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On the span invariant for cubic similarity

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Abstract. Given a real $n \times n$ matrix A, we make some conjectures and prove partial results about the range of the function that maps the n-tuple x into the entrywise kth power of the n-tuple Ax. This is of interest in the study of the Jacobian Conjecture.

1. Introduction. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a map of the following form:

$$F_A(x) := x + (Ax)^{*3},$$

where A is a real $n \times n$ matrix and "$*$" means entrywise power. We say that A is a *Drużkowski matrix* when $\det(\operatorname{Jac} F) = 1$ (cf. e.g. [1]). This paper is the result of trying to understand the image of the map $x \mapsto (Ax)^{*3}$, or, more generally, the image of the map $x \mapsto (Ax)^{*k}$, where $k \in \mathbb{N}$ and A is any square matrix. More precisely, we focus on the "span" of these maps, by which we mean the smallest linear subspace that contains the range of the maps.

We say that two real $n \times n$ matrices A, B are *cubic-similar* if there exists an invertible $n \times n$ matrix T such that

$$F_B(x) = T^{-1} F_A(Tx) \quad \forall x \in \mathbb{R}^n.$$

It is easy to see that cubic similarity is an equivalence relation. Some effort has been made to classify all Drużkowski matrices in low dimension with respect to cubic similarity (cf. [4, 5]). To this end some *invariants of cubic similarity* have been used (cf. e.g. [6]). The dimension of the span of the map $x \mapsto (Ax)^{*3}$,

$$\dim \operatorname{span}(x \mapsto (Ax)^{*3}),$$

is an integer that is easily seen to be such an invariant, but it seems that it has not been used so far.

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Our investigation of this "span invariant" has led us to the following conjecture.

Conjecture 1. *For any square matrix A and integer $k \geq 1$,*

$$\operatorname{span}(x \mapsto (Ax)^{*k}) = \operatorname{range}(AA^T)^{*k}.$$

Since $\operatorname{range} A = \operatorname{range} AA^T$ for all real matrices, by well known elementary facts in linear algebra Conjecture 1 is equivalent to the following one.

Conjecture 2. *For any symmetric positive semidefinite matrix $B \in \mathcal{L}(\mathbb{R}^n)$ and integer $k \geq 1$,*

$$\operatorname{span}(x \mapsto (Bx)^{*k}) = \operatorname{range} B^{*k}.$$

(*The inclusion $\supseteq$ is obvious.*)

We can also write the formula as

$$\operatorname{span}(x \mapsto (Bx)^{*k}) = \operatorname{span}(x \mapsto B^{*k}x),$$

because for a linear map the span and the range coincide. Then a way to state our conjecture is that when B is symmetric and positive semidefinite, $(Bx)^{*k}$ and $B^{*k}x$ are the same as far as the span is concerned.

These conjectures were checked on thousands of random integer matrices with n between 3 and 6 and k between 2 and 5. No counterexample was found. We will show that *the conjectures are true for $n = 2$ and $n = 3$ and arbitrary k*. Our main tool will be the properties of the Kronecker (a.k.a. tensor) product of matrices (see [3], Section 4.2). We are grateful to Prof. Friedland for his tip in the right direction.

2. Examples. One may wonder why the transpose A^T appears in Conjecture 1. Consider the matrix

$$A := \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Then both A and A^{*2} have rank 2, while $(AA^T)^{*2}$ has rank 3. This shows that the (true) inclusion

$$\operatorname{span}(x \mapsto (Ax)^{*2}) \supseteq \operatorname{range} A^{*2}$$

can be strict if A is not symmetric and positive semidefinite.

The symmetric matrix

$$B := \begin{pmatrix} 5 & 2 & 4 \\ 2 & 8 & 4 \\ 4 & 4 & 4 \end{pmatrix}$$

is positive semidefinite and of rank 2, but B^{*2} is positive definite. This shows that the inequality $\operatorname{rank} B^{*k} \geq \operatorname{rank} B$ can be strict.

