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Some finitely generated modules and cohomologies and the Jacobian conjecture

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Abstract. We show that the plane Jacobian conjecture is equivalent to finite generatedness of certain modules.

0. Introduction. Let $f \in \mathbb{C}[\mathbb{C}^2]$ be a primitive polynomial. That is, the fiber

$$V_t := \{(x,y) \in \mathbb{C}^2 : f(x,y) = t\}$$

is an irreducible affine curve for all but a finite number of values of t. It is well known that there exists a finite set $K(f) \subset \mathbb{C}$ so that for each $t \in \mathbb{C} \setminus K(f)$ the affine curve V_t is homeomorphic to a fixed closed Riemann surface Σ punctured at $k \geq 1$ distinct points $\zeta_1, \ldots, \zeta_k$ (see e.g. [Fri]). We assume that $K(f)$ is a minimal set, i.e. for any $t \in K(f)$ the affine curve V_t is not homeomorphic to $\Sigma \setminus \{\zeta_1, \ldots, \zeta_k\}$. It is well known that f is linearizable iff $K(f)$ is an empty set. (We give a short proof of this statement for completeness.)

Associate with f the following partial differential operator:

$$L(u) = -f_y u_x + f_x u_y. \tag{0.1}$$

Here u is a holomorphic function on a domain $X \subset \mathbb{C}^2$. It is known that topological type of V_t, $t \in \mathbb{C} \setminus K(f)$, is reflected in certain properties of L. We bring the following two examples which motivate this paper. Let

$$\mathcal{F} := \mathbb{C}(f) \subset \mathbb{C}(\mathbb{C}^2), \quad \mathcal{F}[\mathbb{C}^2] \subset \mathbb{C}(\mathbb{C}^2),$$

be the field generated by f and the ring $\mathcal{F} \otimes \mathbb{C}[\mathbb{C}^2]$ respectively. Then $L(\mathcal{F}) = \{0\}$ and

$$L : \mathcal{F}[\mathbb{C}^2] \to \mathcal{F}[\mathbb{C}^2] \tag{0.2}$$

is a linear operator over $\mathcal{F}$. In [Fri] we showed that L is Fredholm with $\ker L = \mathcal{F}$ and the dimension of $\operatorname{coker} L$ is equal to the rank of $H_1(V_t, \mathbb{Z})$ for

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any regular fiber $t \notin K(f)$. Let X be a domain in $\mathbb{C}^2$ and denote by $\mathcal{O}_X$ the sheaf of germs of holomorphic functions on X. Then $H^0(X, \mathcal{O}_X)$ is the ring of holomorphic functions on X. We call X *quasi-projective* if X is $\mathbb{C}^2$ minus a finite number of affine algebraic curves. For a quasi-projective domain X let

$$\mathcal{O}_{X,r} := \mathcal{O}_X \cap \mathbb{C}(\mathbb{C}^2), \quad H^0(X, \mathcal{O}_{X,r})$$

be the sheaf of rational functions holomorphic on X and the ring of rational functions holomorphic on X respectively. We assume that X is quasi-projective whenever we use the ring $\mathcal{O}_{X,r}$. Clearly,

$$L : H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X), \tag{0.3}$$

$$L : H^0(X, \mathcal{O}_{X,r}) \to H^0(X, \mathcal{O}_{X,r}) \tag{0.3r}$$

are derivations. Let $\ker_X L$ and $\ker_{X,r} L$ be the kernels of (0.3) and (0.3r) respectively. Obviously, each of these kernels is a ring. We view $H^0(X, \mathcal{O}_X)$ (resp. $H^0(X, \mathcal{O}_{X,r})$) as a $\ker_X L$-module (resp. $\ker_{X,r} L$-module). Then L in (0.3) (resp. (0.3r)) is a $\ker_X L$-homomorphism (resp. $\ker_{X,r} L$-homomorphism). Define the following $\ker_X L$-modules and $\ker_{X,r} L$-modules respectively:

$$\begin{aligned} \mathcal{M}_X &:= H^0(X, \mathcal{O}_X)/L(H^0(X, \mathcal{O}_X)), \\ \mathcal{M}_{X,r} &:= H^0(X, \mathcal{O}_{X,r})/L(H^0(X, \mathcal{O}_{X,r})), \\ \mathcal{M} &:= \mathcal{M}_{\mathbb{C}^2,r} = \mathbb{C}[\mathbb{C}^2]/L(\mathbb{C}[\mathbb{C}^2]). \end{aligned} \tag{0.4}$$

