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Stability of solutions for an abstract Dirichlet problem

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Abstract. We consider continuous dependence of solutions on the right hand side for a semilinear operator equation $Lx = \nabla G(x)$, where $L : D(L) \subset Y \to Y$ (Y a Hilbert space) is self-adjoint and positive definite and $G : Y \to Y$ is a convex functional with superquadratic growth. As applications we derive some stability results and dependence on a functional parameter for a fourth order Dirichlet problem. Applications to P.D.E. are also given.

1. Introduction. We shall prove the stability of solutions to the following family of abstract semilinear problems:

$$Lx = \nabla G_k(x), \tag{1.1}$$

for $k = 0, 1, 2, \ldots$, where L is a self-adjoint mapping defined on a separable real Hilbert space and $D(L)$ with values in a separable real Hilbert space Y, and $G_k : Y \to Y$ is a superquadratic convex mapping for $k = 0, 1, \ldots$. We provide conditions under which from a sequence $\{x_k\}_{k=1}^{\infty}$ solving (1.1) one may choose a subsequence converging weakly to an $\overline{x}$ which is a solution to the problem

$$L\overline{x} = \nabla G_0(\overline{x}).$$

This property is called *stability* of system (1.1).

In [10] a stability result based on a dual variational method from [8], [9] and some ideas from [11], [12] is given. In case G has quadratic growth a problem similar to ours has been considered in [4], for L not necessarily positive definite. However, for a superquadratic nonlinearity the method from [4] does not work since in this case both the action and the dual action functionals are unbounded. We believe that the variational method from [2] may contribute to this research when combined with some stability results from [4].

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Now we specify the assumptions under which we shall consider system (1.1).

(A1) $D(L)$ is dense in Y; $L : D(L) \to Y$ is a selfadjoint and positive definite linear operator, i.e. there exists a constant $\alpha > 0$ such that for all $x \in D(L)$,

$$\langle Lx, x\rangle \geq \alpha \|x\|^2. \tag{1.2}$$

Let $L^{1/2} : D(L^{1/2}) \to Y$ denote the square root operator (see [5]). Then $D(L^{1/2})$ is a Hilbert space, and $L^{1/2}x \in D(L^{1/2})$ for any $x \in D(L)$.

(A2) For any $k = 0, 1, \ldots$, $\nabla G_k : Y \to Y$ is a gradient mapping, its potential $G_k : Y \to \mathbb{R}$ is lower semicontinuous and convex; for all sequences $\{x_n\} \subset D(L)$ strongly convergent in Y the sequence $\{\nabla G_k(x_n)\}$ is weakly convergent; there exists $c \geq 0$ such that $|G_k(0)| \leq c$ for all k; $\nabla G_k(0) \neq 0$ for all k; $\nabla G_k(x) \in D(L^{1/2})$ for any $x \in D(L)$; there exist constants $q \geq q_1 \geq 2$, $a_k, a'_k > 0$, $b_k, b'_k \geq 0$ such that for any $x \in Y$,

$$\|\nabla G_k(x)\| \leq a_k \|x\|^{q-1} + b_k, \qquad G_k(x) \geq a'_k \|x\|^{q_1} + b'_k, \tag{1.3}$$

where the sequences $\{a_k\}$, $\{a'_k\}$, $\{b_k\}$, $\{b'_k\}$ are bounded by a, a', b, b' respectively.

(A3) $D(L^{1/2})$ is compactly imbedded in Y; for any sequence $\{x_n\} \subset D(L)$,

$$\lim_{n\to\infty} \|x_n - \overline{x}\|_{D(L^{1/2})} = 0 \quad \text{iff} \quad \lim_{n\to\infty} \|L^{1/2}x_n - L^{1/2}\overline{x}\|_Y = 0.$$

Equation (1.1) is the Euler–Lagrange equation for the action functional $J_k : D(L^{1/2}) \to \mathbb{R}$ given by

$$J_k(x) = \tfrac{1}{2}\langle L^{1/2}x, L^{1/2}x\rangle - G_k(x).$$

Now we cite conditions ensuring the solvability of system (1.1). Let $m_k > 0$ satisfy

$$\|L^{-1}\|(a_k m_k^{q-1} + b_k) \leq m_k. \tag{1.4}$$

(A4) The sequence m_k does not converge to 0.

