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A class of counterexamples to the Cancellation Problem for arbitrary rings

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Abstract. We present a class of counterexamples to the Cancellation Problem over arbitrary commutative rings, using non-free stably free modules and locally nilpotent derivations.

1. Introduction. The Cancellation Problem for algebraic varieties asks the following question.

Problem 1 (Cancellation Problem, geometric formulation). *Let V be an algebraic variety over a field k and $n \in \mathbb{N}^*$. Does $V \times k \cong k^n$ imply that $V \cong k^{n-1}$?*

This question can be reformulated as follows.

Problem 2 (Cancellation Problem, algebraic formulation). *Let B be an affine domain over a field k and $n \in \mathbb{N}$. Assume that $B[T] \cong_k k[X_1, \ldots, X_n]$. Does it then follow that $B \cong_k k[X_1, \ldots, X_{n-1}]$?*

See also the paper by Kraft ([Kra89]) for background on these and other cancellation problems in algebraic geometry. This paper considers this question not for a field k, but for an arbitrary commutative ring A.

Problem 3 (General Cancellation Problem). *Let A be a commutative ring, B an A-domain, and $n \in \mathbb{N}$. Assume that $B[T] \cong_A A[X_1, \ldots, X_n]$. Does it then follow that $B \cong_A A[X_1, \ldots, X_{n-1}]$?*

This paper shows how to construct a whole class of counterexamples to this problem.

The construction in Section 3 has two ingredients. On the one hand, it uses the existence of commutative rings A with a unimodular row $(a_1, \ldots, a_n)$ over A that cannot be completed to an invertible square matrix. In other words, it uses the existence of commutative rings A for which

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there exists a stably free module of type 1 that is not free. This was in fact also a basic ingredient in a paper by Hochster ([Hoc72]) to construct a counterexample to the Biregular Cancellation Problem. He considered the ring $\mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$ and the unimodular row $(\overline{x},\overline{y},\overline{z})$. On the other hand, our construction uses the notion of locally nilpotent derivations. Section 2 contains a brief overview of the required facts about these derivations.

2. Derivations. Let k be a field of characteristic zero and let A be a commutative k-algebra. A *k-derivation* on A is a k-linear map $D : A \to A$ satisfying the Leibniz rule, $D(ab) = a(Db) + (Da)b$ for all $a, b \in A$. It is said to be *locally nilpotent* if for all $a \in A$ there is an $n \in \mathbb{N}$ such that $D^n(a) = 0$. The kernel of such a derivation D is denoted by A^D. A *slice* of D is an element $s \in A$ such that $D(s) = 1$.

If D is locally nilpotent and $t \in A$, then we can define a map $\phi_t : A \to A$ by $\phi_t(a) := \sum_{i=0}^{\infty}(1/i!)D^i(a)t^i$. If D also has a slice s, this map can be used to easily describe the kernel of D.

PROPOSITION 4 ([Ess93], Proposition 2.1). *Let D be a locally nilpotent derivation on a finitely generated commutative k-algebra $A = k[a_1,\ldots,a_n]$. Assume that D has a slice $s \in A$. Then*

$$A^D = \phi_{-s}(A) = k[\phi_{-s}(a_1),\ldots,\phi_{-s}(a_n)].$$

PROPOSITION 5 ([Wri81], Proposition 2.1). *Let D be a locally nilpotent derivation on a commutative k-algebra A and assume that D has a slice $s \in A$. Then*

(1) $A = A^D[s]$;

(2) *s is algebraically independent over A [and therefore $A = A^D[s]$ is a polynomial ring in one variable over A^D];*

(3) $D = d/ds$.

REMARK 6. Note that if, in the above situation, A is a domain and $\operatorname{trdeg}_k Q(A)$ is finite, it follows that $\operatorname{trdeg}_k Q(A^D) = \operatorname{trdeg}_k Q(A)-1$. In particular, if A is of the form $A = B[X_1,\ldots,X_n]$ for some domain B whose quotient field is of finite transcendence degree over k and $A^D = B[F_1,\ldots,F_{n-1}]$ for certain polynomials $F_1,\ldots,F_{n-1} \in A$, then the F_i are algebraically independent over k.

