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Characteristic values of the Jacobian matrix and global invertibility

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Abstract. Characteristic matrix values (singular values, eigenvalues, and pivots arising from Gaussian elimination) for the Jacobian matrix and its inverse are considered for maps of real n-space to itself with a nowhere vanishing Jacobian determinant. Bounds on these are related to global invertibility of the map. Polynomial maps with a constant nonzero Jacobian determinant are a special case that allows for sharper characterizations.

1. Introduction. Consider a map $F : \mathbb{R}^n \to \mathbb{R}^n$ that is C^1 (that is, one that has continuous first order partial derivatives). Let $x_1, \ldots, x_n$ be a (linear) coordinate system for $\mathbb{R}^n$, and write $F = (F_1, \ldots, F_n)$, where each F_i is a function of $x_1, \ldots, x_n$. Denote by $J(F)(x)$ the Jacobian matrix $\partial F_i/\partial x_j$ at a point $x \in \mathbb{R}^n$. Assume that the Jacobian determinant, $\det(J(F)(x))$, vanishes nowhere. Then for every $x \in \mathbb{R}^n$, the matrix $J(F)(x)$ is a square nonsingular matrix. From the standard inverse function theorem, the map F is a local diffeomorphism (locally invertible with a C^1 inverse). The global invertibility of F is linked to bounds on the singular values of $J(F)(x)$ (a result that dates back to a paper of Hadamard early in the 20th century).

This has led to a conjecture that global invertibility might be a consequence of similar bounds on the eigenvalues of $J(F)(x)$. That (strong, but as yet unrefuted) conjecture, even restricted to polynomial maps, would imply the real case of the Jacobian conjecture, which is the conjecture that if F is polynomial and its Jacobian determinant is a nonzero constant, then F has a polynomial inverse.

A Samuelson map is one whose Jacobian matrix has nowhere vanishing leading principal minors. For such a map, Gaussian elimination (without pivoting—that is, in strict order of the coordinates) naturally introduces

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quantities called pivots, which are the only quantities whose reciprocals are required. The pivots can also be characterized as quotients of successive leading principal minors. For a C^1 Samuelson map, bounds on the pivots similar to those in the singular value and eigenvalue cases yield invertibility also. All three cases bear a striking resemblance, in that the characteristic values in each case have a product whose absolute value is the absolute value of the Jacobian determinant. For *polynomial* maps with a constant nonzero Jacobian determinant, the characteristic values are constant if they are bounded either above or below, and it is known that an injective polynomial local homeomorphism has a global (but perhaps not polynomial) inverse; these facts imply that only the case of bounded eigenvalues is of real interest for polynomial maps (and is an important open question).

REMARK. C^1 differentiability is assumed as a convenience, but the appropriate inverse function theorems etc. hold under somewhat less stringent differentiability assumptions. It is also clear that there are extensions to the case of maps of $\mathbb{C}^n$ to itself, but they do not seem to be worth the effort of formulating them separately.

2. Singular values. For the special case of a real $n \times n$ matrix A, the singular value decomposition yields the following: $A = U\Sigma V$, where U and V are orthogonal matrices, and Σ is a real diagonal matrix with diagonal entries $\sigma_i = \Sigma_{ii}$ that satisfy $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$. The σ_i depend only on the matrix A and are called the *singular values* of A (and written $\sigma_i(A)$ if A needs to be identified). They are the positive square roots of the eigenvalues of AA^{T} and of $A^{\mathrm{T}}A$, arranged in nonascending order. They are continuous functions of the entries of A. The product of the singular values is $\sigma_1 \ldots \sigma_n = |\det(A)|$, the absolute value of the determinant of A. The first (hence largest, though possibly tied for size) singular value, σ_1, is also known as the *spectral norm* of A. Let $\|x\|$ denote the Euclidean norm of a vector $x \in \mathbb{R}^n$. Let $|||A|||$ denote the associated operator norm of a matrix A; that is, the supremum of $\|Ax\|$ taken over all x satisfying $\|x\| = 1$. Then $\sigma_1(A) = |||A|||$. Denote by $\varrho(A)$ the spectral radius of A; that is, the supremum of the absolute values of the eigenvalues of A (which are in general complex numbers). Then $\varrho(A) \leq \sigma_1(A)$. Finally, if A is non-singular, then $\sigma_n(A) > 0$, and $\sigma_{n-i+1}(A^{-1}) = 1/\sigma_i(A)$. A general reference for matrix algebra issues, here and in later sections, is [15].

