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The effect of rational maps on polynomial maps

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Abstract. We describe the polynomials $P \in \mathbb{C}[x,y]$ such that $P(1/v^n, A_1v^n + A_2v^{2n} + \ldots + A_{m-1}v^{n(m-1)} + v^{nm-k}w) \in \mathbb{C}[v,w]$. As applications we give new examples of bad field generators and examples of families of polynomials with smooth and irreducible fibers.

Let $P(x,y) \in \mathbb{C}[x,y]$. Suppose that $[1,a,0]$ is a point at infinity of P. Then there exist rational maps

$$\phi : \mathbb{C}^2 \setminus \{v=0\} \to \mathbb{C}^2, \quad (v,w) \mapsto (x,y),$$
$$x = 1/v^\beta, \quad y = w_0/v^\alpha + w_1/v^{\alpha-1} + \ldots + w/v^{\alpha-k},$$

with $\beta \in \mathbb{N}$ and $\alpha \in \mathbb{Z}$ such that

$$Q_t(v,w) = P \circ \phi - t \in \mathbb{C}[v,w].$$

For example, if $P(x,y) = x^2+y^3$, then $P(1/v^3, -1/v^2+wv^4)-t = -t+3w-3w^2v^6+w^3v^{12}$, and if $P(x,y) = x^2y+x$, one has $P(1/v, -v+wv^2)-t = -t+w$ and $P(wv, 1/v^2) - t = -t + w^2 + wv$. Let us write

$$Q_t(v,w) = -t + q_0(w) + q_1(w)v + \ldots + q_n(w)v^n.$$

One says that the polynomial P is *not good* if there exists a map ϕ such that q_0 is zero or has degree strictly greater than one. In this case, the *critical values at infinity* of P are the roots of the discriminant of $q_0(w)-t$ if $q_0 \not\equiv 0$, and 0 otherwise. The polynomial $P(x,y) = x^2y + x$ is not good and 0 is a critical value at infinity.

The study of these polynomials is very important. In particular, the generically rational polynomials which are not of simple type, the polynomials with only smooth and irreducible fibers which are not variables, potential counterexamples to the jacobian conjecture, are to be found among non-good polynomials.

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In order to better understand the polynomial P, we will study the polynomial Q. It often happens that the polynomial Q is very simple. Moreover, one can reconstruct P from Q.

In this article we will study the map ϕ given by

$$x = 1/v^n, \qquad y = A_1 v^n + A_2 v^{2n} + \ldots + A_{m-1} v^{n(m-1)} + v^{nm-k} w$$

where n, m, k are natural numbers such that $k < n$. We show how to recognize polynomials P such that $P \circ \phi = Q \in \mathbb{C}[v, w]$ and also the polynomials Q which have this property. This is inspired by Peretz [P], who studied the case $n = 1$. We will also give some applications of the main theorem. More applications will be the aim of forthcoming papers, in particular to study generically rational polynomials and polynomials with smooth and irreducible fibers.

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I. MAIN THEOREM

THEOREM 1. *Let n, m, k be natural integers such that $k < n$ and let $P(x, y) \in \mathbb{C}[x, y]$. Let $p(x, y) = x^{m-1} y - A_1 x^{m-2} - \ldots - A_{m-1}$. Then the following assertions are equivalent*:

(i) $P(1/v^n, A_1 v^n + \ldots + A_{m-1} v^{n(m-1)} + v^{nm-k} w) \in \mathbb{C}[v, w]$,

(ii) $P(x, y) \in \mathbb{C}[y, xy, x^2 y - A_1 x, \ldots, x^{m-1} y - A_1 x^{m-2} - \ldots - A_{m-2} x, \ldots$ $\ldots, x^{h_i} p(x, y)^{r_i}, \ldots]$ *where* (h_i, r_i) *runs through*

$$N = \{(h_i, r_i) \in \mathbb{N}^2 \mid 1 \le r_i \le n,\ 0 \le h_i \le n - k,\ (n-k) r_i - n h_i \ge 0\}.$$

