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# The sixtieth anniversary of the Jacobian Conjecture: a new approach

by Arno van den Essen (Nijmegen)

**Abstract.** We investigate an approach of Bass to study the Jacobian Conjecture via the degree of the inverse of a polynomial automorphism over an arbitrary $\mathbb{Q}$-algebra.

**Introduction and notations.** This year we celebrate the 60th anniversary of the Jacobian Conjecture. On the occasion of this event I would like to present a new approach to attack this conjecture. In fact, the approach is not completely new but is a continuation of an idea of Bass [1] which goes back to 1983. This continuation was motivated by some more recent results of Derksen [3] and Furter [6], to which I will come back below.

The main aim of this paper is to give a new impulse to this approach, which hopefully will lead to the solution of the Jacobian Conjecture!

Throughout this paper we use the following notations: $k$ denotes a field, $k[X] := k[X_1, \ldots, X_n]$ the polynomial ring over $k$ and if $F = (F_1, \ldots, F_n) \in k[X]^n$ then $\deg F := \max_i \deg F_i$, where $\deg F_i$ denotes the total degree of $F_i$. Finally, by $\mathrm{JC}(\mathbb{C}, n)$ we denote the *$n$-dimensional Jacobian Conjecture*, i.e. the statement

$$\text{if } F \in \mathbb{C}[X]^n \text{ with } \det JF \in \mathbb{C}^*, \text{ then } \mathbb{C}[F_1, \ldots, F_n] = \mathbb{C}[X].$$

**1. The degree of the inverse of a polynomial automorphism.** To start my story let us go back some twenty years. Then the first significant result on the Jacobian Conjecture was obtained by Stuart Wang in [9] who showed that the Jacobian Conjecture is true in case $F : k^n \to k^n$ is a polynomial map with $\deg F \le 2$ and $\mathrm{char}\, k \ne 2$. In fact, he even showed that in case $k$ is a UFD with $2 \ne 0$ the Jacobian Conjecture (i.e. its obvious generalisation, with $\mathbb{C}$ replaced by $k$) holds. At the end of his paper

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he makes the following *Degree Conjecture*: if $k$ is a UFD with $2 \neq 0$ and $F \in \operatorname{Aut}_k k[X]$ with $\deg F \leq 2$, then $\deg F^{-1} \leq 2^{n-1}$. This conjecture was remarkable at that time since it contrasted with an earlier conjecture of Sathaye which, as Wang writes, states that the degree of the inverse of a polynomial automorphism is in general not bounded.

Wang's conjecture was proved in the field case around 1980 by Rusek and Winiarski [8] and simultaneously by Gabber (see [2]). In fact, they proved a more general result.

PROPOSITION 1.1 (Rusek, Winiarski, Gabber). *Let $k$ be a field and $F \in \operatorname{Aut}_k k[X]$. Then* $\deg F \leq (\deg F)^{n-1}$.

This most probably finished the Sathayer conjecture. I write "most probably" since I do not know what the exact meaning of "in general" was, namely one can ask: what happens if one replaces $k$ by an arbitrary commutative ring $R$?

The first partial answer is

PROPOSITION 1.2. *If $R$ is a reduced ring, i.e. $R$ has no non-zero nilpotent elements, then* $\deg F^{-1} \leq (\deg F)^{n-1}$ *for all* $F \in \operatorname{Aut}_R R[X]$.

*Proof.* Write $G = (G_1, \ldots, G_n)$ instead of $F^{-1}$.

(i) If $R$ is a domain, embed $R$ in its quotient field and apply Proposition 1.1.

(ii) To prove the general case let $1 \leq i \leq n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ with all $\alpha_j \geq 0$ such that $|\alpha| > (\deg F)^{n-1}$. It suffices to show that $c_\alpha^{(i)} = 0$, where $c_\alpha^{(i)}$ is the coefficient of the monomial $X^\alpha$ in $G_i$. Therefore let $p$ be a prime ideal in $R$ and consider the maps $\overline{F}$ and $\overline{G}$, obtained by reducing the coefficients of the $F_i$ and $G_j$ modulo $p$. Put $\overline{R} := R/p$. Then we deduce from (i) that

$$\deg \overline{G} \leq (\deg \overline{F})^{n-1} \leq (\deg F)^{n-1}.$$

So $\overline{c_\alpha^{(i)}} = 0$, i.e. $c_\alpha^{(i)} \in p$. Since this holds for all prime ideals $p$ in $R$ we deduce that $c_\alpha^{(i)} \in \bigcap p = (0)$, since $R$ is reduced. ∎

So the next question to consider is: what happens if $R$ does have non-zero nilpotent elements?