The 4×4 matrix

$$A := \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 4 & -1 \\ 1 & 1 & 3 & 0 \end{pmatrix}$$

has rank 2, no two rows (or columns) are parallel, and the span of $x \mapsto (Ax)^{*2}$ coincides with the range of $(AA^T)^{*2}$ and has dimension 3. Similarly the 5×5 matrix

$$A := \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 2 & 1 \end{pmatrix}$$

has rank 2, no two rows (or columns) are parallel, and the span of $x \mapsto (Ax)^{*3}$ coincides with the range of $(AA^T)^{*3}$ and has dimension 4. Thus it is not true that the dimension of the span of $x \mapsto (Ax)^{*k}$ is the same as the number of rows (or columns) that are left if we delete the zero rows (or columns) and identify any two remaining rows (or columns) that are parallel to each other.

It is not true that if B is symmetric and with positive determinant then B^{*k} necessarily has positive determinant. For example the symmetric matrix

$$B := \begin{pmatrix} 31 & 24 & 48 \\ 24 & 12 & 48 \\ 48 & 48 & 48 \end{pmatrix}$$

has positive determinant but B^{*2} has negative determinant.

3. A bound on the dimension of the span

Definition. Given $x, y \in \mathbb{R}^n$ (instead of $\mathbb{R}$ any other field is fine) let us define the binary operation

$$x * y := (x_1y_1, x_2y_2, \dots, x_ny_n), \qquad \text{where } x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n).$$

This "multiplication" is of course commutative, associative, and also distributive with respect to the ordinary sum: $x * (y + z) = x * y + x * z$ for all $x, y, z \in \mathbb{R}^n$. We can also define the power with integer exponent by induction as

$$x^{*0} := (1, 1, \dots, 1), \qquad x^{*(k+1)} := x^{*k} * x.$$

Proposition 3. *Let A be a matrix of rank r. Then* $\mathrm{span}(x \mapsto (Ax)^{*k})$ *is generated by all "monomials" $x_{(1)}^{*i_1} * x_{(2)}^{*i_2} * \dots * x_{(r)}^{*i_r}$ where the column vectors $x_{(j)}$ are a basis of the range of A and the indices $i_j \geq 0$ range over* *all r-tuples of integers such that $i_1 + \ldots + i_r = k$. As a consequence*,

$$\dim \operatorname{span}(x \mapsto (Ax)^{*k}) \le \binom{k-1+\operatorname{rank} A}{k}$$

(*binomial coefficient*).

Proof. Consider the mapping $\varphi_k : \mathbb{R}^n \to \mathbb{R}^n$ defined as $\varphi_k(x) := x^{*k}$. We claim that the image under φ_k of an r-dimensional linear subspace of $\mathbb{R}^n$ is contained in a linear subspace of dimension at most $\binom{k-1+r}{r}$. If this is proved, then we only need to apply it with the range of A as the linear subspace.

Let $x_{(1)}, \ldots, x_{(r)}$ be a base of the subspace and $\lambda_1, \ldots, \lambda_r$ be scalars. Let us expand the value of φ_k on the linear combination of the vectors through Leibniz's formula:

$$\begin{aligned}(Ax)^{*k} &= (\lambda_1 x_{(1)} + \ldots + \lambda_k x_{(r)})^{*k} \\ &= \sum_{i_1+\ldots+i_r=k,\, i_j \ge 0} \underbrace{c_{i_1\ldots i_r} \cdot \lambda_1^{i_1} \ldots \lambda_r^{i_r}}_{\text{scalar}} \cdot \underbrace{x_{(1)}^{*i_1} * \ldots * x_{(r)}^{*i_r}}_{\text{vector}}\end{aligned}$$

(where $c_{i_1\ldots i_r}$ is a scalar that just happens to be $k!/(i_1! \ldots i_r!)$). Hence $(Ax)^{*k}$ is a linear combination of vectors from the set

$$\{x_{(1)}^{*i_1} * \ldots * x_{(r)}^{*i_r} \mid i_1 \ldots, i_r \ge 0,\ i_1 + \ldots + i_r = k\}.$$

These vectors are at most as many as the number of possible monomials of degree k in r variables, or, what is the same, the number of r-tuples $(i_1, \ldots, i_r) \in \mathbb{N}^r$ such that $i_1 + \ldots + i_r = k$. It is well known that this number is $\binom{r+k-1}{k}$. ■

LEMMA 4. *If B is symmetric and positive semidefinite with rank r, then there exists an invertible principal minor of dimension r.*

Proof. Let B be $n \times n$, and C be a matrix whose $n - r$ columns form a basis of the kernel of B. Up to a reshuffling of the coordinates, we can assume that the last $n - r$ rows of C form an invertible submatrix. The quadratic form $\varphi(x) := x \cdot Bx$ vanishes only on the range of C. Hence its restriction to the subspace of the first $n-r$ coordinates is positive definite. ■