The term “Hadamard’s Theorem” is very ambiguous. First, there are quite a few significant theorems in different areas of mathematics that are often thus identified. In the context of global inverses, the original Hadamard Theorem is the one proved in [14]. Today, a reference to this theorem often refers to a purely topological variant, such as

THEOREM 2.1 (Variant of Hadamard's Theorem on Inverses). *If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a local homeomorphism, then it is a global homeomorphism if, and only if, it is a proper map.*

Here a *proper map* is one for which $F^{-1}(K)$ is compact for each compact $K \subset \mathbb{R}^n$. Generalizations of this theorem are standard fare in point set topology.

What Hadamard actually proved is that if a suitable norm of the inverse of the Jacobian matrix is bounded above, then the map is invertible.

A modern variant is Plastock's Theorem.

THEOREM 2.2 (Plastock). *If F is C^1 with a nowhere vanishing Jacobian determinant and with $\int_0^\infty u(r)\,dr = \infty$, where $u(r)$ is the infimum of $1/|||J(F)^{-1}(x)|||$ over all x with $\|x\| = r$, then F is globally invertible.*

Note that $J(F)^{-1}$ is the inverse of the Jacobian matrix $J(F)$, and not the Jacobian matrix of the inverse map. The original proof is in [20]; another proof can be found in [18, Chap. IV]. This clearly yields the original Hadamard Theorem. In fact, as remarked by Patrick Rabier in [21], Hadamard's assumed condition amounts to the largest singular value of the inverse of $J(F)(x)$ remaining bounded away from 0 as $x \to \infty$. Under that condition $\int_0^\infty u(r)\,dr$ must be ∞. Another condition equivalent to the original Hadamard assumption is that the singular values of $J(F)(x)$ are bounded away from 0. An independent proof of the fact that this implies invertibility can be found in [6].

Plastock's condition is sufficient, but not necessary. Rabier provides a version of Hadamard's Theorem with a necessary and sufficient condition in [22, Thm. 5.3]:

THEOREM 2.3 (Rabier). *If F is C^1 with nowhere vanishing Jacobian determinant and for each compact $K \subset \mathbb{R}^n$ there exists a constant C_K such that $|||J(F)^{-1}(x)||| < C_K$ for $x \in F^{-1}(K)$, then F is a diffeomorphism onto $\mathbb{R}^n$. The converse holds as well.*

Reformulating this in terms of $\sigma_n(J(F)(x)) = 1/\sigma_1(J(F)^{-1}(x))$ yields

THEOREM 2.4. *If F is C^1, then $\sigma_n(J(F)(x))$ is bounded away from 0 on each inverse image of a compact set if, and only if, F has a global inverse.*

REMARK. This short statement takes into account the fact that $\sigma_n \neq 0$ implies that the matrix is nonsingular. In particular, if the stated condition is satisfied, the Jacobian determinant of F must be nowhere vanishing. A special case, obviously, is the one in which $\sigma_n(J(F)(x))$ is globally bounded away from 0. Because of the difficulty of identifying the inverse images of compact sets in a general setting, that special case is the one which will see the most use. Furthermore, it allows one to deal only with the behavior of

F in a neighborhood of infinity; for if $\sigma_n(J(F)(x))$ is bounded away from 0 on the complement of a compact set K, then it is automatically globally bounded away from 0, because the continuity of $\sigma_n(J(F)(x))$ implies that it is bounded away from 0 on K. For instance, a map that is nonsingular and linear in a neighborhood of infinity has a global inverse.

To conclude, observe that $\delta = |\det(J(F))|$ is the product of the singular values. If σ_n is bounded away from 0 on some set, then so is δ. Conversely, if δ is bounded away from 0 on a set, and σ_1 (the spectral norm) is also bounded on that set, then σ_n is necessarily bounded away from 0 there. So

Corollary 2.5. *If F is C^1 and the spectral norm and determinant of $J(F)$ are, respectively, bounded and bounded away from* 0 *on each inverse image of a compact set, then F has a global inverse.*

Remark. Corollary 1(a) to Plastock's Theorem in Chap. IV of [18] is this result in the case where the bounds are global (a single bound for each of σ_1 and δ on all of $\mathbb{R}^n$). Part (b) deals with the similar case in which the determinant is bounded away from 0 and the condition number κ is bounded. (Recall that $\kappa(A) = \|A\| \cdot \|A^{-1}\|$ for a given matrix norm; here the choice is the spectral norm.)