For each k we put

$$\widetilde{X}_k = \{v \in D(L) : \|v\|_Y \leq m_k\}.$$

We shall further restrict the sets $\widetilde{X}_k$ in order to apply the duality results from [2]. Let X^k be a subset of $\widetilde{X}_k$ such that for each $x \in X^k$,

$$L\widetilde{x} = \nabla G_k(x) \text{ implies } \widetilde{x} \in X^k.$$

Putting $X^k = \widetilde{X}_k$ we observe that for each k there exists a nonempty set X^k.

The following corollary easily follows from the main theorem in [2].

Corollary 1.1. *Assume* (A1)–(A3). *Then for each k there exists an $x_k \in X^k$ such that*

$$Lx_k = \nabla G_k(x_k).$$

2. Stability result. This section contains the main result of the paper which provides conditions under which the family of problems (1.1) is stable.

Theorem 2.1. *Assume* (A1)–(A3) *and that for any $x \in Y$ there is a subsequence k_i such that*

$$\lim_{i\to\infty} \nabla G_{k_i}(x) = \nabla G_0(x)$$

weakly in Y. Then there exists a sequence $\{x_k\}_{k=1}^{\infty}$ of solutions to (1.1) *and $\overline{x} \in D(L)$ such that*

$$\lim_{k\to\infty} x_k = \overline{x} \qquad \text{weakly in } D(L^{1/2}), \text{strongly in } Y.$$

Moreover

$$L\overline{x} = \nabla G_0(\overline{x}).$$

Proof. From Corollary 1.1 it follows that for each k there exists $x_k \in D(L)$ such that $Lx_k = \nabla G_k(x_k)$. For $b > 0$ we define

$$S_b = \{x \in \widetilde{X}_k : J_k(x) \le b,\ k = 1, 2, \ldots\}.$$

Due to (A2) this set is nonempty for sufficiently large b. Indeed, for any $\widetilde{x} \in \widetilde{X}_k$ and corresponding $x \in \widetilde{X}_k$, and for some constant γ,

$$\gamma\|L^{1/2}\widetilde{x}\| \le \gamma\|\nabla G_k(x)\|\,\|\widetilde{x}\| \le \gamma(a_k\|x\|^{q-1} + b_k)\|\widetilde{x}\| \le \gamma(am_k^{q-1} + b_k)m_k.$$

This estimate combined with the growth conditions on G_k leads to the conclusion. By construction it follows that S_b is weakly compact in $D(L^{1/2})$.

We show that G_k and ∇G_k are uniformly bounded on S_b. Indeed, from (1.3) and (A2) it follows that

$$\begin{aligned} G_k(x) &\le \langle \nabla G_k(x), x\rangle + G_k(0) \le \|\nabla G_k(x)\|\,\|x\| + G_k(0) \\ &\le a_k\|x\|^q + b_k\|x\| + G_k(0) \le a\|x\|^q + b\|x\| + C. \end{aligned}$$

By the above and by (A3) there exists a constant γ such that $G_k(x) \le \gamma$ for any $x \in S_b$. A similar reasoning applies to ∇G_k.

We choose from $\{x_k\}_{k=1}^{\infty}$ a subsequence weakly converging in $D(L^{1/2})$ which we denote again by $\{x_k\}_{k=1}^{\infty}$. This sequence now converges strongly in Y to $\overline{x}$ by (A3). Let $\{k_i\}$ be a subsequence such that $\lim_{i\to\infty} \nabla G_{k_i}(\overline{x}) = \nabla G_0(\overline{x})$ weakly. We denote all the resulting subsequences by the subscript k for simplicity.