Here we get a first surprise: consider $n = 1$ and $R := \mathbb{C}_m := \mathbb{C}[T]/(T^m)$, where $m \geq 2$. So $\varepsilon := \overline{T}$ satisfies $\varepsilon^m = 0$ and $\varepsilon^{m-1} \neq 0$. Define $F = X + \varepsilon X^2$ (so $F$ is quadratic!).

CLAIM. $F \in \operatorname{Aut}_{\mathbb{C}_m} \mathbb{C}_m[X]$ *and* $\deg F^{-1} = m$.

To get $F^{-1}$ we just have to solve for $X$ the quadratic equation $F(X) = Y$, i.e. $\varepsilon X^2 + X = Y$. Every highschool student can do this and one finds

$$X = \frac{-1 + (1 + 4\varepsilon Y)^{1/2}}{2\varepsilon} = \sum_{i=1}^{m} 2\binom{1/2}{i}(4\varepsilon)^{i-1}Y^i.$$

So indeed $F \in \mathrm{Aut}_{\mathbb{C}_m} \mathbb{C}_m[X]$ and $\deg F^{-1} = m$ (this in spite of the fact that $\deg F = 2$ !).

CONCLUSION. If we do admit non-zero nilpotent elements in the coefficient ring, Sathaye was not wrong after all, or to put it more precisely: for $d \geq 2$, there does not exist a positive integer $C(n,d)$ such that $\deg F^{-1} \leq C(n,d)$ for all $F \in \mathrm{Aut}_R R[X]$ with $\deg F \leq d$ and all $\mathbb{Q}$-algebras $R$.

Now you may wonder: what does all of this have to do with the Jacobian Conjecture? The answer is given by the following results:

THEOREM 1.3 ([2]). *Let* $n \geq 1$. *If* $\mathrm{JC}(\mathbb{C}, n)$ *is true then the following statement*, *denoted by* $\mathrm{UB}(n)$, *is true as well*:

$\mathrm{UB}(n)$ *For every* $d \geq 1$ *there exists a positive integer* $C(n,d)$ *such that for any* $\mathbb{Q}$*-algebra* $R$ *and any* $F \in \mathrm{Aut}_R R[X]$ *with* $\deg F \leq d$ *and* $\det JF = 1$ *we have* $\deg F^{-1} \leq C(n,d)$.

The point in $\mathrm{UB}(n)$ is that one only considers $R$-automorphisms $F$ of $R[X]$ having $\det JF = 1$ (or equivalently, $\det JF \in R^*$, the group of units of $R$). This is really a restriction, namely from the chain rule one easily deduces that if $F \in \mathrm{Aut}_R R[X]$ then $\det JF \in R[X]^*$. However, if $R$ has non-zero nilpotent elements then $R^* \subsetneq R[X]^*$. Our example $F = X + \varepsilon X^2$ also illustrates this point:

$$\det JF = 1 + 2\varepsilon X \in R[X]^* \setminus R^*.$$

Apparently, the existence of such a uniform bound $C(n,d)$ is a necessary condition for the Jacobian Conjecture to be true.

However, there is more: it was observed by Hyman Bass in [1] around 1983 that the condition $\mathrm{UB}(n)$ is also sufficient:

THEOREM 1.4 (Bass). *Let* $n \geq 1$. *Then* $\mathrm{UB}(n)$ *and* $\mathrm{JC}(\mathbb{C}, n)$ *are equivalent.*

So the Jacobian Conjecture, if true, is reduced to finding a positive integer $C(n,d)$ such that $\deg F^{-1} \leq C(n,d)$ for all $F \in \mathrm{Aut}_R R[X]$ with $\deg F \leq d$ and $\det JF = 1$, independent of the $\mathbb{Q}$-algebra $R$!