3. Eigenvalues. Denote by λ_i the n eigenvalues of an $n \times n$ matrix (real or complex), arranged in (some) nonascending order of their absolute values. (The relative order of repeated eigenvalues, or of eigenvalues of the same absolute value, is not significant.) Write $\lambda_i(A)$ to indicate the matrix, if necessary. Then $|\lambda_1| \geq \ldots \geq |\lambda_n| \geq 0$ and $|\lambda_1| \ldots |\lambda_n| = |\det(A)|$. And if A is nonsingular then $|\lambda_n|(A) > 0$ and $|\lambda_{n-i+1}|(A^{-1}) = 1/|\lambda_i|(A)$.

There are numerous results in the literature relating spectral conditions (conditions on eigenvalues) for the Jacobian matrix of a map $F : \mathbb{R}^n \to \mathbb{R}^n$ to invertibility of F. One class of results concerns the Markus–Yamabe conjecture [17]: a fixed point and stability (all eigenvalues have strictly negative real parts at each point) imply global asymptotic stability for the solutions of the ordinary differential equation $dx/dt = F(x)$, and consequently the injectivity of F. This conjecture is now known to be true for $n = 2$ (cf. [11, 12, 13]), and false for $n \geq 3$ (cf. [2, 7]). For additional general results relating relating spectral conditions to injectivity and invertibility see [21, 24].

Conditions relating to bounds on the eigenvalues, similar to the conditions on singular values in the preceding section, appear to be rare in the literature. However, the following significant conjecture is made in [6]:

Conjecture 3.1 (Chamberland). *If F is C^1 and the eigenvalues of $J(F)(x)$ are globally bounded away from* 0 *(that is, $|\lambda_n(J(F)(x))| > C > 0$, for some C that does not depend on x), then F is injective.*

The Jacobian conjecture has a known reduction to the case of polynomial maps that have constant eigenvalues. This suggests that verifying the above conjecture for a given n might lead to a proof of the real Jacobian conjecture for that n, since polynomial maps of $\mathbb{R}^n$ to itself are bijective if they are injective [3, 9]. One delicate issue is that it is not known whether a bijective polynomial map of $\mathbb{R}^n$ to itself with a constant nonzero Jacobian determinant necessarily has a polynomial inverse. That is, the inverse certainly exists, but is it polynomial? This is stated as one of the conclusions of a theorem in [1], but it is widely recognized that there is a gap in the proof (specifically, an algebraically closed field may be needed in the proof of (c)$\Rightarrow$(b) in Thm. 2.1). A less delicate issue involves the fact that the reduction to the case of constant eigenvalues requires increasing the value of n. However, the validity of the conjecture for every n would definitely imply the Jacobian conjecture in every dimension for every field of characteristic zero. This is because the complex case is just a special case of the real case, with $\mathbb{C}^n$ considered as $\mathbb{R}^{2n}$, and bijective polynomial maps of $\mathbb{C}^n$ to itself are indeed known to have polynomial inverses (this is what is known as the birational case of the Jacobian conjecture, first proved in [16], but see also [1, 23]). Furthermore, it is known that the complex case implies the general case (any field of characteristic zero) [1].

As a first class of special cases of the Chamberland conjecture, one can take those C^1 maps with a *normal* Jacobian matrix, that is, maps for which $J(F)(x)$ commutes with its transpose. The reason this is significant is that for a normal matrix $\sigma_i = |\lambda_i|$ [15, p. 417], and the results of the previous section can be applied. Symmetric, skew symmetric, orthogonal, and various other special types of matrices are normal, so the class in question contains at least some interesting examples; among them, all gradient maps for C^2 functions. The normality condition need only hold in a neighborhood of infinity in order for the Chamberland condition on eigenvalues to imply that σ_n is globally bounded away from 0. We record this observation as a theorem:

THEOREM 3.2. *If $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and its Jacobian matrix has eigenvalues globally bounded away from* 0 *and there is a compact subset of $\mathbb{R}^n$ outside of which the Jacobian matrix is normal*, *then F has a global inverse.*

REMARK. Note that the conclusion in the above theorem is actually stronger than the injectivity called for in the Chamberland conjecture.