We will now prove that

$$L\overline{x} = \nabla G_0(\overline{x}).$$

By convexity of G_k we get for any $x \in Y$,

$$\langle \nabla G_k(x_k) - \nabla G_k(x), x_k - x \rangle \geq 0.$$

Hence

$$\langle Lx_k + (\nabla G_0(x) - \nabla G_k(x)) - \nabla G_0(x), x_k - x \rangle \geq 0.$$

Since $x_k \to \overline{x}$ strongly in Y and $\nabla G_k(x) \rightharpoonup \nabla G_0(x)$ weakly we easily get

$$\langle (\nabla G_0(x) - \nabla G_k(x)) - \nabla G_0(x), x_k - x \rangle \to 0.$$

Moreover $\langle Lx_k, -x \rangle \to \langle L\overline{x}, -x \rangle$ since L is selfadjoint. It remains to prove that

$$\langle Lx_k, x_k \rangle \to \langle L\overline{x}, \overline{x} \rangle.$$

Let F be the spectral measure defined by L. We put

$$P^+ = \int_{\alpha/2}^{\infty} 1 \, dF(\lambda).$$

Then P^+ commutes with L. Using the inclusion $P^+ D(L) \subset D(L)$ and since the operator $P^+ L^{-1/2}$ is compact we get

$$\begin{aligned} \langle Lx_k, x_k \rangle &= \langle LP^+ x_k, P^+ x_k \rangle = \langle P^+ Lx_k, P^+ x_k \rangle \\ &= \langle P^+ L^{-1/2} p_k, P^+ L^{-1/2} p_k \rangle \to \langle P^+ L^{-1/2} \overline{p}, P^+ L^{-1/2} \overline{p} \rangle = \langle L\overline{x}, \overline{x} \rangle. \end{aligned}$$

Hence

$$\langle L\overline{x} - \nabla G_0(x), \overline{x} - x \rangle \geq 0 \tag{2.1}$$

for any $x \in D(L)$.

Now we apply Minty's trick, i.e. we consider the points $\overline{x} + tx$, where $x \in D(L)$ and $t > 0$. By the above inequality we obtain

$$\langle L\overline{x} - \nabla G_0(\overline{x} + tx), x \rangle \leq 0.$$

Since the function

$$[-1, 1] \ni t \mapsto G_0(\overline{x} + tx) \in \mathbb{R}$$

is convex it follows that its derivative

$$[-1, 1] \ni t \mapsto \langle \nabla G_0(\overline{x} + tx), x \rangle \in \mathbb{R}$$

is continuous at any point $t \in [-1, 1]$. Hence

$$0 \geq \lim_{t \to 0} \langle L\overline{x} - \nabla G_0(\overline{x} + tx), x \rangle = \langle L\overline{x} - \nabla G_0(\overline{x}), x \rangle$$

for any $x \in D(L)$. Since $D(L)$ is dense in Y, this means that

$$\langle L\overline{x} - \nabla G_0(\overline{x}), x \rangle \leq 0.$$

By the above and (2.1) we get $L\overline{x} = \nabla G_0(\overline{x})$. ∎

Remark 1. Reasoning similarly to the last part of the proof of the main theorem of [4] we may prove that

$$J(\overline{x}) = \inf_{x \in X^0} J(x) = \inf_{p \in X^{d,0}} J_D(p) = J_D(\overline{p}),$$

where $J_D : D(L^{1/2}) \to \mathbb{R}$ is defined by $J_D(p) = G^*(L^{1/2}) - \frac{1}{2}\langle p, p\rangle$ (cf. [2]), G^* denotes the Fenchel–Young dual of the convex functional G (see [1]) and $X^{d,0} = L^{1/2}(X^0)$.

3. Applications. The above results are of use when considering superlinear Dirichlet problems. They also apply when G_k are sublinear although in that case the assumptions on L are too restrictive. In this section we give applications to some concrete problems. We assume throughout that (A4) holds and we shall check each time that (A1)–(A3) are satisfied.

3.1. *Stability for a fourth order Dirichlet problem.* As an example we consider the following Dirichlet problem, for $k \in \mathbb{N}$:

$$\begin{gathered} \frac{d^4}{dt^4}x - \frac{d^2}{dt^2}x + x = \nabla G_k(t, x), \\ x(0) = x(\pi) = \dot{x}(0) = \dot{x}(\pi) = 0, \end{gathered} \tag{3.1}$$

