral equation.* Now we consider the integral equation (1.2). Put $B_1 = \{x \in C(I_a, E) : \|x\| \leq \|\varphi(\cdot)\| + l\}$, where l is some

positive number. We define the operator G_x by

$$G_x(t) = \varphi(t) + \int_0^t f(t, s, x(s)) ds \quad \text{for } t \in I_a \text{ and } x \in B_1,$$

where the integral is the HL integral.

Moreover, let $G = \{G_x : x \in B_1\}$.

THEOREM 3.4. *Assume that for each continuous function $x : I_a \rightarrow E_1$, $f(\cdot, s, x(s))$ is HL integrable, f is a Carathéodory function and*

$$(3.5) \quad \alpha(f(t, s, X)) \leq h(t, s, \alpha(X)) \quad \text{for each } X \subset B, 0 \leq s \leq t \leq a,$$

where h is a Kamke function. Suppose that G is equicontinuous, equibounded and uniformly ACG_* on I_a . Then there exists a solution of the problem (1.2) on I_c for some $0 < c \leq a$ with continuous initial function φ .

Proof. By equicontinuity and equiboundedness of G there exists some number c ($0 < c \leq a$) such that $\|\int_0^t f(t, s, x(s)) ds\| \leq l$ for $t \in I_c$ and $x \in B_1$.

Let $V = \overline{\text{conv}}(\{x\} \cup G_x(V))$ for some bounded countable set $V \subset B_1$.

Analogously to the proof of Theorem 3.2 we prove that $v(t) \leq 2 \int_0^t f(t, s, v(s)) ds$ for $t \in I_c$. Hence applying now a theorem on differential inequalities we get $v(t) = \alpha(V(t)) = 0$. By Arzelà–Ascoli’s theorem, V is relatively compact. So, by Theorem 3.1, G_x has a fixed point which is a solution of the problem (1.1).

Now, let $r(K)$ be the spectral radius of the integral operator K defined by $Ku(t) = \int_0^t k(t, s)u(s) ds$, $u \in B_1$, $t \in I_a$.

For a real Banach space E we have the following theorem.

THEOREM 3.5. *Assume that for each continuous function $x : I_a \rightarrow E$, $f(\cdot, s, x(s))$ is HL integrable, f is a Carathéodory function and there exists a function $k : I_a \times I_a \rightarrow \mathbb{R}_+$ such that $k(t, \cdot)$ is continuous and*

$$(3.6) \quad \alpha(f(t, J, X)) \leq \sup_{s \in J} k(t, s)\alpha(X) \quad \text{for each } X \subset E \text{ and } J \subset I_a.$$

Suppose that G is equicontinuous, equibounded and uniformly ACG_* on I_a . Moreover, let $r(K) < 1$. Then there exists a solution of the problem (1.2) on I_c for some $0 < c \leq a$ with continuous initial function φ .

Proof. By equicontinuity and equiboundedness of G there exists some number c ($0 < c \leq a$) such that $\|\int_0^t f(t, s, x(s)) ds\| \leq l$ for $t \in I_c$ and $x \in B_1$.

Let $V = \overline{\text{conv}}(\{x\} \cup G_x(V))$ for some bounded countable set $V \subset B_1$.

Analogously to the proof of Theorem 3.3 we divide the interval $[0, t]$ (for fixed $t \in I_c$) into m parts $0 = t_0 < t_1 < \dots < t_m = t$, where $t_i = it/m$,

$i = 0, 1, \dots, m$, and we prove that $v(t) \leq \int_0^t k(t, s)v(s) ds$ for $t \in I_c$. Because this inequality holds for every $t \in I_c$ and $r(K) < 1$, by applying a theorem on integral inequalities, we conclude that $\alpha(V(t)) = 0$ for $t \in I_c$. Hence Arzelà–Ascoli’s theorem proves that the set V is relatively compact. Consequently, by Theorem 3.1, G_x has a fixed point which is a solution of the problem (1.2).

REMARK. Let S be the set of solutions of the problems (1.1) and (1.2) and $S_n \subset S$ be a sequence of solutions. As $S_n = F(S_n)$, by repeating the calculation in the proof of Theorems 3.2 (or Theorem 3.3) and 3.4 (or Theorem 3.5), where $V = S_n$, we can show that S is compact.

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