Let

$$B := \mathbb{C} \setminus K(f), \qquad Y := f^{-1}(B) = \mathbb{C}^2 \setminus \bigcup_{t \in K(f)} V_t. \tag{0.5}$$

In [Dim] Dimca proved that $\mathcal{M}_{Y,r}$ (resp. $\mathcal{M}_Y$) is a finitely generated free $\ker_{Y,r} L$-module (resp. $\ker_{X,r} L$-module) whose rank is equal to the rank of $H_1(V_t, \mathbb{Z})$, $t \in \mathbb{C} \setminus K(f)$. The following problem arises naturally:

Problem 1. Let $f \in \mathbb{C}[\mathbb{C}^2]$ be a primitive polynomial. When is $\mathcal{M}$ a finitely generated module over $\mathbb{C}[f]$ ($= \ker_{\mathbb{C}^2,r} L$)?

We now briefly summarize the results of our paper. In §1 we show that for $f = x^m y^n$, $(m, n) = 1$, the module $\mathcal{M}$ is finitely generated. Theorem 1 claims that if $\mathcal{M}$ is a finitely generated $\mathbb{C}[f]$-module and f has no critical points, then the monodromy action on $H_1(V_t, \mathbb{Z})$, $t \notin K(f)$, is trivial. Let $B(a, R) \subset \mathbb{C}^2$ be an open Euclidean ball of radius R centered at a. We show that the results of Theorem 1 hold if $\mathcal{M}_{B(0,R)}$ is a finitely generated $\ker_{B(0,R)} L$-module for R big enough.

Assume that $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ is a polynomial map with a nonzero constant Jacobian $L(g) = \text{const} \neq 0$. (We call such a pair (f, g) a *Jacobian* *pair.*) The celebrated Jacobian conjecture claims that F is an automorphism of $\mathbb{C}^2$. See for example [B-C-W], [Dru] and [Ess]. It is known that F is a diffeomorphism iff $\mathcal{M}$ is a trivial module [Ste], [K-S]. We show that if each fiber V_t is irreducible and $\mathcal{M}$ is finitely generated then F is an automorphism. Kaliman's result shows that the plane Jacobian conjecture is equivalent to the statement that for any Jacobian pair the module $\mathcal{M}$ is a finitely generated $\mathbb{C}[f]$-module.

In §2 we show that $\mathcal{M}_X$ is isomorphic to certain first cohomology associated with (0.3). This cohomology has a Stein cover [G-R], consisting of a countable set $\{W_i\}_{i\in\mathbb{N}}$ of open sets covering $\mathbb{C}^2$. On each W_i the cohomology is trivial. Clearly, closure $B(0,R)$ can be covered by a finite cover from $\{W_i\}_{i\in\mathbb{N}}$. Thus the Jacobian conjecture is reduced to the finiteness problem of the above cohomology. It is our hope that this finiteness can be proved by a careful study of the patching of a finite Stein cover for $\overline{B}(0,R)$.

1. Finitely generated modules

LEMMA 1. *Let $f = x^m y^n$, where m,n are coprime positive integers. Then $\mathcal{M}$ is a finitely generated module over the ring $\mathcal{R} = \mathbb{C}[f]$. The number of minimal generators is mn. Moreover $\mathcal{M}$ is a free module iff $m = n = 1$.*

Proof. Clearly,

$$L(x^p y^q) = (mq - np)x^{m+p-1}y^{n+q-1}.$$

Thus $L(\mathbb{C}[\mathbb{C}^2])$ does not have monomials $x^a y^b$ of the following type:

$$\begin{gathered} a \le m-1, \qquad b \le n-1, \\ a = lm-1, \qquad b = ln-1, \qquad l = 2,\ldots \end{gathered} \tag{1.1}$$

That is, $\mathcal{M}$ is a $\mathbb{C}$-vector space generated by the vectors $x^a y^b$, with (a,b) satisfying (1.1). Note that for $l \ge 2$ we have the equality

$$x^{lm-1}y^{ln-1} = (x^m y^n)^{l-1} x^{m-1} y^{n-1}.$$

Hence the $\mathcal{R}$-module $\mathcal{M}$ is generated by nm monomials given by the first condition of (1.1). Note that the monomial $x^{m-1}y^{n-1}$ generates a free $\mathcal{R}$-submodule $\mathcal{M}$. For any other monomial $x^a y^b$ which satisfies the first condition of (1.1), we find that $fx^a y^b$ is a zero element in $\mathcal{M}$. Hence $\mathcal{M}$ is free iff $m = n = 1$. As the monomials given in the first part of (1.1) are linearly independent over $\mathbb{C}$ it follows that mn is the minimal number of generators of $\mathcal{M}$. ∎