THEOREM 2. (i) *If one of the assertions of Theorem* 1 *is true, let*

$$P(1/v^n, A_1 v^n + \ldots + A_{m-1} v^{n(m-1)} + v^{nm-k} w) = Q(v, w),$$

then $Q(v, w)$ *is in* $\mathbb{C}[\ldots, v^{(n-k) r_i - n h_i} w^{r_i}, \ldots, A_{m-1} v^n + v^{2n-k} w, \ldots, A_1 v^n + \ldots + A_{m-1} v^{n(m-1)} + v^{nm-k} w]$ *where* (h_i, r_i) *runs through* N.

(ii) *Conversely, if* $Q(v, w)$ *is in* $\mathbb{C}[\ldots, v^{(n-k) r_i - n h_i} w^{r_i}, \ldots, A_{m-1} v^n + v^{2n-k} w, \ldots, A_1 v^n + \ldots + A_{m-1} v^{n(m-1)} + v^{nm-k} w]$ *then setting*

$$v^{(n-k) r_i - n h_i} w^{r_i} = x^{h_i} p^{r_i}$$

and

$$A_3 v^n + \ldots + A_{m-1} v^{n(m-3)} + v^{n(m-2)-k} w = x^2 y - A_1 x - A_2,$$

$$A_2 v^n + \ldots + A_{m-1} v^{n(m-2)} + v^{n(m-1)-k} w = xy - A_1$$

one gets a polynomial $P(x, y) \in \mathbb{C}[x, y]$ *such that*

$$P(1/v^n, A_1 v^n + \ldots + A_{m-1} v^{n(m-1)} + v^{nm-k} w) \in \mathbb{C}[v, w].$$

One has to prove that in Theorem 1, (i) implies (ii). The rest is easy. We use R. Peretz' ideas [P]. Let $P(x,y) \in \mathbb{C}[x,y]$ satisfy condition (i) of Theorem 1,

$$P(x,y) = \sum a_{i,j} x^i y^j.$$

Let

$$P_+(x,y) = \sum_{0\le i\le j} a_{i,j} x^i y^j \quad \text{and} \quad P_-(x,y) = \sum_{0\le j<i} a_{i,j} x^i y^j.$$

Then

$$P_+(1/v^n, A_1 v^n + \ldots + A_{m-1} v^{n(m-1)} + v^{nm-k} w) = \sum_{0\le i\le j} a_{i,j} v^{n(j-i)} (A_1 + \ldots + A_{m-1} v^{n(m-2)} + v^{n(m-1)-k} w)^j.$$

It follows that

$$P_+(1/v^n, A_1 v^n + \ldots + A_{m-1} v^{n(m-1)} + v^{nm-k} w) \in \mathbb{C}[v,w].$$

Moreover,

$$P_+(x,y) = \sum_{0\le i\le j} a_{i,j} x^i y^j = \sum_{0\le i\le j} a_{i,j} (xy)^i y^{j-i}.$$

Then $P_+(x,y) \in \mathbb{C}[y,xy]$. Since P and P_+ satisfy condition (i), so does P_-. Moreover,

$$\begin{aligned} P_-(1/v^n, A_1 + \ldots + A_{m-1} v^{n(m-1)} + v^{nm-k} w) &= \sum_{0\le j<i} a_{i,j} v^{n(j-i)} (A_1 + \ldots + A_{m-1} v^{n(m-2)} + v^{n(m-1)-k} w)^j \\ &= \sum_{0\le j<i} a_{i,j} \sum_{l_1=0}^{j} \binom{j}{l} \\ &\quad \times A_1^{j-l_1} v^{n(j-i)+nl_1} (A_2 + \ldots + A_{m-1} v^{n(m-3)} + v^{n(m-2)-k} w)^{l_1} \\ &= \sum_{0\le j<i} a_{i,j} \sum_{l_1=0}^{j} \cdots \sum_{l_{m-1}=0}^{l_{m-2}} \binom{j}{l} \cdots \binom{l_{m-2}}{l_{m-1}} \\ &\quad \times A_1^{j-l_1} \ldots A_{m-1}^{l_{m-2}-l_{m-1}} v^{n(j-i+l_1+\ldots+l_{m-2})+(n-k)l_{m-1}} w^{l_{m-1}}. \end{aligned}$$