The next question which arises immediately is: what, in case the Jacobian Conjecture is true, is a natural candidate for $C(n,d)$? (From now on we denote by $C(n,d)$ the *smallest* upper bound as in $\mathrm{UB}(n)$.)

Before addressing this question one may wonder: does the statement $\mathrm{UB}(n)$ look much easier than $\mathrm{JC}(\mathbb{C}, n)$?

At first glance one is inclined to say NO, because of various reasons. For example:

• Instead of understanding automorphisms over $\mathbb{C}$, one has to study automorphisms over all $\mathbb{Q}$-algebras $R$.

• Amongst all automorphisms over $R$ (whose Jacobian determinant is a unit in $R[X]$) one has to characterize those automorphisms whose Jacobian determinant equals 1.

• One has to be able to compare $F$ with $F^{-1}$.

All of this seems to vote against studying the statement UB$(n)$: this might be the reason why it remained untouched since 1983. However, in the remainder of this paper we will show how the above objections can be overruled.

**2. Bass' theorem revisited.** The first of the above objections is that one has to study automorphisms over all $\mathbb{Q}$-algebras $R$. The following beautiful argument due to Harm Derksen ([3]) overcomes this objection: it shows that one only has to study automorphisms over the simplest $\mathbb{Q}$-algebras having nilpotent elements, namely the $\mathbb{C}$-algebras $\mathbb{C}_m$, $m \geq 2$, as introduced in §1. More precisely, let us formulate the following statement.

$\overline{\mathrm{UB}}(n)$ For every $d \geq 1$ there exists a positive integer $\overline{C}(n,d)$ such that for any $\mathbb{C}$-algebra $\mathbb{C}_m$, $m \geq 2$, and every $F \in \mathrm{Aut}_{\mathbb{C}_m} \mathbb{C}_m[X]$ with $\deg F \leq d$ and $\det JF = 1$ we have $\deg F^{-1} \leq \overline{C}(n,d)$.

Theorem 2.1 (Derksen). *$\overline{\mathrm{UB}}(n)$ implies* JC$(\mathbb{C},n)$ (*and hence these statements are equivalent by Theorem* 1.2).

*Proof.* Let $F \in \mathbb{C}[X]^n$ with $\det JF = 1$ and $\deg F \leq d$. Let $G \in \mathbb{C}[[X]]^n$ be its formal inverse and denote by $G_{(i)}$ its homogeneous component of degree $i$. We will show that $G_{(l)} = 0$ for all $l > \overline{C}(n,d)$. Therefore introduce a new variable $T$ and define

$$F^T := T^{-1}F(TX) = X + TF_{(2)} + T^2F_{(3)} + \ldots + T^{d-1}F_{(d)}$$

and

$$G^T := T^{-1}G(TX) \in \mathbb{C}[T][[X]]^n.$$

One easily verifies that $\det J_X F^T = \det JF(TX) = 1$. Furthermore,

$$F^T \circ G^T = X = G^T \circ F^T \tag{1}$$

(composition as formal power series in $X$). Now let $l > \overline{C}(n,d)$. Reducing mod $T^l$ we deduce from (1) that

$$\overline{F^T} \in \mathrm{Aut}_{\mathbb{C}_l} \mathbb{C}_l[X] \quad \text{with inverse} \quad \overline{G^T} = X + G_{(2)}\overline{T} + \ldots + G_{(l)}\overline{T}^{l-1}.$$

Also, $\det J_X \overline{F^T} = 1$ and $\deg \overline{F^T} \le d$. So by $\overline{\mathrm{UB}}(n)$ we get

$$\deg \overline{G^T} \le \overline{C}(n,d). \tag{2}$$

However, $\overline{G^T} = X + G_{(2)}\overline{T} + \ldots + G_{(l)}\overline{T}^{l-1}$. So if $G_{(l)} \neq 0$ then, using $\overline{T}^{l-1} \neq 0$, we get $\deg \overline{G^T} = l > \overline{C}(n,d)$, contradicting (2). So $G_{(l)} = 0$ for all $l > \overline{C}(n,d)$, i.e. $G$ is a polynomial map as desired. ∎