Let $f \in \mathbb{C}[\mathbb{C}^2]$ be a primitive polynomial. Then $Y \to B$ is a fiber bundle with a fiber V_t homeomorphic to $\Sigma \setminus \{\zeta_1,\ldots,\zeta_k\}$. Therefore

$$\chi(B) = 1 - |K(f)|, \qquad \chi(\Sigma\setminus\{\zeta_1,\ldots,\zeta_k\}) = 2 - 2\,\mathrm{gen} - k,$$
$$\chi(Y) = \chi(\Sigma\setminus\{\zeta_1,\ldots,\zeta_k\})\chi(B) = (2\,\mathrm{gen} + k - 2)(|K(f)| - 1).$$

Here $\chi(W)$ is the Euler characteristic of a CW complex W and gen is the genus of Σ. If $K(f) = 0$ then $Y = \mathbb{C}^2$ and $\chi(Y) = 1$. Hence each fiber V_t is a smooth irreducible affine curve which is homeomorphic to $\mathbb{C}$. The Abhyankar–Moh theorem [A-M] implies that f is linearizable. That is, there exists a polynomial automorphism $F = (f, g)$. In this case the module $\mathcal{M}$ is trivial.

We recall some basic notions and results on the monodromy of the regular fiber V_τ, $\tau \in B$. We closely follow our exposition in [Fri]. By choosing a canonical basis in $H_1(V_\tau, \mathbb{Z})$ one obtains the representation

$$\phi : \pi_1(B, \tau) \to \operatorname{Aut} H_1(V_\tau, \mathbb{Z})$$

of the fundamental group $\pi_1(B, \tau)$ in $\operatorname{Aut} H_1(V_\tau, \mathbb{Z})$. This representation is called the *monodromy* of $H_1(V_\tau, \mathbb{Z})$. More precisely, let an element $\alpha \in \pi_1(B, \tau)$ be represented by a closed continuous path $\alpha : [0, 1] \to B$ starting at τ. Then for any given element $[\gamma]$ in the homology class of V_τ we can define a unique continuation

$$[\gamma](t) \in H_1(V_{\alpha(t)}, \mathbb{Z}), \qquad t \in [0, 1],$$
$$[\gamma](0) = [\gamma], \qquad [\gamma](1) = \phi(\alpha)(\gamma).$$

The above continuation is called the *Hurewicz connection*. The monodromy ϕ of f is called *trivial* if ϕ is a trivial homomorphism. Let $\mathcal{H}^1(V_t)$, $t \in B$, be the first regular holomorphic cohomology of V_t. Any class $[\omega] \in \mathcal{H}^1(V_t)$ is represented by a holomorphic 1-form ω on V_t, which has rational singularities in the closure of V_t. We have $[\omega] = [\theta]$ iff $\omega - \theta = df$, where f is a holomorphic function on V_t which is rational on the closure of V_t. Note that

$$\mathcal{H}^1(V_t) \sim \mathbb{C} \otimes H^1(V_t, \mathbb{Z}), \qquad t \in B.$$

Hence monodromy acts dually on $\mathcal{H}^1(V_\tau)$. Let $\mathcal{H}^1_{\mathrm{fix}}(V_\tau) \subset \mathcal{H}^1(V_\tau)$ be the subspace of all cohomology elements which are fixed by the monodromy action. Denote by $\mathrm{fix}^1(f)$ the dimension of $\mathcal{H}^1_{\mathrm{fix}}(V_\tau)$. Let $\delta(f, t)$, $t \in \mathbb{C}$, be the number of irreducible components of V_t minus 1. Clearly, $\delta(f, t) = 0$, $t \in B$. Let

$$\delta(f) := \sum_{t \in \mathbb{C}} \delta(f, t) = \sum_{t \in K(f)} \delta(f, t).$$

It is shown in [A-C-D] and in [Fri] that

$$\mathrm{fix}^1(f) = \delta(f).$$

Hence if each V_t is irreducible then $\mathrm{fix}^1(f) = 0$.