Since P_- satisfies condition (i), one has $n(j-i+l_1+\ldots+l_{m-2})+(n-k)l_{m-1} \ge 0$. Let

$$Q_-(v,w) = P_-(1/v^n, A_1 + \ldots + A_{m-1} v^{n(m-1)} + v^{nm-k} w).$$

Then

$$Q_-(v,w) = \sum_{ni/(nm-k)\leq j<i} a_{i,j} \sum_{l_1=0}^{j} \dots \sum_{l_{m-1}=0}^{l_{m-2}} \binom{j}{l} \dots \binom{l_{m-2}}{l_{m-1}} \times A_1^{j-l_1} \dots A_{m-1}^{l_{m-2}-l_{m-1}} v^{n(j-i+l_1+\ldots+l_{m-2})+(n-k)l_{m-1}} w^{l_{m-1}}$$

where the l_i's satisfy $n(j-i+l_1+\ldots+l_{m-2})+(n-k)l_{m-1}\geq 0$. Write $Q_-(v,w)=Q_-^1(v,w)+Q_-^2(v,w)$ such that

$$\begin{aligned} Q_-^1(v,w) = &\sum_{ni/(nm-k)\leq j<i} a_{i,j} \\ &\times \sum_{l_1=0}^{j} \dots \sum_{l_{m-2}=0}^{l_{m-3}} \binom{j}{l} \dots \binom{l_{m-3}}{l_{m-2}} A_1^{j-l_1} \dots A_{m-2}^{l_{m-3}-l_{m-2}} \\ &\times v^{n(j-i+l_1+\ldots+l_{m-2})} \\ &\times \sum_{l_{m-1}=0}^{l_{m-2}} \binom{l_{m-2}}{l_{m-1}} A_{m-1}^{l_{m-2}-l_{m-1}} v^{(n-k)l_{m-1}} w^{l_{m-1}} \end{aligned}$$

where the summation over l_i, $i\in\{1,\ldots m-2\}$, is taken for $j-i+l_1+\ldots+l_{m-2}\geq 0$, and

$$\begin{aligned} Q_-^2(v,w) = &\sum_{ni/(nm-k)\leq j<i} a_{i,j} \sum_{l_1=0}^{j} \dots \sum_{l_{m-2}=0}^{l_{m-3}} \\ &\times \sum_{l_{m-1}\geq (n/(n-k))(i-j-l_1-\ldots-l_{m-2})} \binom{j}{l} \dots \binom{l_{m-3}}{l_{m-2}} \binom{l_{m-2}}{l_{m-1}} \\ &\times A_1^{j-l_1} \dots A_{m-2}^{l_{m-3}-l_{m-2}} \\ &\times A_{m-1}^{l_{m-2}-l_{m-1}} v^{n(j-i+l_1+\ldots+l_{m-2})} (v^{n-k}w)^{l_{m-1}} \end{aligned}$$

where the summation is over the l_i's, $i\in\{1,\ldots,m-2\}$, such that $j-i+l_1+\ldots+l_{m-2}<0$. In $Q_-^2(v,w)$, replace v^{-n} by x and $v^{n-k}w$ by $x^{m-1}y-A_1x^{m-2}-\ldots-A_{m-1}=p(x,y)$. One gets a polynomial