4. Pivots.

Let A be an $n \times n$ matrix (real or complex). Let $\mu_1, \ldots, \mu_n$ denote the leading principal minors of A (writing $\mu_i(A)$ if A needs to be identified). That is, μ_i is the determinant of the $i \times i$ submatrix of A formed from entries in the first i rows and first i columns. Then A has a decomposition $A = LDU$, in which the $n \times n$ matrices L, D, and U are, respectively, a unit lower triangular matrix, a diagonal matrix with nonzero diagonal elements, and a unit upper triangular matrix, if, and only if, each μ_i is nonzero, in which case the decomposition is unique [15, Thm. 3.5.2]. The qualifier "unit" for the triangular matrices means that the diagonal elements of those matrices are all 1.

If the leading principal minors are all nonzero, then the entries on the diagonal of the (diagonal) matrix D are $\mu_1, \mu_2/\mu_1, \mu_3/\mu_2, \ldots, \mu_n/\mu_{n-1}$. These ratios are called *pivots*, because they are the quantities whose reciprocals are required in performing elementary row and column operations to reduce the matrix to the identity matrix. Suppose that A is such a matrix; since $\mu_n = \det(A)$ it follows that A is nonsingular. Denote the pivots, arranged in nonascending order of absolute value, by $\pi_1, \ldots, \pi_n$. By construction $|\pi_1| \geq \ldots \geq |\pi_n| > 0$ and $|\pi_1| \ldots |\pi_n| = |\det(A)|$. Since $A = LDU$, it follows that $A^{-1} = U^{-1}D^{-1}L^{-1}$. If we consider the coordinate system in reverse order, $x_n, x_{n-1}, \ldots, x_1$, then U^{-1} is lower triangular and L^{-1} is upper triangular. If P is the permutation matrix that reverses the order of coordinates, then the previous is really just the identity $PA^{-1}P = PU^{-1}P(PD^{-1}P)PL^{-1}P$ (note that P^2 is the identity matrix, so that $P^{-1} = P$). Since the diagonal elements of D^{-1} are the reciprocals of those of D, and the ordering of the pivots is by nonascending absolute value, it follows that $|\pi_{n-i+1}|(PA^{-1}P) = 1/|\pi_i|(A)$.

A map $F : \mathbb{R}^n \to \mathbb{R}^b$ is called *Samuelson* if the leading principal minors of $J(F)(x)$ are all nonzero for each point $x \in \mathbb{R}^n$. The same definition, with $\mathbb{R}$ replaced by k, a field of characteristic zero, is used in the more general setting of maps $k^n \to k^n$, provided the Jacobian matrix has a sensible definition (e.g. for polynomial maps). An important result about (real) Samuelson maps is

Theorem 4.1 (Campbell). *If $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and Samuelson, and the pivots are globally bounded away from 0, then F has a global inverse.*

The proof [5] is a variant, using bounds rather than rationality, of the analogous result for (real) rational maps [4]. Actually, more is established. Namely, if one considers the pivots in their original order, rather than as arranged in nonascending order of size, then if the first $n-1$ of them are bounded away from 0 the map is injective, and if the last one is as well, then the map is a homeomorphism and hence has a global inverse. The conclusion can also be strengthened; the map can be factored as the composition of n C^1 maps, each of which changes only a single coordinate; see [4] for details.

5. Polynomial maps. If F is a polynomial map, bounds on the characteristic values can be used to deduce their constancy. This was noted for the case of eigenvalues in [8]. The key observation is that a polynomial that is globally bounded (has a uniform bound on all of $\mathbb{R}^n$) is necessarily constant.

THEOREM 5.1. *Suppose that the Jacobian determinant of a polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$ is a nonzero constant. Let $\{\alpha_1, \ldots, \alpha_n\}$ be one of the sets of characteristic values of the Jacobian matrix considered above (singular values, absolute values of the eigenvalues, or absolute values of the pivots for a Samuelson map). Then the following are equivalent:*

(1) *the α_i are (jointly) globally bounded,*
(2) *the α_i are (jointly) globally bounded away from* 0,
(3) *the α_i are all constant.*