The arguments in [Fri, Lemma 3.2] yield that there exists a rational 1-form ω on $\mathbb{C}^2$, which is holomorphic on Y, such that, in Y,

$$\omega = s\,dx + t\,dy, \qquad df \wedge \omega = dx \wedge dy. \tag{1.2}$$

Moreover, if f does not have critical points then s, t can be chosen to be polynomials. It follows that for $t \in B$, any cohomology class in $\mathcal{H}^1(V_t)$ can be given by the restriction $h\omega|_{V_t}$ for some $h \in \mathbb{C}[\mathbb{C}^2]$. Moreover, for any $h \in \mathbb{C}[\mathbb{C}^2]$, $L(h)\omega|_{V_t}$ is an exact 1-form on V_t. See [Fri] or [Dim]. Hence $\mathcal{H}^1(V_t)$ is isomorphic to $\mathcal{M}_Y|_{V_t}$. The derivation d/dt of a cohomology element in $\mathcal{H}^1(V_t)$ is given by the Gauss–Manin connection, which is dual to the Hurewicz connection. It is given by the following differential map:

$$M : \mathcal{M}_Y \to \mathcal{M}_Y, \qquad M(h) = th_x - sh_y + o(\omega)h, \qquad o(\omega) = \frac{d\omega}{dx \wedge dy} = t_x - s_y. \tag{1.3}$$

Let $\gamma \subset V_t$, $t \in B$, be a smooth closed path. Let $B(t, r) \subset B$ be an open disk centered at t with a small radius r. Extend γ to a continuous family of smooth closed curves $\gamma(z) \subset V_z$, $z \in B(t, r)$. Then (1.3) yields

$$\frac{d}{dz} \int_{\gamma(z)} h\omega = \int_{\gamma(z)} M(h)\omega. \tag{1.4}$$

(In [Fri] we prove (1.4) in the special case where $\omega = dg$ and the Jacobian of the map $F = (f, g)$ is equal to 1.)

THEOREM 1. *Let $f \in \mathbb{C}[\mathbb{C}^2]$ be a polynomial without critical points. Assume that $\mathcal{M}$ is a finitely generated $\mathbb{C}[f]$-module. Then the monodromy of f is trivial.*

Proof. Let $h_1, \ldots, h_n$ be generators of $\mathcal{M}$. View $M(h_i)$ and $o(\omega)h_i$ as elements of $\mathcal{M}$. As $\mathcal{M}$ is finitely generated we get

$$M(h_i) + o(\omega)h_i = \sum_{j=1}^{n} a_{ji}(f)h_j, \qquad i = 1, \ldots, n.$$

Thus, $A(t) = (a_{ij}(t))_1^n$ is an $n \times n$ matrix with polynomial entries. Consider the following linear ODE system for $x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}}$ in a complex variable t:

$$\frac{dx}{dt} = -A(t)x, \qquad x(\tau) = \xi \in \mathbb{C}^n. \tag{1.5}$$

Since $A(t)$ is entire on $\mathbb{C}$ there exists a unique entire solution $x(t)$ of (1.5). Let $h := (h_1, \ldots, h_n)^{\mathrm{T}}$. Consider the holomorphic 1-form on $\mathbb{C}^2$,

$$\theta := \sum_{i=1}^{n} x_i(f)h_i\omega. \tag{1.6}$$

Let $\theta_t := \theta|_{V_t}$ be viewed as an element of $\mathcal{H}^1(V_t)$. Then for $t \in B$ let $d\theta_t/dt \in \mathcal{H}^1(V_t)$ be the derivative of θ_t with respect to the Gauss–Manin connection. The definition of M and (1.5) yield

$$\frac{d\theta_t}{dt} = (x'(t))^{\mathrm{T}}(h|_{V_t}) + (x(t))^{\mathrm{T}}(M(h)|_{V_t}) = -x(t)^{\mathrm{T}}A^{\mathrm{T}}(u|_{V_t}) + x(t)^{\mathrm{T}}A(t)^{\mathrm{T}}(u|_{V_t}) = 0. \tag{1.7}$$

Hence the monodromy action fixes the cohomology element θ_τ. From the proof of Lemma 3.2 in [Fri] it follows that any element in the cohomology $\mathcal{H}^1(V_\tau)$ is represented by $h\omega|_{V_\tau}$, $h \in \mathbb{C}[\mathbb{C}^2]$. As $\mathcal{M}$ is generated by $h_1, \ldots, h_n$ over the ring $\mathbb{C}[f]$, it follows that the monodromy action fixes any element in cohomology. Hence the monodromy action fixes any element in homology. ■

We do not know if the converse to Theorem 1 holds.