$$\begin{aligned} P_-^2(x,y) = &\sum_{ni/(nm-k)\leq j<i} a_{i,j} \sum_{l_1=0}^{j} \dots \sum_{l_{m-2}=0}^{l_{m-3}} \\ &\times \sum_{l_{m-1}\geq (n/(n-k))(i-j-l_1-\ldots-l_{m-2})} \binom{j}{l} \dots \binom{l_{m-3}}{l_{m-2}} \binom{l_{m-2}}{l_{m-1}} \\ &\times A_1^{j-l_1} \dots A_{m-2}^{l_{m-3}-l_{m-2}} A_{m-1}^{l_{m-2}-l_{m-1}} x^{i-j-l_1-\ldots-l_{m-2}} p(x,y)^{l_{m-1}}, \end{aligned}$$

the sum being taken over the l_i's, $i\in\{1,\ldots,m-2\}$, such that $j-i+$

$l_1+\ldots+l_{m-2}<0$. But $l_{m-1}\geq (n/(n-k))(i-j-l_1-\ldots-l_{m-2})\geq i-j-l_1-\ldots-l_{m-2}$. Write

$$i-j-l_1-\ldots-l_{m-2}=(n-k)q+h$$

with $h<n-k$, and $l_{m-1}=nq+r$. One has $nh\leq (n-k)r$. Then

$$x^{i-j-l_1-\ldots-l_{m-2}}p(x,y)^{l_{m-1}}=(x^{n-k}p(x,y)^n)^q x^h p(x,y)^r$$

If $r<n$, the pair (h,r) is in N. If $r\geq n$, we write $x^hp(x,y)^r=x^hp(x,y)^n\times p(x,y)^{r-n}$. Then $P_-^2(x,y)\in\mathbb{C}[y,xy,x^2y-A_1x,\ldots,x^{m-1}y-A_1x^{m-2}-\ldots-A_{m-2}x,\ldots,x^{h_i}p(x,y)^{r_i},\ldots]$ where (h_i,r_i) runs through N.

Now we come back to

$$\begin{aligned}Q_-^1(v,w)=&\sum_{ni/(nm-k)\leq j<i}a_{i,j}\\&\times\sum_{l_1=0}^{l_1=j}\cdots\sum_{l_{m-2}=0}^{l_{m-3}}\binom{j}{l}\cdots\binom{l_{m-3}}{l_{m-2}}A_1^{j-l_1}\ldots A_{m-2}^{l_{m-3}-l_{m-2}}\\&\times v^{n(j-i+l_1+\ldots+l_{m-2})}\\&\times\sum_{l_{m-1}=0}^{l_{m-2}}\binom{l_{m-2}}{l_{m-1}}A_{m-1}^{l_{m-2}-l_{m-1}}v^{(n-k)l_{m-1}}w^{l_{m-1}},\end{aligned}$$

the summation being taken over the l_i's, $i\in\{1,\ldots,m-2\}$, such that $j-i+l_1+\ldots+l_{m-2}\geq 0$. Then

$$\begin{aligned}Q_-^1(v,w)=&\sum_{ni/(nm-k)\leq j<i}a_{i,j}\\&\times\sum_{l_1=0}^{j}\cdots\sum_{l_{m-2}=0}^{l_{m-3}}\binom{j}{l}\cdots\binom{l_{m-3}}{l_{m-2}}A_1^{j-l_1}\ldots A_{m-2}^{l_{m-3}-l_{m-2}}\\&\times v^{n(j-i+l_1+\ldots+l_{m-2})}(A_{m-1}+v^{n-k}w)^{l_{m-2}},\end{aligned}$$

the summation being taken over the l_i's, $i\in\{1,\ldots,m-2\}$, such that $j-i+l_1+\ldots+l_{m-2}\geq 0$. Again we split $Q_-^1(v,w)=Q_-^{1,1}(v,w)+Q_-^{1,2}(v,w)$ where

$$\begin{aligned}Q_-^{1,1}(v,w)=&\sum_{ni/(nm-k)\leq j<i}a_{i,j}\\&\times\sum_{l_1=0}^{j}\cdots\sum_{l_{m-3}=0}^{l_{m-4}}\binom{j}{l}\cdots\binom{l_{m-4}}{l_{m-3}}A_1^{j-l_1}\ldots A_{m-3}^{l_{m-4}-l_{m-3}}\end{aligned}$$