**3. Some interesting automorphisms.** Now let us return to the question: what is a natural candidate for $C(n,d)$? By Proposition 1.2 we know that in case $R$ is a reduced ring, we have $\deg F^{-1} \le d^{n-1}$ for all $F \in \mathrm{Aut}_R\, R[X]$ with $\deg F \le d$. Furthermore, the bound $d^{n-1}$ is sharp as follows easily from the example

$$(X_1 + X_2^d, X_2 + X_3^d, \ldots, X_{n-1} + X_n^d, X_n).$$

Therefore it seems reasonable to hope that $C(n,d) = d^{n-1}$. However, in January 1996 Jean-Philippe Furter found the following counterexample ([6]).

EXAMPLE 3.1. Let $n = 2$, $R = \mathbb{C}_2$ and $\varepsilon = \overline{T}$. Define

$$F = (X + \varepsilon X^3, (1 - 3\varepsilon X^2)Y + X^2).$$

Then $F \in \mathrm{Aut}_{\mathbb{C}_2} \mathbb{C}_2[X,Y]$, $\det JF = 1$, $\deg F = 3$. However, $F^{-1} = (X - \varepsilon X^3, (1 + 3\varepsilon X^2)Y - (X^2 + \varepsilon X^4))$, so $\deg F^{-1} = 4 > 3^{2-1} = 3$.

After this example was found, Furter, together with Fournié and Pinchon and much help of a computer, were able to show in [5] that $C(2,3) = 9$. In order to guess some formula for $C(2,d)$, $d \ge 3$, it would be rather interesting to know $C(2,4)$.

To get some feeling for what $C(2,d)$ might be I looked at the Jacobian equations and the formulas for $F^{-1}$ in the paper [5] and was able to give an explicit example of an automorphism of degree 3 whose inverse has the maximal degree 9.

EXAMPLE 3.2. Let $R = \mathbb{C}_7$, $\varepsilon = \overline{T}$ and define $F = (F_1, F_2)$ by

$$F_1 = X - \frac{4}{3}\varepsilon^3 X^2 - 2\varepsilon XY + \frac{64}{27}\varepsilon^6 X^3 + \frac{8}{3}\varepsilon^4 X^2 Y + 4\varepsilon^2 XY^2 + Y^3,$$
$$F_2 = Y + \frac{8}{3}\varepsilon^3 XY + \varepsilon Y^2.$$

Then $F \in \mathrm{Aut}_R\, R[X,Y]$, $\det JF = 1$, $\deg F = 3$ and $\deg F^{-1} = 9$.

Then there was another surprise: looking at this example I observed that all monomials involved an $\varepsilon$, except the terms $X + Y^3$ in $F_1$ and $Y$ in $F_2$. Consequently, if we define $F_* := F \circ (X - Y^3, Y)$, we see that $F_*$ is of the form $(X + \varepsilon(\ldots), Y + \varepsilon(\ldots))$. Then I computed $\deg F_*$ and $\deg F_*^{-1}$ and found to my own surprise that both degrees are equal (to 9)!

Of course, the above idea of constructing $F_*$ can be easily generalised: namely, let $F \in \operatorname{Aut}_{\mathbb{C}_m} \mathbb{C}_m[X]$ with $\det JF = 1$. Let $\overline{F} \in \operatorname{Aut}_{\mathbb{C}} \mathbb{C}[X]$ be obtained by reducing $F$ mod $\varepsilon$ $(= \overline{T})$. Define

$$F_* := F \circ \overline{F}^{-1}.$$

Then $F_* \in \operatorname{Aut}_{\mathbb{C}_m} \mathbb{C}_m[X]$, $\det JF_* = 1$ and $\overline{F}_* = X$, i.e. $F = X + \varepsilon(\ldots)$. Now the point is that it suffices to find a uniform bound for the degrees of all $F_*^{-1}$. More precisely, define

$\overline{\mathrm{UB}}_*(n)$ For every $d \geq 1$ there exists a positive integer $C_*(n,d)$ such that for any $\mathbb{C}$-algebra $\mathbb{C}_m$, $m \geq 1$, and any $F \in \operatorname{Aut}_{\mathbb{C}_m} \mathbb{C}_m[X]$ satisfying $\deg F \leq d$, $\det JF = 1$ and $\overline{F} = X$, the degree of $F^{-1}$ is bounded by $C_*(n,d)$, i.e. independent of $m$.