COROLLARY 1. *Under the assumptions of Theorem* 1,

$$\operatorname{rank} H_1(V_\tau, \mathbb{Z}) = \dim \mathcal{H}^1(V_\tau) = \operatorname{fix}^1(f) = \delta(f), \qquad \tau \in B. \tag{1.8}$$

COROLLARY 2. *Let $f \in \mathbb{C}[\mathbb{C}^2]$ be a polynomial without critical points. Assume that $\mathcal{M}$ is a finitely generated $\mathbb{C}[f]$-module and each fiber V_t, $t \in \mathbb{C}$, is irreducible. Then f is linearizable.*

Proof. If a regular fiber V_τ, $\tau \in B$, is $\mathbb{C}$ then f is linearizable [A-M]. Assume to the contrary that V_τ is not $\mathbb{C}$. Then $\mathcal{H}^1(V_\tau)$ is nontrivial contrary to Corollary 1 ($\delta(f) = 0$). ■

We now show that (1.8) holds under milder conditions. Let $D(a, r)$ be an open disk of radius r centered at a in $\mathbb{C}$. For any set $S \subset \mathbb{C}^n$ let $\overline{S}$ be the closure of S. We first establish the following lemma:

LEMMA 2. *Let $f \in \mathbb{C}[\mathbb{C}^2]$ be a polynomial without critical points. Assume that $D(a, r) \supset K(f)$. Then there exists $B(0, R)$, $R = R(r)$, with the following property. Let*

$$L(u) = 0, \qquad u \in H^0(B(0, R), \mathcal{O}_{B(0,R)}). \tag{1.9}$$

Then there exists $v \in H^0(D(a, r), \mathcal{O}_{D(a,r)})$ such that

$$u(x, y) = v(f(x, y)), \qquad \forall (x, y) \in B(0, R) \cap f^{-1}(D(a, r)). \tag{1.10}$$

Proof. Fix a point $P = (x_0, y_0)$. Since P is not a critical point of f there exists a linear function $g = bx + cy$ so that $L(g)(P) \neq 0$. Hence the polynomial map $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ is a dominating polynomial map, which is a local diffeomorphism at P. That is, there exists $\varrho_P > 0$ such that

$$F : B(P, \varrho_P) \to F(B(P, \varrho_P)) \tag{1.11}$$

is a diffeomorphism. In particular, $F(B(P, \varrho_P))$ is simply connected. Assume furthermore that ϱ_P is small enough so that there exists $\varepsilon_P > 0$ such that

$$\begin{aligned} &f(B(P, \varrho_P)) \supset D(f(P), \varepsilon_P), \\ &W_P := B(P, \varrho_P) \cap f^{-1}(D(f(P), \varepsilon_P)) \quad \text{is connected}, \\ &B(P, \varrho_P) \cap V_t = W_P \cap V_t \quad \text{is connected}, \qquad t \in D(f(P), \varepsilon_P). \end{aligned} \tag{1.12}$$

Assume that

$$L(u)=0,\quad u\in H^0(B(P,\varrho_P),\mathcal{O}_{B(P,\varrho_P)}). \tag{1.13}$$

Introducing new variables $s=f$, $t=g$ we deduce that $u_g=0$ (see e.g. [Fri]). Since each $W_P\cap V_t$ is connected for $t\in D(f(P),\varepsilon_P)$, there exists $v_P\in H^0(D(f(P),\varepsilon_P),\mathcal{O}_{D(f(P),\varepsilon_P)})$ so that

$$u(x,y)=v_P(f(x,y)),\quad (x,y)\in W_P. \tag{1.14}$$

Thus $\{W_P\}_{P\in\mathbb{C}^2}$ is an open cover of $\mathbb{C}^2$. Assume (1.9). Then for each $P\in B(0,r)$ we have the function v_P. We view the set v_P, $P\in B(0,R)$, as the set of germs of analytic functions on $E:=f(B(0,R))$. Choosing a path $\alpha:[0,1]\to B(0,R)$ and considering the family v_P along this path, we obtain an analytic continuation of v_P in E along the path $f\circ\alpha$. Since $B(0,R)$ is a semialgebraic set and V_t is an algebraic set, $B(0,R)\cap V_t$ has a finite number of connected components for any $t\in\mathbb{C}$. Suppose that $B(0,R)\cap V_t$ is a nonempty connected set. Continuing v_P along paths lying on $B(0,R)\cap V_t$ we deduce that all v_P, $P\in B(0,R)\cap V_t$, give rise to the same germ of analytic function v_t in the neighborhood of t. Let $\kappa>0$ be small enough so that

$$K_\kappa(f):=\bigcup_{t\in K(f)} D(t,\kappa)\subset D(a,r).$$

Let $S:=\overline{D}(a,r)\setminus K_\kappa(f)$. Morse theory, e.g. the arguments in [Fri, §1], yields that there exist $R\gg 1$ so that $B(0,R)\cap V_t$ is a nonempty connected set for any $t\in S$. Choose $g=bx+cy$ so that the map $F=(f,g):\mathbb{C}^2\to\mathbb{C}^2$ is proper. ($f=0$ and $g=0$ do not have common points at the line at infinity.) Assume that