$$\times\, v^{n(j-i+l_1+\ldots+l_{m-3})} \sum_{l_{m-2}=0}^{l_{m-3}} \binom{l_{m-3}}{l_{m-2}}$$
$$\times\, A_{m-2}^{l_{m-3}-l_{m-2}} v^{nl_{m-2}} (A_{m-1}+v^{n-k}w)^{l_{m-2}},$$

the sum being taken over the l_i, $i \in \{1,\ldots,m-3\}$, such that $j-i+l_1+\ldots+l_{m-3} \geq 0$, and

$$Q_-^{1,2}(v,w) = \sum_{ni/(nm-k)\leq j<i} a_{i,j} \sum_{l_1=0}^{j} \cdots \sum_{l_{m-3}=0}^{l_{m-4}}$$
$$\times \sum_{l_{m-2}\geq i-j-l_1-\ldots-l_{m-3}} \binom{j}{l} \cdots \binom{l_{m-3}}{l_{m-2}}$$
$$\times A_1^{j-l_1} \ldots A_{m-2}^{l_{m-3}-l_{m-2}} v^{n(j-i+l_1+\ldots+l_{m-2})} (A_{m-1}+v^{n-k}w)^{l_{m-2}},$$

the summation being taken over the l_i, $i \in \{1,\ldots,m-2\}$, such that $j-i+l_1+\ldots+l_{m-3} < 0$. Replace v^{-n} by x and $A_{m-1}+v^{n-k}w$ by $p(x,y)+A_{m-1}$. Then

$$v^{n(j-i+l_1+\ldots+l_{m-2})}(A_{m-1}+v^{n-k}w)^{l_{m-2}} = x^{i-j-l_1-\ldots-l_{m-3}}(x^{m-2}y-\ldots-A_{m-2})^{l_{m-2}}.$$

We write

$$x^{i-j-l_1-\ldots-l_{m-3}}(x^{m-2}y-\ldots-A_{m-2})^{l_{m-2}} = p(x,y)^{i-j-l_1-\ldots-l_{m-3}}(x^{m-2}y-\ldots-A_{m-2})^{l_{m-2}-(i-j-l_1-\ldots-l_{m-3})}.$$

Step by step, the result follows.

Remark. In the case where $n=1$, D. Wright [W] studied $\mathrm{Spec}(A)$ for

$$A = \mathbb{C}[y, xy, x^2y - A_1x, \ldots, x^{m-1}y - A_1x^{m-2} - \ldots - A_{m-2}x, \ldots, x^my - A_1x^{m-1} - \ldots - A_{m-1}x].$$

II. APPLICATIONS

1. First we will study a simple case of a map ϕ which is already famous. Let us consider the map

$$\phi : \mathbb{C}^2 \setminus \{v=0\} \to \mathbb{C}^2, \quad (v,w) \mapsto (1/v^2, -v^2+v^3w).$$

Applying Theorem 1, one knows that $Q = P\circ\phi \in \mathbb{C}[v,w]$ is equivalent to $P(x,y) \in \mathbb{C}[y, xy, x(xy+1)^2]$ and $Q(v,w) \in \mathbb{C}[w^2, vw, v^3w - v^2]$. One goes from Q to P replacing w^2 by $x(xy+1)^2$, vw by $xy+1$ and v^3w-v^2 by y.

This map occurs in two well known examples.

(a) *Briançon's example.* Briançon's example [ACL] is the first known one of a polynomial with smooth and irreducible fibers. It is defined by

$$s = xy + 1, \quad p = sx + 1, \quad u = s^2 + y,$$
$$f = p^2 u + a_1 ps + a_0 s + t$$

with $a_0 = -1/3$ and $a_1 = -5/3$. Let us consider f depending on the parameters a_0 and a_1. One sees that s, ps and p^2u belong to $\mathbb{C}[y, xy, x(xy+1)^2]$. Then $f_1(v,w) = f(1/v^2, -v^2 + v^3 w) \in \mathbb{C}[v,w]$ and

$$f_1(v,w) = v^3 w + (3w^2 - 1)v^2 + (a_1 w + a_0 w + 3w^3 - 2w)v + w^4 - w^2 + t + a_1 w^2.$$