PROPOSITION 3.3. *$\overline{\mathrm{UB}}_*(n)$ implies $\overline{\mathrm{UB}}(n)$.*

*Proof.* Let $F \in \operatorname{Aut}_{\mathbb{C}_m} \mathbb{C}_m[X]$ with $\det JF = 1$ and $\deg F \leq d$. Put $F_* := F \circ \overline{F}^{-1}$. So $F^{-1} = \overline{F}^{-1} \circ F_*^{-1}$, whence

$$\deg F^{-1} \leq \deg \overline{F}^{-1} \cdot \deg F_*^{-1} \leq (\deg F)^{n-1} C_*(n, \deg F_*). \tag{3}$$

Since $F_* = F \circ \overline{F}^{-1}$ we have

$$\deg F_* \leq \deg F \cdot (\deg F)^{n-1} = (\deg F)^n. \tag{4}$$

From (3) and (4) we get

$$\deg F^{-1} \leq (\deg F)^{n-1} C_*(n, (\deg F)^n) \leq d^{n-1} C_*(n, d^n),$$

which implies $\overline{\mathrm{UB}}(n)$. ∎

After my surprising discovery of the equality of $\deg F_*$ and $\deg F_*^{-1}$, I tried to see if this was an accident. I tested Furter's example: again equalities of the degrees! Still not convinced I computed a new example in dimension 2 and degree 4. I found the following:

EXAMPLE 3.4. Let $R = \mathbb{C}_{13}$, $\varepsilon = \overline{T}$ and define $F = (F_1, F_2)$ by

$$\begin{aligned} F_1 &= X - 2\varepsilon^4 X^2 - 2\varepsilon XY - 4\varepsilon^5 X^2 Y \\ &\quad + 24\varepsilon^{12} X^4 + 16\varepsilon^9 X^3 Y + 24\varepsilon^6 X^2 Y^2 + 8\varepsilon^3 XY^3 + Y^4, \\ F_2 &= Y + 4\varepsilon^4 XY + \varepsilon Y^2 - \frac{16}{3}\varepsilon^{11} X^3 + 16\varepsilon^8 X^2 Y + 8\varepsilon^5 XY^2 + \frac{4}{3}\varepsilon^2 Y^3. \end{aligned}$$

Then $F \in \operatorname{Aut}_R R[X,Y]$, $\det JF = 1$, $\deg F = 4$ and $\deg F^{-1} = 16$.

Again I computed $\deg F_*$ and $\deg F_*^{-1}$ and found equalities of their degrees!! Of course, if this were always true, one would have $C_*(2,d) = d$, which by Proposition 3.3 and Theorem 2.1 would imply $\mathrm{JC}(\mathbb{C}, 2)$. So I made the following conjecture:

CONJECTURE B. *$C_*(2,d) = d$, i.e. if $F \in \operatorname{Aut}_{\mathbb{C}_m} \mathbb{C}_m[X,Y]$ with $\det JF = 1$ and $\overline{F} = (X,Y)$, then $\deg F^{-1} = \deg F$.*

**4. The nilpotency subgroup.** To investigate Conjecture B we will study the $\mathbb{C}_m$-automorphisms $F$ of $\mathbb{C}_m[X]$ satisfying $\overline{F} = X$. Therefore we recall some results of [4].

Let $A$ be a $\mathbb{Q}$-algebra. Then a $\mathbb{Q}$-linear map $l : A \to A$ is called *locally nilpotent* if for every $a \in A$ there exists a positive integer $q$ such that $l^q(a) = 0$. For such a map define $\exp l : A \to A$ by the formula

$$\exp l(a) := \sum_{i \geq 0} \frac{1}{i!} l^i(a).$$

If furthermore $l$ is a derivation on $A$, in which case we write $D$ instead of $l$, then $\exp D$ is a $\mathbb{Q}$-automorphism of $A$ with inverse $\exp(-D)$ (see, for example, Proposition 2.1.1 in [4]). Such an automorphism of $A$ is called an *exponential automorphism*. To decide if a given ring homomorphism $f : A \to A$ is an exponential automorphism, define $E : A \to A$ by $E := f - 1_A$.