$$F^{-1}(\overline{D}(a,r)\times\{0\})\subset B(0,R).$$

Let $\tau\in S$. Then $B(0,R)\cap V_\tau$ is connected, hence v defines a unique holomorphic germ v_τ. We claim that v_τ has an analytic continuation along any smooth closed path in $\alpha\subset D(a,r)$. Furthermore, this continuation terminates with the germ v_τ. Let L be the line $g=0$. Then $F|_L=f|_L$ is a proper map. View α as a closed path in $D(a,r)\times\{0\}$. Then one can lift α to L. Since $F|_L$ may have a finite number of critical points, we may have a finite number of possible continuous liftings of α, which are piecewise smooth. Let $\gamma:[0,1]\to L$ be one of these liftings. Our assumptions yield that $\gamma([0,1])\subset B(0,R)$. Continue $v_\tau=v_{\gamma(0)}$ along $\gamma([0,1])$. Since $\gamma(1)\in B(0,R)\cap V_\tau$ it follows that $v_{\gamma(1)}=v_\tau$. Hence the analytic continuation of v_τ in $D(a,r)$ gives rise to $v\in H^0(D(a,r),\mathcal{O}_{D(a,r)})$ and (1.10) holds. ■

We remark that a more careful analysis shows the validity of Lemma 2 under the assumptions that $f\in\mathbb{C}[\mathbb{C}^2]$ has a finite number of critical points. (We are not using this remark.)

THEOREM 2. *Let $f \in \mathbb{C}[\mathbb{C}^2]$ be a polynomial without critical points. Assume that $\mathcal{M}_{B(0,R)}$ is a finitely generated* $\ker_{B(0,R)}$ *L-module for each $R > 0$. Then the monodromy of f is trivial.*

Proof. Fix $D(a,r)$ which contains $K(f)$. Let S be as defined in the proof of Lemma 2. Assume that κ is small enough so that S is homotopic to B. Choose $R \gg 1$ so that $B(0,R)$ satisfies the assumptions in the proof of Lemma 2. Furthermore, for each $t \in S$, $V_t \cap B(0,R)$ is a connected set which is homeomorphic to V_t. Assume that the $\ker_{B(0,R)}$ L-module $\mathcal{M}_{B(0,R)}$ is generated by $u_1, \ldots, u_N \in H^0(B(0,R), \mathcal{O}_{B(0,R)})$. Let ω be the 1-form defined by (1.2). As in the proof of Theorem 1 it follows that $u_1\omega|_{V_\tau}, \ldots, u_N\omega|_{V_\tau}$, $\tau \in S$, span $\mathcal{H}^1_{\mathrm{hol}}(V_\tau \cap B(0,R))$, the space of holomorphic 1-forms modulo the exact forms on $V_\tau \cap B(0,R)$. We see that $\pi_1(S,\tau)$ acts on $\mathcal{H}^1_{\mathrm{hol}}(V_\tau \cap B(0,R))$ (the monodromy action). Combine Lemma 2 with the proof of Theorem 1 to deduce that this action is trivial. As $\mathcal{H}^1_{\mathrm{hol}}(V_\tau \cap B(0,R))$ is isomorphic to $\mathcal{H}^1(V_\tau)$ we deduce our theorem. ■

2. Cohomologies. Let $f \in \mathbb{C}[\mathbb{C}^2]$ be a polynomial. Let $\mathcal{P}$ be the following sheaf over $\mathbb{C}^2$: Each stalk s of $\mathcal{P}_{(x_0,y_0)}$ is given by

$$s(x,y) = \psi(f(x,y) - f(x_0,y_0)), \tag{2.1}$$

where ψ is the germ of a holomorphic function in one variable t at 0. It is straightforward to check that $\mathcal{P}$ is a sheaf.

LEMMA 3. *Let $f \in \mathbb{C}[\mathbb{C}^2]$ be a polynomial without critical points. Then the following sequence of sheaf maps on $\mathbb{C}^2$ is exact*:

$$0 \to \mathcal{P} \xrightarrow{\text{inclusion}} \mathcal{O}_{\mathbb{C}^2} \xrightarrow{L} \mathcal{O}_{\mathbb{C}^2} \to 0. \tag{2.2}$$