One sees that f is a non-good polynomial with critical values $t = 0$ and $t = (a_1 - 1)^2/4$. It also has 2 critical points. Let us consider the rational map ϕ_1, of degree two,

$$\mathbb{C}^2 \setminus \{v_1 = w_1\} \xrightarrow{\sigma} \mathbb{C}^2 \setminus \{v = 0\} \xrightarrow{\phi} \mathbb{C}^2, \qquad (v_1, w_1) \mapsto (v, w) \mapsto (x, y),$$

where σ is the automorphism $v = v_1 - w_1$, $w = w_1$.

Define $f_2(v_1, w_1) = f_1 \circ \sigma$. Then

$$f_2(v_1, w_1) = v_1^3 w_1 - v_1^2 + (a_0 + a_1) v_1 w_1 - a_0 w_1^2 + t.$$

This polynomial is non-degenerate and commode, hence it is tame [B]. It has no critical values at infinity; its global Milnor number is 5 and can be computed using Kouchnirenko's theorem. One singular point is $(0,0)$, which lies on $v_1 = w_1$ and is sent to infinity by ϕ_1, and four others are sent to the two critical points of f. To get rid of these two critical points one has to put the four critical points of f_2 on the line $v_1 = w_1$. This gives two possible values $a_0 = -1/3$, $a_1 = -5/3$ and $a_0 = -1/9$, $a_1 = -7/9$.

It is easy to see that $f = c$ is an irreducible fiber if and only if $f = c$ is not divisible by a power of x and $f_2 = c$ is not divisible by a power of $v_1 - w_1$. Then the irreducibility of all the fibers $f = c$ is very easy to check.

Starting with f_2, using the automorphism $v = v_1 - w_1^3$, $w = w_1$, which is an automorphism of $\mathbb{C}[w_1^2, v_1 w_1, v_1^3 + v_1^3 w_1]$, and replacing w_1^2 by $x(xy+1)^2$, $v_1 w_1$ by $xy + 1$ and $v_1^3 w_1 - v_1^2$ by y, one gets a new polynomial of degree 15. Now if we send the critical points of f_2 to the curve $v_1 = w_1^3$, we will again get a polynomial with smooth and irreducible fibers. This can be achieved with $a_0 = -1/4$, $a_1 = \sqrt{3} - 1/4$.

The other example we want to discuss is due to Pinchuk.

(b) *Pinchuk's example.* Pinchuk [Pi] found an example of a map (f, g) from $\mathbb{R}^2$ to $\mathbb{R}^2$ whose jacobian does not vanish in $\mathbb{R}^2$ and which is not injective. The two polynomials f and g satisfy

$$(*) \qquad f_1 = f(1/v^2, v^2 + v^3 w) \in \mathbb{C}[v,w], \qquad g_1 = g(1/v^2, v^2 + v^3 w) \in \mathbb{C}[v,w].$$

Using again the automorphism

$$v = v - w, \qquad w = w$$

one gets

$$\begin{aligned} f_2 &= v^2 + v^3 w + vw, \\ g_2 &= -75w^4v^6 - 270v^5w^3 - \tfrac{1}{4}w(1460w + 75w^3)v^4 \\ &\quad - \tfrac{1}{4}w(300w^2 + 680)v^3 - \tfrac{1}{4}w(392w - 24w^3)v^2 + 8w^3v - w^4. \end{aligned}$$

The jacobian of the rational map is 1, hence the jacobian of (f_2, g_2) is also a sum of squares. But the two polynomials f_2 and g_2 have a critical point at the origin (because they belong to $\mathbb{C}[w^2, vw, v^2 + v^3w]$), thus their jacobian vanishes at the origin. This proves two things: first we will never get a jacobian equal to 1 starting with a map satisfying $(*)$, and, as the jacobian of (f_2, g_2) always vanishes, there exists a real sequence (x_k, y_k) going to infinity such that the jacobian of (f, g) goes to 0 as $k \to \infty$. This is compatible with Conjecture 2 of [CM].