For more information on locally nilpotent derivations see, for instance, [Ess00], Chapter 1, or [Now94].

3. Counterexamples. Let k be a field of characteristic zero and let A be a finitely generated commutative k-algebra without zero divisors. Let $(a_1,\ldots,a_n)$ be a unimodular row over A, say $b_1,\ldots,b_n \in A$ with $b_1a_1 +$

$\ldots + b_n a_n = 1$, and assume that it cannot be completed to an invertible square matrix.

Let $A[X]$ denote the polynomial ring $A[X_1, \ldots, X_n]$ over A in n variables. Now define a k-derivation $D : A[X] \to A[X]$ by

$$D := b_1 \frac{\partial}{\partial X_1} + \ldots + b_n \frac{\partial}{\partial X_n}.$$

This derivation is locally nilpotent and has a slice, namely $s := a_1 X_1 + \ldots + a_n X_n$. Letting $B := A[X]^D$ be the kernel of the derivation, we deduce from Proposition 5 that $A[X] = B[s]$, a polynomial ring over B in s, and from Proposition 4 that $B = A[X_1 - b_1 s, \ldots, X_n - b_n s]$.

Notation 7. For $F \in A[X]$ and $j \in \mathbb{N}$ we denote by $F_{(j)}$ the homogeneous part of F of degree j. So $F = F_{(0)} + F_{(1)} + \ldots + F_{(d)}$ where $d := \deg F$.

Lemma 8. *Let $F_1, \ldots, F_{n-1} \in A[X]$ and assume $B = A[F_1, \ldots, F_{n-1}]$. Take $f_i := F_{i(1)}$ (i.e. the linear part of F_i). Then $B = A[f_1, \ldots, f_{n-1}]$.*

Proof. "$\subseteq$" We may assume, without loss of generality, that the polynomials $F_1, \ldots, F_{n-1}$ do not have a constant term. Now consider $X_i - b_i s \in B = A[F_1, \ldots, F_{n-1}]$. Then there is a polynomial $p(T_1, \ldots, T_{n-1}) \in A[T_1, \ldots, T_{n-1}]$ such that $X_i - b_i s = p(F_1, \ldots, F_{n-1})$. Then

$$\begin{aligned} X_i - b_i s &= (p(F_1, \ldots, F_{n-1}))_{(1)} && \text{(because } X_i - b_i s \text{ is linear)} \\ &= (p_{(1)}(F_1, \ldots, F_{n-1}))_{(1)} && \text{(because } F_1, \ldots, F_{n-1} \text{ have} \\ & && \ \text{no constant term)} \\ &= p_{(1)}(F_{1(1)}, \ldots, F_{n-1(1)}) && \text{(because } p_{(1)} \text{ is linear)} \\ &= p_{(1)}(f_1, \ldots, f_{n-1}) \in A[f_1, \ldots, f_{n-1}]. \end{aligned}$$

"$\supseteq$" Because $F_i \in B = A[X]^D$, every homogeneous part $F_{i(j)}$ of F_i is also in B. In particular, $f_i \in B$. ∎

Lemma 9. *Let $f_1, \ldots, f_m \in A[X]$ be linear polynomials. Then*

$$A[f_1, \ldots, f_m] \cap AX_1 \oplus \ldots \oplus AX_n = Af_1 + \ldots + Af_m$$

[i.e. every polynomial expression $p(f_1, \ldots, f_m)$ in the f_i which is linear in the X_i is in fact an A-linear combination of the f_i].

Proof. "$\subseteq$" Take $p(T_1, \ldots, T_m) \in A[T_1, \ldots, T_m]$ and let $g := p(f_1, \ldots, f_m)$ be a polynomial expression in the f_i. Assume that g is in fact linear in the X_i. Then, using essentially the same argument as in the proof of the previous lemma, we get

$$\begin{aligned} g &= (p(f_1, \ldots, f_m))_{(1)} = (p_{(1)}(f_1, \ldots, f_m))_{(1)} \\ &= p_{(1)}(f_1, \ldots, f_m) \in Af_1 + \ldots + Af_n. \end{aligned}$$

"$\supseteq$" This is obvious. ■

LEMMA 10. *Let $f_1, \ldots, f_{n-1} \in A[X]$ be linear polynomials and assume that $B = A[f_1, \ldots, f_{n-1}]$. Then*

$$As \oplus Af_1 \oplus \ldots \oplus Af_{n-1} = AX_1 \oplus \ldots \oplus AX_n$$

[*i.e. every linear polynomial in $A[X]$ can be written in a unique way as an A-linear combination of $s, f_1, \ldots, f_{n-1}$*].