PROPOSITION 4.1 ([4], Proposition 2.1.3). *Let $f : A \to A$ be a ring homomorphism. Then $f$ is an exponential automorphism of $A$ if and only if $E$ is locally nilpotent. Furthermore, if $E$ is locally nilpotent then the map $D : A \to A$ defined by*

$$D(a) = \sum_{i \geq 1} (-1)^{i+1} \frac{E^i(a)}{i} \quad \textit{for all } a \in A$$

*is a locally nilpotent derivation on $A$ and $f = \exp D$.*

*Proof.* (i) If $f$ is an exponential automorphism, then $f = \exp D$ for some locally nilpotent derivation $D$ on $A$. Hence $E = f - 1_A = D + D/2! + \ldots$, which readily implies that $E$ is locally nilpotent.

(ii) Conversely, suppose that $E$ is locally nilpotent. From the definition of $D$ it follows that $D$ is locally nilpotent as well. Since $D = \log(1_A + E)$ we get $\exp D = 1_A + E = f$. So it remains to show that $D$ is a derivation on $A$. Therefore observe that $\exp D = f$ implies that $\exp nD = f^n$ is a ring homomorphism for all $n \geq 1$. Then the desired result follows from

LEMMA 4.2. *Let $D : A \to A$ be a locally nilpotent $\mathbb{Q}$-linear map. Then $D$ is a derivation on $A$ if and only if $\exp nD$ is a ring homomorphism for all $n \geq 1$.*

*Proof.* (i) If $D$ is a derivation on $A$ then $nD$ is a locally nilpotent derivation on $A$, which implies that $\exp nD$ is a ring automorphism of $A$ for all $n \geq 1$.

(ii) Conversely, suppose that $\exp nD$ is a ring homomorphism for all $n \geq 1$. Let $a, b \in A$. We need to show that $D(ab) = aD(b) + D(a)b$. Therefore introduce a new variable $T$ and consider the polynomial ring $A[T]$. Extend

$D$ to a $\mathbb{Q}$-linear map on $A[T]$ by defining

$$D\Big(\sum a_i T^i\Big) = \sum D(a_i)T^i.$$

Define $a(T) = \exp TD(a)$, $b(T) = \exp TD(b)$ and $c(T) = \exp TD(ab)$. Since $\exp nD$ is a ring homomorphism for all $n \geq 1$ we deduce that $a(n)b(n) = c(n)$ for all $n \geq 1$, whence $a(T)b(T) = c(T)$. Considering the coefficient of $T$ on both sides of the last equality we get $aD(b) + D(a)b = D(ab)$, as desired. ∎

Now let $R$ be a commutative $\mathbb{Q}$-algebra. The *nilpotency subgroup* of $\mathrm{Aut}_R R[X]$, denoted by $N(R,n)$, consists of all $F$ of the form

$$(X_1 + g_1, \ldots, X_n + g_n) \tag{5}$$

where each $g_i$ is a nilpotent element of $R[X]$ or equivalently belongs to $\eta R[X]$, where $\eta$ is the nilradical of $R$.

Indeed, we will show that each map of the form described in (5) is an $R$-automorphism of $R[X]$. In fact, it turns out to be an exponential automorphism. More precisely:

Proposition 4.3 ([4], Proposition 2.1.13). *$F \in N(R,n)$ if and only if $F = \exp D$ for some locally nilpotent $R$-derivation of $R[X]$ satisfying $\overline{D} = 0$ ($\overline{D}$ is obtained from $D$ by reducing its coefficients mod $\eta$).*

*Proof.* If $F = \exp D$ with $D$ a locally nilpotent $R$-derivation satisfying $\overline{D} = 0$ then obviously $F \in N(R,n)$. Conversely, let $F \in N(R,n)$. Put $A := R[X]$ and $E := F - 1_A$. By Proposition 4.1 we need to show that $E$ is locally nilpotent. So let $a \in A$. We must prove that $E^p(a) = 0$ for some $p \geq 1$. Therefore replacing $R$ by the subalgebra of $R$ generated by all coefficients appearing in $a$ and $F$ we may assume that $R$ is noetherian and hence that $\eta^m = 0$ for some $m \geq 1$.