Remark. Peretz, as well as Wright, uses the ring $\mathbb{C}[y, xy, x^2y+x]$ instead of $\mathbb{C}[y, xy, x(xy+1)^2]$ which is contained in the previous one. In fact, the ring which appears in Theorem 1 is contained in the ring $A = \mathbb{C}[y, xy, x^2y - A_1x, \ldots, x^my - A_1x^{m-1} - \ldots - A_{m-1}x]$ studied by Peretz and Wright, for which Wright [W] settled Conjecture 3.2, which says that there is no pair of polynomials in this ring with non-zero constant jacobian. Wright proved the conjecture in the case where A_1 is non-zero. There are non-good polynomials which are not contained in any of the rings studied by Wright. An example is Jan's polynomial [J]:

$$\begin{aligned} f := x(x^5y^3 + 1)^3 + y(x^2y + 1)^8 - x^{16}y^9 + 4xy + 6x^2y \\ + 19x^3y^2 + 8x^4y^2 + 36x^5y^3 + 34x^7y^4 + 16x^9y^5. \end{aligned}$$

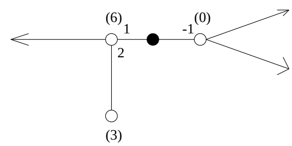
2. Bad field generators. A *field generator* is a polynomial whose generic fiber is rational (a generically rational polynomial). If f is a field generator, there exist $g \in \mathbb{C}(x, y)$ such that $\mathbb{C}(f, g) = \mathbb{C}(x, y)$. One says that f is a *bad field generator* if there does not exist $g \in \mathbb{C}[x, y]$ such that $\mathbb{C}(f, g) = \mathbb{C}(x, y)$. One can recognize that a polynomial f is a bad field generator by the fact that the generic fiber is rational and for any rational map ϕ such that $Q = f \circ \phi \in \mathbb{C}[v, w]$,

$$Q(v, w) = q_0(w) + q_1(w)v + \ldots + q_n(w)v^n,$$

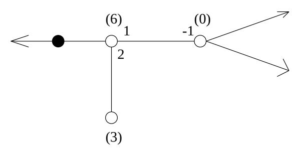
the polynomial q_0 has degree strictly greater than 1 if it is not zero. Until now, two examples of bad field generators have been known. The first one was discovered by Jan [J]; its degree is 25. Later Russell [R] found an example of degree 21 and showed that this is the lowest possible degree. Let us build new examples, based on Russell's. Start with

$$Q(v,w) = w^3 + v^2w^2 + vw + t.$$

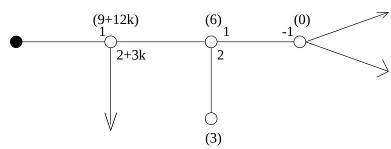
Then $Q \in \mathbb{C}[w^3, vw, -v^3 + v^4w]$. Consider the two automorphisms of this ring $w = w_1 - v_1^2, v = v_1$ (we get Q_1) and $w_1 = w_2, v_1 = v_2 + w_2^{2+3k}$. Let Q_2 be the compositum. Now if we replace $-v_2^3 + v_2^4w_2$ by y, v_2w_2 by $xy - 1$ and w_2^3 by $x(xy-1)^3$, we get a polynomial f which is a bad field generator. (The case $k = 0$ is Russell's polynomial.) To see this, it is useful to look at the splice diagrams at infinity of the fibers of the polynomials occurring here. Splice diagrams are explained in [N]; they give a picture of the branches at infinity of a curve. The splice diagram at infinity of the generic fiber of Q is



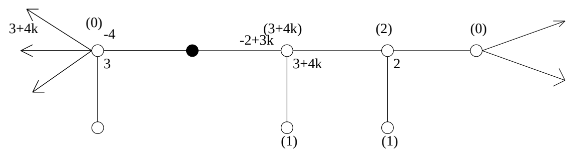
The splice diagram at infinity of the generic fiber of Q_1 is



The splice diagram at infinity of the generic fiber of Q_2 is



and the splice diagram at infinity of the generic fiber of f is



which shows that the polynomial f is a bad field generator.