Proof. We first show that $As + Af_1 + \ldots + Af_{n-1} = AX_1 \oplus \ldots \oplus AX_n$.

"$\subseteq$" This is obvious.

"$\supseteq$" Take $g \in AX_1 \oplus \ldots \oplus AX_n$. Then $Dg \in A$ and therefore we have $D(g - (Dg)s) = Dg - (D^2g)s - (Dg)(Ds) = Dg - Dg = 0$. So

$$\begin{aligned} g - (Dg)s \in B \cap AX_1 \oplus \ldots \oplus AX_n &= A[f_1, \ldots, f_{n-1}] \cap AX_1 \oplus \ldots \oplus AX_n \\ &= Af_1 + \ldots + Af_{n-1} \qquad \text{(by Lemma 9)} \end{aligned}$$

and hence $g \in As + Af_1 + \ldots + Af_{n-1}$.

To see that $As + Af_1 + \ldots + Af_n$ is in fact a direct sum, take $\mu, \lambda_1, \ldots, \lambda_{n-1} \in A$ and assume that $\mu s + \lambda_1 f_1 + \ldots + \lambda_{n-1} f_{n-1} = 0$. Applying D to both sides yields $\mu = 0$, so $\lambda_1 f_1 + \ldots + \lambda_{n-1} f_{n-1} = 0$. The f_i, however, are even algebraically independent (by Remark 6) and therefore $\lambda_1 = \ldots = \lambda_{n-1} = 0$. ■

THEOREM 11. *We have $B \not\cong_A A[X_1, \ldots, X_{n-1}]$, even though $B[s] = A[X_1, \ldots, X_{n-1}][X_n]$.*

Proof. Assume that $B \cong_A A[X_1, \ldots, X_{n-1}]$. Then $B = A[F_1, \ldots, F_{n-1}]$ for certain polynomials $F_1, \ldots, F_{n-1} \in A[X]$ and by Lemma 8 even $B = A[f_1, \ldots, f_{n-1}]$ for certain linear polynomials $f_1, \ldots, f_{n-1} \in A[X]$. Now Lemma 10 implies that

$$As \oplus Af_1 \oplus \ldots \oplus Af_{n-1} = AX_1 \oplus \ldots \oplus AX_n,$$

say $f_i = \lambda_{i1} X_1 + \ldots + \lambda_{in} X_n$ (and $s = a_1 X_1 + \ldots + a_n X_n$). This is an equality between free A-modules of rank n and the base transformation matrix is

$$\begin{pmatrix} a_1 & \ldots & a_n \\ \lambda_{11} & \ldots & \lambda_{1n} \\ \vdots & & \vdots \\ \lambda_{n-1,1} & \ldots & \lambda_{n-1,n} \end{pmatrix}.$$

This is an invertible matrix and hence the unimodular row $(a_1, \ldots, a_n)$ has been completed to an invertible matrix, which contradicts the assumption. ■

So, every coordinate ring A of an affine variety that has a unimodular row that cannot be completed to an invertible matrix, gives rise to a counterexample to the General Cancellation Problem.

Over the real numbers, we recover Hochster's example mentioned in the Introduction. Over the complex numbers, one can consider the "generic" example $A = \mathbb{C}[a,b,c,x,y,z]/(ax+by+cz-1)$. The unimodular row $(\overline{x},\overline{y},\overline{z})$ cannot be completed to an invertible square matrix. This was shown by Raynaud [Ray68] using homological methods and, in a more general setting, by Suslin [Sus82] (Theorem 2.8) using K-theory.

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