Now let $h \in R[X]$. Since each $g_i \in \eta R[X]$ the same holds for $E(h) = h(X_1 + g_1, \ldots, X_n + g_n) - h(X_1, \ldots, X_n)$. So

$$E(R[X]) \subset \eta R[X]. \tag{6}$$

Since $E$ is $R$-linear, applying $E$ to (6) gives $E^2(R[X]) \subset \eta^2 R[X]$.

Repeating this argument we finally arrive at $E^m(R[X]) \subset \eta^m R[X] = 0$, as desired. Finally, the formula for $D$ given in Proposition 4.1 together with (6) gives $\overline{D} = 0$. ∎

The next step is to characterize amongst the elements of $N(R,n)$ those $F$'s which satisfy $\det JF = 1$.

Theorem 4.4. *Let $F \in N(R,n)$ be of the form $\exp D$, where $D$ is a locally nilpotent $R$-derivation of $R[X]$ satisfying $\overline{D} = 0$. Then $\det JF = 1$ if and only if $\operatorname{div} D = 0$, where $\operatorname{div} D = \sum \partial_i(DX_i)$.*

The proof of this result is based on the following result of Nowicki. Let $D$ be any $R$-derivation on $R[X]$ and let $\exp TD : R[X] \to R[X][[T]]$ be defined by the usual formula

$$\exp TD(g) = \sum_{i\geq 0} \frac{T^i}{i!} D^i(g) \quad \text{for all } g \in R[X].$$

Then $\exp TD$ is a ring homomorphism (see [4], Proposition 1.2.14). To simplify notations we write $J_X(\exp TD)$ for $(\partial \exp TD(X_i)/\partial X_j)_{1\leq i,j\leq n}$.

THEOREM 4.5 (Nowicki, [7]). *Define* $B_0, B_1, \ldots$ *in* $R[X]$ *by*

$$\det J_X(\exp TD) = \sum_{p\geq 0} \frac{1}{p!} B_p T^p.$$

*Then* $B_0 = 1$ *and* $B_{p+1} = dB_p + D(B_p)$ *for all* $p \geq 0$, *where* $d := \operatorname{div} D$.

*Proof of Theorem 4.4.* (i) Suppose $d := \operatorname{div} D = 0$. Then by Nowicki's theorem $B_p = 0$ for all $p \geq 1$, whence $\det J_X(\exp TD) = 1$. So $\det J_X F = 1$.

(ii) Now assume that $F = \exp D$, where $\overline{D} = 0$ and $\det JF = 1$. Put $d = \operatorname{div} D$ and suppose that $d \neq 0$. As in the proof of Proposition 4.3 we may assume that $R$ is noetherian and $\eta^m = 0$ for some $m \geq 1$. Since $\overline{D} = 0$ and $d \neq 0$, there exists $r \geq 1$ such that $d \in \eta^r R[X] \setminus \eta^{r+1} R[X]$. By Nowicki's theorem

$$\det J_X(\exp TD) = \sum_{p\geq 0} \frac{1}{p!} B_p T^p$$

with $B_0 = 1$ and $B_{p+1} = dB_p + D(B_p)$ for all $p \geq 0$. By induction on $p$ it follows that $B_p \in \eta^{r+p-1} R[X]$ for all $p \geq 1$. Consequently,

$$1 = \det J_X(\exp D) = \sum_{p\geq 0} \frac{1}{p!} B_p = 1 + d + B \quad \text{where } B \in \eta^{r+1} R[X].$$

Hence $d = -B \in \eta^{r+1} R[X]$, a contradiction. ∎

As an immediate consequence of Theorem 2.1, Propositions 3.3 and 4.3 and Theorem 4.4 we get

THEOREM 4.6. $\mathrm{JC}(\mathbb{C}, n)$ *is equivalent to the following statement. For every* $d \geq 1$ *there exists a positive integer* $C_*(n, d)$ *such that for every* $m \geq 1$ *and every* $D \in \operatorname{Der}_{\mathbb{C}_m} \mathbb{C}_m[X]$ *with* $\operatorname{div} D = 0$ *and* $\overline{D} = 0$ *we have*: *if* $\deg \exp D \leq d$, *then* $\deg \exp(-D) \leq C_*(n, d)$.