Proof. Clearly, $L(\psi(f(x,y) - f(x_0,y_0))) = 0$. Assume that u is a germ of a holomorphic function in (x,y) in the neighborhood of (x_0,y_0) such that $L(u) = 0$. Then the proof of Lemma 2 yields that u is in $\mathcal{P}$. That is, $\mathcal{P}$ is the kernel of L on the sheaf $\mathcal{O}_{\mathbb{C}^2}$. It is left to show that $L(\mathcal{O}_{\mathbb{C}^2}) = \mathcal{O}_{\mathbb{C}^2}$. This is equivalent to the local solution of $L(v) = u$ at (x_0,y_0) for any holomorphic germ u at (x_0,y_0). As in the proof of Lemma 2 choose $g = bx + cy$ so that $L(g)(x_0,y_0) \neq 0$. Then $L(v) = u$ becomes the equation $v_g = u/L(g)$ in the coordinates (f,g) (see e.g. [Fri]). Clearly, one can find a local solution to this equation by integrating with respect to g. Pull back by $F = (f,g)$ to obtain a local solution. ■

Let Y be a topological space with a given open cover $\mathbf{V} = \{V_i\}_{i \in I}$. Let $\mathcal{S}$ be a given sheaf. Then $\mathbf{V}$ is called an *acyclic covering* for $\mathcal{S}$ if

$$H^q(V_{i_1} \cap \ldots \cap V_{i_p}, \mathcal{S}) = 0, \quad q > 0, \text{ for any } i_1, \ldots, i_p \in I.$$

The Leray Theorem states that [G-H, 0.3]

$$H^*(\mathbf{V}, \mathcal{S}) = H^*(Y, \mathcal{S}).$$

LEMMA 4. *Let $f \in \mathbb{C}[\mathbb{C}^2]$ be a polynomial with no critical points. Then there exists a countable covering $\mathbf{U}$ of $\mathbb{C}^2$ which is acyclic for the sheafs $\mathcal{P}, \mathcal{O}_{\mathbb{C}^2}$.*

Proof. For each $P = (x_0, y_0)$ choose an open set W_P as defined in (1.12). For each integer $p \geq 1$, choose a finite cover $\mathbf{U}_p$ of $\overline{B}(0,p) \setminus B(0,p-1)$ out of the cover

$$\{W_P\}_{P \in \overline{B}(0,p) \setminus B(0,p-1)}.$$

Let $\mathbf{U} := \bigcup_{p \geq 1} \mathbf{U}_p$. Without loss of generality we may assume that each $W_P \in \mathbf{U}$ has a nonempty intersection with a finite number of elements of $\mathbf{U}$. That is, there exists a discrete countable set $T \subset \mathbb{C}^2$ so that $\mathbf{U} = \{W_P\}_{P \in T}$. As each W_P is a Stein manifold it follows that $\mathbf{U}$ is an acyclic cover for $\mathcal{O}_{\mathbb{C}^2}$. Consider W_P with $P = (x_0, y_0)$. Use the local diffeomorphism F defined in the proof of Lemma 2 to deduce that $H^*(W_P, \mathcal{P}_{W_P}) = 0$. Hence $\mathbf{U}$ is an acyclic cover for $\mathcal{P}$. ∎

COROLLARY 3. *Let $f \in \mathbb{C}[\mathbb{C}^2]$ be a polynomial with no critical points. Assume that $X \subset \mathbb{C}^2$ is a Stein manifold. Then*

$$\mathcal{M}_X \sim H^1(X, \mathcal{P}_X). \tag{2.3}$$

Proof. Consider the exact sequence of cohomology groups corresponding to the exact sequence (2.2) [G-H, 0.3], using a countable acyclic cover $\mathbf{U}_X := \{W_P\}_{P \in T(X)}$ of X as constructed in the proof of Lemma 4:

$$\begin{aligned} 0 \to H^0(X, \mathcal{P}_X) \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X) \\ \to H^1(X, \mathcal{P}_X) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to \ldots \end{aligned}$$

As the inclusion $H^0(X, \mathcal{P}_X) \to H^0(X, \mathcal{O}_X)$ is an injection and $H^1(X, \mathcal{O}_X) = 0$ we obtain (2.3). ∎

We now explain the isomorphism map in (2.3). Let $u \in H^0(X, \mathcal{O}_X)$. For each $P \in T(X)$ we find a local solution

$$L(v_P) = u, \quad v_P \in \mathcal{O}_X(W_P), \ P \in T(X). \tag{2.4}$$

Assume that $W_P \cap W_Q \neq \emptyset$. Our arguments show that

$$h_{P,Q} := v_P - v_Q \in \mathcal{P}(W_P \cap W_Q), \quad P, Q \in T(X). \tag{2.5}$$