where

(A5) $\nabla G_k : [0,\pi]\times\mathbb{R} \to \mathbb{R}$, $G_k : [0,\pi]\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function, G_k is convex with respect to the second variable; $\nabla G_k(t, 0) \neq 0$ and there exists a constant c such that $|G_k(t, 0)| \le c$ for all k and for a.e. t; there exist constants $q \ge q_1 \ge 2$, $a_k, a'_k > 0$ and functions $b_k \in L^1([0,\pi],\mathbb{R}^+)$, $b'_k \in L^1([0,\pi],\mathbb{R})$ for $k = 0, 1, 2, \ldots$ such that for any $x \in \mathbb{R}$ and for a.e. $t \in [0,\pi]$,

$$\|\nabla G_k(t, x)\| \le a_k|x|^{q-1} + b_k(t), \qquad G_k(t, x) \ge a'_k|x|^{q_1} + b'_k(t),$$

where the sequences $\{a_k\}$, $\{a'_k\}$ and the functions b_k, b'_k for $k = 0, 1, 2, \ldots$ are bounded by a, a', b, b' respectively.

Hence $Lx = \frac{d^4}{dt^4}x - \frac{d^2}{dt^2}x + x$ for $x \in H^2_0(0,\pi) \cap H^4(0,\pi)$. The operator L satisfies assumption (A1) and due to the Poincaré inequality also (A3) holds. Here $Y = L^2(0,\pi)$. We observe that since $x \in H^2_0(0,\pi)\cap H^4(0,\pi)$ the growth conditions are valid without assuming that $x \in L^q(0,\pi)$. This follows since each x is actually absolutely continuous. But in contrast to the quadratic case it does not follow by the Krasnosel'skiĭ theorem [3] that $\nabla G_k(\cdot, x(\cdot))$ is demicontinuous as required by (A2). Thus an additional assumption has to be made.

The announced theorem reads

THEOREM 3.1. *Assume* (A5) *and that for each x there exists a subsequence k_i such that*

$$\nabla G_{k_i}(\cdot, x(\cdot)) \rightharpoonup \nabla G_0(\cdot, x(\cdot)) \quad \text{as } i \to \infty$$

weakly in Y, and moreover, for all k and for all sequences x_n strongly convergent in Y,

$$\nabla G_k(\cdot, x_n(\cdot)) \rightharpoonup \nabla G_k(\cdot, x(\cdot)) \quad \text{as } n \to \infty$$

weakly in Y. Then for each $k \in \mathbb{N}$ there exists a solution x_k to problem (3.1) *and there exists $\overline{x} \in D(L)$ such that*

$$\lim_{k\to\infty} x_k = \overline{x} \quad \text{strongly in } Y.$$

Moerover

$$\frac{d^4}{dt^4}\overline{x}(t) - \frac{d^2}{dt^2}\overline{x}(t) + \overline{x}(t) = \nabla G_0(t, \overline{x}(t)).$$

3.2. *Dependence on parameters.* We now consider a similar problem but we concentrate on continuous dependence on parameters. Consider the Dirichlet problem

$$\begin{aligned} &\frac{d^4}{dt^2}x(t) - \frac{d^2}{dt^2}x(t) + x(t) = \nabla G(t, x(t), u(t)),\\ &x(0) = x(\pi) = \dot{x}(0) = \dot{x}(\pi) = 0, \end{aligned} \tag{3.2}$$

where $u : [0, \pi] \to \mathbb{R}^m$ is a functional parameter from the set

$$L_M = \{u : [0, \pi] \to \mathbb{R}^m : u \text{ is measurable, } u(t) \in M \text{ a.e.}\}$$

and $M \subset \mathbb{R}^m$ is a given bounded set. Again $Lx = \frac{d^4}{dt^4}x - \frac{d^2}{dt^2}x + x$ for $x \in H_0^2(0, \pi) \cap H^4(0, \pi)$. As before we assume

(A6) $\nabla G : [0, \pi] \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, $G : [0, \pi] \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function, G is continuously differentiable and convex with respect to the second variable and there exist constants $q \geq q_1 \geq 2$, $a_1, a_1' > 0$ and functions $b_1 \in L^1([0, \pi], \mathbb{R}^+)$, $b_1' \in L^1([0, \pi], \mathbb{R})$ such that for any $x \in \mathbb{R}$, $u \in M$ and a.e. $t \in [0, \pi]$,

$$\|\nabla G(t, x, u)\| \leq a_1|x|^{q-1} + b_1(t), \qquad G(t, x, u) \geq a_1'|x|^{q_1} + b_1'(t). \tag{3.3}$$

We have the following direct consequence of Theorem 2.1.