Proof. Consider the case of eigenvalues: $\alpha_i = |\lambda_i|$. The λ_i are roots of the characteristic polynomial of $J(F)$. The coefficients of the characteristic polynomial are polynomials in $x_1, \ldots, x_n$. If the $|\lambda_i|$ are jointly globally bounded, then so are the coefficients of the characteristic polynomial, since they are elementary symmetric functions of the eigenvalues. But then the coefficients of the characteristic polynomial are constant. Thus it has only finitely many roots. By the continuity of the roots of a polynomial, the eigenvalues are continuous functions assuming only finitely many values, and hence constant. (The fact that the eigenvalues need not be distinct does not affect the argument, despite the fact that the eigenvalues are not really "functions" of the coefficients at a multiple root; their absolute values, arranged in nonascending order, are actual, continuous, functions of the position $x \in \mathbb{R}^n$.) This is essentially the argument in [8]. This proves that (1) implies (3) for eigenvalues. The converse is obvious. For the equivalence of (2) and (3) consider the absolute values of the eigenvalues of $J(F)^{-1}$; they are the reciprocals of the $|\lambda_i|$, arranged backwards. But $J(F)^{-1}$ is also a matrix with polynomial entries, since the determinant of $J(F)$ is assumed to be a nonzero constant.

For the case of singular values, observe that they are the square roots of the eigenvalues of $J(F)J(F)^T$, another matrix with polynomial entries and a polynomial inverse. So the same argument applies.

For the case of pivots, consider the pivot values in their original order (rather than rearranged in nonascending order of absolute value). Then $\pi_1 = \mu_1$, a polynomial. So if π_1 is bounded, it is constant. Then $\pi_2 = \mu_2/\mu_1$. If π_1 and π_2 are bounded, then μ_1 is a constant, so π_2 is not just a rational function, but actually a polynomial. Continuing, if all the π_i are bounded, then they are all constant. For the case in which the pivots are bounded away from 0, observe that the absolute values of their reciprocals are the absolute values of the pivots of $PJ(F)^{-1}P$, where P is the permutation matrix that reverses the order of the coordinates. But that is clearly also a matrix with polynomial entries. ∎

REMARK. From the proof, condition (1), even without the assumption that the Jacobian determinant is a nonzero constant, implies (2), (3) and thus that the Jacobian determinant is indeed constant (and hence a nonzero constant or identically zero). For context, the recent example of Pinchuk [19] should be mentioned. Pinchuk constructs a class of polynomial maps of $\mathbb{R}^2$ to itself which have a nowhere vanishing (but nonconstant) Jacobian determinant, and which are not injective. There are also plentiful examples of polynomial Samuelson maps with nonconstant Jacobian determinant that are (necessarily, by [4]) bijective, but whose inverses are not polynomial; for instance, the map $F(x) = x + x^3$ from $\mathbb{R}$ to itself.

The case of constant singular values is relatively uninteresting.

THEOREM 5.2. *If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map whose Jacobian matrix has constant singular values, then F is an affine* (*linear + constant*) *map.*

Proof. The result is true even for the case in which some of the singular values are (identically) 0. Since any two vector space norms on a finite dimension real vector space are equivalent, the fact that the spectral norm of $J(F)(x)$ is bounded implies that the ordinary Euclidean norm of $J(F)(x)$, considered as a vector in $\mathbb{R}^{n^2}$, is also bounded. This implies the boundedness of every element of the Jacobian matrix, so the Jacobian matrix has constant entries, and thus F is affine. ∎

This in turn shows that the case of normal Jacobian matrices and constant eigenvalues is also uninteresting.

COROLLARY 5.3. *If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map whose Jacobian matrix has constant eigenvalues and is normal on some open set, then F is an affine* (*linear + constant*) *map.*

Proof. Normality is an algebraic condition that is true globally if it holds on an open set. Thus the singular values will be constant. For that matter, the constant eigenvalue condition need hold only locally as well, and for the same reason. ∎

For the case of a Samuelson map with constant pivots, one has the following result [10], where k is an arbitrary field of characteristic zero.

THEOREM 5.4 (van den Essen and Parthasarathy). *If $F : k^n \to k^n$ is a polynomial Samuelson map whose pivots are constant, then F can be written as a composition $F = D \circ E^{(n)} \circ \cdots \circ E^{(1)}$, where D is the linear map defined by the diagonal matrix whose entries are the pivots π_i* (*in their original order*) *and $E^{(i)}$ is an elementary polynomial automorphism of the form*

$$E_j^{(i)}(x_1,\ldots,x_n) = \begin{cases} x_j & \text{for } j \neq i, \\ x_i + h_i & \text{for } j = i, \end{cases}$$

where h_i is a polynomial independent of x_i.

In sum, bounds on the characteristic values of the Jacobian matrix for polynomial maps lead to well understood situations, except in the constant eigenvalue case. This underlines the importance of the Chamberland conjecture.

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