Hence $u \in L(H^0(X, \mathcal{O}_X))$ iff the above cocycle in $H^1(\mathbf{U}_X, \mathcal{P}_X)$ is trivial. Note that the coset $u + L(H^0(X, \mathcal{O}_X))$ corresponds to the same cocycle (2.5) in $H^1(\mathbf{U}_X, \mathcal{P}_X)$. This gives the injection

$$\iota : \mathcal{M}_X \to H^1(\mathbf{U}_X, \mathcal{P}_X).$$

Corollary 3 implies that ι is surjective. That is, for a cocycle

$$h_{P,Q} \in \mathcal{P}_X(W_P \cap W_Q), \qquad P, Q \in T(X),$$

there exists $u \in H^0(X, \mathcal{O}_X)$ with local solutions (2.4) so that the above cocycle is the cocycle (2.5). Note that $H^1(X, \mathcal{P}_X)$ is an $H^0(X, \mathcal{O}_X)$-module.

Fix $R \geq 0$. Let

$$T(R) := \{P \in T : W_P \cap \overline{B}(0, R) \neq \emptyset\}. \tag{2.6}$$

As we pointed out in the proof of Lemma 4 we may assume that $T(R)$ is a finite set. Let

$$X_R := \bigcup_{P \in T(R)} W_P. \tag{2.7}$$

Then X_R is a Stein manifold with a finite acyclic cover $\{W_P\}_{P \in T(R)}$. Combine Corollary 2 with the arguments of the proof of Theorem 2 to obtain

THEOREM 3. *Let $f \in \mathbb{C}[\mathbb{C}^2]$ have no critical points. Let X_R be defined by* (2.6)–(2.7). *If $H^1(X_R, \mathcal{P}_{X_R})$ is a finitely generated module for each $R > 0$ then the monodromy of f is trivial.*

Assume that $F = (f, g)$ is a Jacobian pair with $L(g) = 1$. Then the 1-form ω defined in (1.2) is given by dg. Furthermore, M defined in (1.3) is given by

$$M := g_y \frac{\partial}{\partial x} - g_x \frac{\partial}{\partial y}.$$

As $ML = LM$ we deduce that for any domain X,

$$M : \mathcal{M}_X \to \mathcal{M}_X. \tag{2.8}$$

Without loss of generality we can assume that for each W_P, which is defined in (1.12), $F|_{W_P}$ is a diffeomorphism. The following observation may be useful in the study of the plane Jacobian conjecture:

PROPOSITION 1. *Let $F = (f, g)$ be a Jacobian pair with $L(g) = 1$. Assume that $X \subset \mathbb{C}^2$ is a Stein manifold. Then the isomorphism* (2.3) *induces the isomorphism between the action of M given by* (2.8) *and the action*

$$M : H^1(X, \mathcal{P}_X) \to H^1(X, \mathcal{P}_X) \tag{2.8$'$}$$

given by

$$h_{P,Q} \mapsto M(h_{P,Q}), \qquad h_{P,Q} \in \mathcal{P}(W_P \cap W_Q). \tag{2.8$''$}$$

Proof. Let $u \in \mathcal{P}(W_P)$. Push forward by F, take the derivative with respect to g and pull back by F to deduce that $M(u) \in \mathcal{P}(W_P)$. Similarly,

$$h_{P,Q} \in \mathcal{P}(W_P \cap W_Q) \Rightarrow M(h_{P,Q}) \in \mathcal{P}(W_P \cap W_Q).$$

Assume that the cocycle corresponding to an acyclic covering of X by a countable cover $\{W_P\}_{P\in T(X)}$ is a trivial cocycle. As $ML = LM$ it follows that

$$M(h_{P,Q}) \in \mathcal{P}(W_P \cap W_Q), \qquad P, Q \in T(X),$$

is a trivial cocycle. Hence (2.8$''$) defines the action (2.8$'$).

Corollary 3 yields that any cocycle in $H^1(X, \mathcal{P}_X)$ is of the form (2.5), where each v_P, $P \in T(X)$, satisfies (2.4). Clearly,

$$M(u) = M(L(v_P)) = L(M(v_P)), \qquad P \in T(X).$$

Corollary 3 yields that the cocycle

$$M(v_P - v_Q) \in \mathcal{P}(W_P \cap W_Q), \qquad P, Q \in T(X),\ P \neq Q,$$

determines the unique coset $M(u) + L(H^0(X, \mathcal{O}_X))$. Hence the action (2.8) is isomorphic to the action (2.8$'$). ∎

Note that for any $s \in \mathcal{P}_{(x_0,y_0)}$ of the form (2.1) we have

$$M(s) = \psi'(f(x,y) - f(x_0,y_0)).$$

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