PROPOSITION 5. *If A, B are symmetric and positive definite matrices of the same dimension, then $A * B$ is also positive definite.*

Proof. Consider the mapping $A \otimes B$ of $\mathbb{R}^n \otimes \mathbb{R}^n$ into itself defined as $(A \otimes B)(x \otimes y) := (Ax) \otimes (By)$ on the tensors $x \otimes y$, and extended by linearity to the whole of $\mathbb{R}^n \otimes \mathbb{R}^n$. Let e_i be the ith element of the canonical basis of $\mathbb{R}^n$. If on $\mathbb{R}^n \otimes \mathbb{R}^n$ we fix the basis $\{e_i \otimes e_j : i, j = 1, \ldots, n\}$, the matrix representing the linear map $A \otimes B$ is called the Kronecker product of the two matrices A and B (cf. e.g. [3]). In terms of components, it is the $n^2 \times n^2$ matrix C defined as

$$c_{n(i-1)+h,n(j-1)+k} = a_{i,j} b_{h,k} \quad \text{for } h,k = 1,\dots,n,$$

or, equivalently, as

$$c_{r,s} := a_{i,j} b_{h,k},$$

where

$$i := \left\lfloor \frac{r-1}{n} \right\rfloor + 1, \quad h := r - n(i-1),$$
$$j := \left\lfloor \frac{s-1}{n} \right\rfloor + 1, \quad k := s - n(j-1).$$

It is known ([3], Corollary 4.2.13) that if A and B are symmetric and positive definite, then also C is symmetric and positive definite. Now note that $A*B$ can be obtained from $A\otimes B$ by deleting all rows and all columns except those of positions $0+1$, $n+2$, $2n+3$, $\dots$,$(n-1)n+n$. In other words, $A*B$ is the matrix representing the restriction of $A\otimes B$ to the subspace with basis $\{e_i \otimes e_i : i = 1,\dots,n\}$. Thus $A*B$ is a principal minor of the symmetric positive definite matrix C. We conclude that $A*B$ is positive definite. ∎

PROPOSITION 6. *If B is symmetric and positive semidefinite, then*

$$\operatorname{rank} B^{*k} \ge \operatorname{rank} B.$$

Equality holds whenever either B is invertible or when $\operatorname{rank} B = 1$.

Proof. Consider an invertible principal minor of B of dimension $\operatorname{rank} B$. In particular the minor is symmetric and positive definite. By taking the kth powers of its entries we get a symmetric positive definite minor of B^{*k}. Hence the rank of B^{*k} is not less than the rank of B. ∎

Summing up, for the dimension of the span the following inequalities hold for any $n\times n$ real matrix A:

$$\operatorname{rank} AA^T \le \operatorname{rank}(AA^T)^{*k} \le \dim\operatorname{span}(x \mapsto (Ax)^{*k}) \le \binom{k-1+\operatorname{rank} A}{k}.$$

4. Dimensions 2 and 3

PROPOSITION 7. *If a real 2×2 matrix B is symmetric and positive semidefinite then for all $k \in \mathbb{N}$,*

$$\operatorname{span}(x \mapsto (Bx)^{*k}) = \operatorname{range} B^{*k}.$$

Proof. If B has rank 2 then B^{*k} is invertible. If B has rank 1 it is obvious that both the span and the range of B^{*k} have dimension 1. ∎

PROPOSITION 8. *Suppose that $B \in \mathcal{L}(\mathbb{R}^3)$ is symmetric, positive semidefinite, of rank 2, and no column is a multiple of another. Then for all $k \ge 2$*

the matrix B^{*k} *is invertible*, *and*

$$\operatorname{span}(x \mapsto (Bx)^{*k}) = \operatorname{range} B^{*k}.$$

Proof. Upon rearranging the coordinates we can assume that

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & \lambda b_{1,1} + \mu b_{1,2} \\ b_{1,2} & b_{2,2} & \lambda b_{1,2} + \mu b_{2,2} \\ \lambda b_{1,1} + \mu b_{1,2} & \lambda b_{1,2} + \mu b_{2,2} & \lambda^2 b_{1,1} + 2\lambda\mu b_{1,2} + \mu^2 b_{2,2} \end{pmatrix},$$