3. Families of polynomials with smooth and irreducible fibers. In [ACL] one can find infinitely many polynomials with smooth and irreducible fibers. But no family of such polynomials is presented there. Now we give examples of such families.

Let us first find a family in degree 9. We start with

$$f = v^2 w + a_1 v + A_1 v w + A_2 w + A_3 w^2 + t$$

where a_1 and A_3 are non-zero. The polynomial f is commode and non-degenerate, we have $\mu = 3$. We will put the three critical points on the line $v = w + a_2$. This is easy to do, because the critical points satisfy

$$(1) \qquad (2v + A_1)(v^2 + A_1 v + A_2) - 2a_1 A_3 = 0,$$

$$(2) \qquad 2A_3 w + v^2 + A_1 v + A_2 = 0.$$

If all the critical points are on the line $v = w + a_2$, all the roots of $P := (2v+A_1)(v^2+A_1v+A_2)-2a_1A_3$ are roots of $Q := 2A_3(v-a_2)+v^2+A_1v+A_2$. If $P = Q_1 Q$ with Q_1 dividing Q, the condition is satisfied. In fact, it is the only possibility which ensures a_1 non-zero. We get

$$a_1 = -16A_3^2, \quad A_1 = -2a_2 - 4A_3, \quad A_2 = a_2^2 + 4a_2 A_3 - 8A_3^2.$$

Now we consider $f_1(v, w) = f(v - w - a_2, w)$ and

$$F(x, y) = f_1(1/x, x^2 y - a_1 x).$$

Then the polynomial F is a polynomial of degree 9 with smooth and irreducible fibers:

$$\begin{aligned} F := {} & x^6y^3 + 48x^5y^2A_3^2 + (-3A_3y^2 + 768yA_3^4)x^4 \\ & + (4096A_3^6 + 2y^2 - 96A_3^3y)x^3 + (40yA_3^2 - 768A_3^5)x^2 \\ & + (128A_3^4 - 4A_3y)x - 16A_3^2a_2 + y - 64A_3^3. \end{aligned}$$

One notices that F only depends on A_3.

Now one can also put the critical points on the curve $v = w^2 + w + a_2$. It is easier because now the condition is that the polynomial P divides the polynomial $Q := (v^2 + A_1 v + A_2)^2 - 2A_3(v^2 + A_1 v + A_2) + 4A_3^2 a_2$. One gets $a_1 = 4A_3$, $A_1 = -2a_2$, $A_2 = 2a_3 A_3 + a_2^2$. Now we consider $f_1(v, w) = f(v - w^2 - w - a_2, w)$ and

$$F(x, y) = f_1(1/x, x^2 y - a_1 x).$$

The polynomial F is of degree 15 with smooth and irreducible fibers:

$$\begin{aligned} F := {} & x^{10}y^5 - 20x^9y^4A_3 + (160y^3A_3^2 + 2y^4)x^8 \\ & + (-32y^3A_3 - 640y^2A_3^3)x^7 + (y^3 + 192y^2A_3^2 + 1280yA_3^4)x^6 \\ & + (2y^3 - 512yA_3^3 - 12y^2A_3 - 1024A_3^5)x^5 \\ & + (48yA_3^2 - 19y^2A_3 + 512A_3^4)x^4 + (56yA_3^2 - 64A_3^3 + 2y^2)x^3 \\ & + (-10yA_3 - 48A_3^3)x^2 + 8A_3^2x + 4a2A_3 + y. \end{aligned}$$

Again it only depends on A_3. More examples of polynomials satisfying Theorem 1 can be found in [CN].

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