**5. Some remarks on the two-dimensional Jacobian Conjecture.** According to Theorem 4.6, in order to understand the two-dimensional Jacobian Conjecture we need to study $\exp D$ where $D$ is a derivation on $\mathbb{C}_m[X, Y]$ satisfying $\overline{D} = 0$ and $\operatorname{div} D = 0$. It is well known that the last two conditions are equivalent to $D$ being of the form

$$D_f := f_Y \partial_X - f_X \partial_Y$$

where $f \in \mathbb{C}_m[X,Y]$ satisfies $\bar{f} = 0$. This $f$ is uniquely determined by $D$ if we assume (as we may) that $f(0,0) = 0$. Using Theorem 4.6 we get

PROPOSITION 5.1. $\mathrm{JC}(\mathbb{C},2)$ *is equivalent to the following statement*: *For every* $d \geq 1$ *there exists a positive integer* $C_*(d)$ *such that for all* $m \geq 1$ *and all* $f \in \mathbb{C}_m[X,Y]$ *with* $\bar{f} = 0$ *we have*: *if* $\deg \exp D_f \leq d$, *then* $\deg \exp(-D_f) \leq C_*(d)$.

Also, we can reformulate Conjecture B as follows:

CONJECTURE B. *Let* $m \geq 1$ *and* $f \in \mathbb{C}_m[X,Y]$ *with* $\bar{f} = 0$. *Then* $\deg \exp D_f = \deg \exp(-D_f)$.

Obviously by Proposition 5.1 a positive solution to Conjecture B would imply $\mathrm{JC}(\mathbb{C},2)$. However, in March 1998 Stefan Maubach found the first family of counterexamples to Conjecture B! A little later the following example was given by Charles Cheng:

EXAMPLE 5.2. Let $f = \varepsilon X^3 + \varepsilon^2 X^3 Y - \frac{3}{10}\varepsilon^3 X^5$, where $\varepsilon^4 = 0$. Then

$$\exp D_f = (X + \varepsilon^2 X^3, Y - 3\varepsilon X^2 - 3\varepsilon^2 X^2 Y + 3\varepsilon^3 X^4),$$
$$\exp(-D_f) = (X - \varepsilon^2 X^3, Y + 3\varepsilon X^2 + 3\varepsilon^2 X^2 Y).$$

So $\deg \exp D_f = 3$ and $\deg \exp(-D_f) = 4$.

On the other hand, this example satisfies $\deg_Y f \leq 1$. Consequently, $\exp D_f(X) \in \mathbb{C}[\varepsilon][X]$. So $\exp D_f$ belongs to the family of $R$-automorphisms $F = (F_1, F_2)$ satisfying $\det JF = 1$ and $F_1 \in R[X]$. For such $F$'s Furter showed in [6], Proposition 3, that $\deg F^{-1} \leq 4(\deg F)^4$.

To conclude this paper I present a modified version of Conjecture B.

CONJECTURE B′. *If* $\deg \exp D_f \leq d$ *then there exists* $\varphi \in \mathrm{Aut}_{\mathbb{C}_m} \mathbb{C}_m[X,Y]$ *with* $\deg \varphi \leq d$ *and such that* $\deg \exp D_{\varphi(f)} = \deg \exp -D_{\varphi(f)}$.

It is not difficult to verify that for all examples given in §3, Conjecture B′ is verified. Furthermore, the importance of this conjecture comes from the fact that it implies $\mathrm{JC}(\mathbb{C},2)$: namely, we just observe that $\exp D_{\varphi(f)} = \varphi \circ \exp D_f \circ \varphi^{-1}$ and then use an argument similar to the one given in the proof of Proposition 3.3.

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Department of Mathematics
University of Nijmegen
Toernooiveld 1
6525 ED Nijmegen, The Netherlands
E-mail: essen@sci.kun.nl

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