where $b_{1,1} > 0$, $b_{2,2} > 0$ and $b_{1,1}b_{2,2} - b_{1,2}^2 > 0$. Then

$$\det B^{*2} = -2(b_{1,2}^2 - b_{1,1}b_{2,2})^3\lambda^2\mu^2.$$

When $\lambda\mu \neq 0$, B^{*2} is invertible, and our claim is proved for $k = 2$. Suppose that it is true for a given $k \geq 2$. Then $B^{*(k+1)}$ is the principal submatrix with indices 1, 4, 6 of the Kronecker product $B^{*k} \otimes B$, which is symmetric and positive semidefinite. The kernel of $B^{*k} \otimes B$ is given by the set of the Kronecker products $(v_1, v_2, v_3) \otimes (\lambda, \mu, -1)$, where (v_1, v_2, v_3) ranges over all of $\mathbb{R}^3$. In components,

$$(v_1, v_2, v_3) \otimes (\lambda, \mu, -1) = (v_1\lambda,\ v_2\lambda,\ v_3\lambda,\ v_1\mu,\ v_2\mu,\ v_3\mu,\ -v_1,\ -v_2,\ -v_3).$$

Now the kernel of the principal submatrix $B^{*(k+1)}$ is the intersection of the kernel of the full matrix $B^{*k} \otimes B$ with the subspace of components 1, 4 and 6. But $v_2\lambda = v_3\lambda = v_1\mu = v_3\mu = -v_1 = -v_2 = 0$ if and only if $v_1 = v_2 = v_3 = 0$, because $\lambda\mu \neq 0$. Hence $B^{*(k+1)}$ is invertible. Finally, from the general inclusions

$$\mathbb{R}^3 \supseteq \operatorname{span}(x \mapsto (Bx)^{*k}) \supseteq \operatorname{range} B^{*k}$$

and since B^{*k} is invertible, we deduce that the inclusions are actually equalities. ∎

Proposition 9. *If* $B \in \mathcal{L}(\mathbb{R}^3)$ *is symmetric and positive semidefinite then for all* $k \geq 2$ *we have*

$$\operatorname{span}(x \mapsto (Bx)^{*k}) = \operatorname{range} B^{*k}.$$

Proof. The only case left is when $\operatorname{rank} B = 2$ and one column is a multiple of another. Upon rearranging the coordinates we can assume that

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & \lambda b_{1,1} \\ b_{1,2} & b_{2,2} & \lambda b_{1,2} \\ \lambda b_{1,1} & \lambda b_{1,2} & \lambda^2 b_{1,1} \end{pmatrix},$$

where $b_{1,1} > 0$, $b_{2,2} > 0$ and $b_{1,1}b_{2,2} - b_{1,2}^2 > 0$. Let $b_{(1)}, b_{(2)}$ be the first two columns of B. Recall that $\operatorname{span}(x \mapsto (Bx)^{*k})$ is generated by the "monomials" $b_{(1)}^{*r} * b_{(2)}^{*(k-r)}$ where $0 \leq r \leq k$. We want to prove that all these monomials are actually linear combinations of the two vectors $b_{(1)}^{*k}$ and $b_{(2)}^{*k}$.

In fact, since the submatrix

$$\widehat{B} := \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{1,2} & b_{2,2} \end{pmatrix}$$

is symmetric and positive definite, also $\widehat{B}^{*k}$ is invertible, and there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \begin{pmatrix} b_{1,1} \\ b_{1,2} \end{pmatrix}^{*k} + \beta \begin{pmatrix} b_{1,2} \\ b_{2,2} \end{pmatrix}^{*k} = \begin{pmatrix} b_{1,1} \\ b_{1,2} \end{pmatrix}^{*r} * \begin{pmatrix} b_{1,2} \\ b_{2,2} \end{pmatrix}^{*(k-r)}.$$

If we multiply the first row of this identity by λ^k we get

$$\alpha(\lambda b_{1,1})^k + \beta(\lambda b_{1,2})^k = (\lambda b_{1,1})^r (\lambda b_{1,2})^{r-k}.$$

Combining the two equalities we get $\alpha b_{(1)}^{*k} + \beta b_{(2)}^{*k} = b_{(1)}^{*r} * b_{(2)}^{*(k-r)}$, which is what we needed. ∎

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