THEOREM 3.2. *Assume that* (A6) *holds and that $\{u_k\}_{k=1}^\infty$, $u_k \in L_M$, is a sequence such that $u_k \to \overline{u}$ in Y. Moreover, assume that for all sequences x_n strongly convergent in Y,*

$$\nabla G(\cdot, x_n(\cdot), u_k(\cdot)) \rightharpoonup \nabla G(\cdot, x(\cdot), u_k(\cdot)) \quad \text{as } n \to \infty$$

weakly in Y for any $u_k \in L_M$. Then for each $k \in \mathbb{N}$ there exists a solution x_k to problem (3.2) *and $\overline{x} \in D(L)$ such that*

$$\lim_{k\to\infty} x_k = \overline{x} \quad \text{strongly in } Y.$$

Moreover
$$\frac{d^4}{dt^4}\overline{x}(t) - \frac{d^2}{dt^2}\overline{x}(t) + \overline{x}(t) = \nabla G(t, \overline{x}(t), \overline{u}(t)).$$

Proof. By (3.3) and by the generalization of the Krasnosel'skiĭ Theorem [3], it follows that for all $x \in Y$ we have

$$\nabla G(\cdot, x(\cdot), u_n(\cdot)) \to \nabla G(\cdot, x(\cdot), \overline{u}(\cdot)) \quad \text{as } n \to \infty$$

weakly in Y. Hence Theorem 2.1 applies with $G_k(\cdot, x(\cdot)) = G(\cdot, x(\cdot), u_k(\cdot))$. ∎

3.3. *Applications to P.D.E..* The method we have just developed also has applications to partial differential equations of second order and of higher orders. But the differential operator must be linear while the nonlinearity is superquadratic.

Namely we consider the following problem:

$$\begin{aligned} &\Delta x(y) = \nabla G_k(y, x(y)), \\ &x|_{\partial\Omega} = 0. \end{aligned} \tag{3.4}$$

(A7) Ω is a region in $\mathbb{R}^n$ having a regular boundary; $\nabla G_k : \Omega \times \mathbb{R} \to \mathbb{R}$, $G_k : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, G_k is convex with respect to the second variable; $\nabla G_k(y, 0) \neq 0$ and there exists a constant c such that $|G_k(y, 0)| \leq c$ for all k and for a.e. y; there exist constants $q \geq q_1 \geq 2$, $a_k, a'_k > 0$ and functions $b_k \in L^1(\Omega, \mathbb{R})$, $b'_k \in L^1(\Omega, \mathbb{R})$ for $k = 0, 1, 2, \ldots$ such that for any $x \in \mathbb{R}$ and a.e. $t \in \Omega$,

$$\|\nabla G_k(y, x)\| \leq a_k |x|^{q-1} + b_k, \qquad G_k(y, x) \geq a'_k |x|^{q_1} + b'_k,$$

where the sequences $\{a_k\}$, $\{a'_k\}$ and the functions b_k, b'_k are bounded by a, a', b, b' respectively.

Here $Y = L^2(\Omega, \mathbb{R})$. Again we must assume that $\nabla G_k(\cdot, x(\cdot))$ is demicontinuous. By Theorem 2.1 we obtain

THEOREM 3.3. *Assume* (A7) *and that for any $x \in Y$ there exists a subsequence k_i such that*

$$\nabla G_{k_i}(\cdot, x(\cdot)) \rightharpoonup \nabla G_0(\cdot, x(\cdot)) \quad \textit{as } i \to \infty$$

weakly in Y, and moreover, for all sequences x_n strongly convergent in Y,

$$\nabla G_k(\cdot, x_n(\cdot)) \rightharpoonup \nabla G_k(\cdot, x(\cdot)) \quad \textit{as } n \to \infty$$

weakly in Y. Then for each $k \in \mathbb{N}$ there exists a solution x_k to problem (3.4) *and $\overline{x} \in D(L)$ such that*

$$\lim_{k\to\infty} x_k = \overline{x} \quad \textit{strongly in } Y.$$

Moreover

$$\Delta\overline{x}(y) = \nabla G_0(y, \overline{x}(y)).$$

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