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Extension of separately holomorphic functions—a survey 1899–2001

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Dedicated to Professor Józef Siciak in honour of his 70th birthday

Abstract. This note is an attempt to describe a part of the historical development of the research on separately holomorphic functions.

1. Introduction. First, let us fix some notations we will need in this survey article. Let $N \in \mathbb{N}$, and let $A_j \subset D_j \subset \mathbb{C}^{k_j}$, D_j a domain, $j = 1, \dots, N$. The set

$$X := \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N) := \bigcup_{j=1}^{N} A_1 \times \dots \times A_{j-1} \times D_j \times A_{j+1} \times \dots \times A_N$$

is called the *N-fold cross* associated to the N pairs (A_j, D_j).

Observe that different pairs may lead to the same cross set; e.g. if $N-1$ of the A_j's coincide with the corresponding D_j's, then $X = D_1 \times \dots \times D_N$.

Moreover, let $M \subset X$ ($M = \emptyset$ is allowed). For $(a_1, \dots, a_N) \in A_1 \times \dots \times A_N$ and $1 \le j \le N$, we define the *fiber* of M over $(a_1, \dots, \widehat{a}_j, \dots, a_N)$ (1) as

$$M_{(a_1,\dots,\widehat{a}_j,\dots,a_N)} := \{z_j \in D_j \colon (a_1, \dots, z_j, \dots, a_N) \in M\}.$$

We will always assume that all the fibers $M_{(a_1,\dots,\widehat{a}_j,\dots,a_N)}$ are closed in D_j.

Our aim is to study separately holomorphic functions. Recall that a function

$$f \colon \mathbb{X}(A_1 \times \dots \times A_N; D_1 \times \dots \times D_N) \setminus M \to \mathbb{C}$$

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(1) $(a_1, \dots, \widehat{a}_j, \dots a_N) := (a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_N)$.

is called *separately holomorphic on* $X \setminus M$ if

$$\forall_{(a_1,\dots,a_N)\in A_1\times\dots\times A_N}\ \forall_{1\le j\le N}:\quad f(a_1,\dots,a_{j-1},\cdot,a_{j+1},\dots,a_N)\in\mathcal{O}(D_j\setminus M_{(a_1,\dots,\widehat{a}_j,\dots,a_N)}).$$

We will write $f\in\mathcal{O}_{\mathrm{s}}(X\setminus M)$.

Problem. Let N, A_j, D_j, and M be as above. We are interested in the following question: Under what conditions on these sets there exist a (pseudoconvex) domain $\widehat{X}\subset\mathbb{C}^{k_1+\dots+k_N}$, $X\subset\widehat{X}$, and a relatively closed (in $\widehat{X}$) set $\widehat{M}\subset\widehat{X}$, $\widehat{M}\cap X\subset M$, such that

$$\forall_{f\in\mathcal{O}_{\mathrm{s}}(X\setminus M)}\ \exists_{\widehat{f}\in\mathcal{O}(\widehat{X}\setminus\widehat{M})}:\quad \widehat{f}|_{X\setminus M}=f,\ \widehat{f}\text{ uniquely determined.}$$

After having fixed the main notions, I wish to invite the reader for a trip over the last 100 years in order to show how the problem of separate holomorphicity has been developed from its very beginning at the end of the 19th century up to the present.

2. First period 1899–1967 characterized by $N=2$ and $A_1=D_1$, i.e. $\mathbb{X}(A_1,A_2;D_1,D_2)=D_1\times D_2$. At the end of the 19th century due to the Cauchy integral representation the following equivalence was well known:

Theorem 1. *Let $D\subset\mathbb{C}^n$ be a domain and let $f\colon D\to\mathbb{C}$. Then the following properties are equivalent*:

(1) *f is complex differentiable at any point of D*;
(2) *f is locally given by a convergent power series*;
(3) *$f\in\mathcal{C}(D)\cap\mathcal{O}_{\mathrm{s}}(D)$.*

Here $\mathcal{O}_s(D)$ is understood in the usual sense, i.e. for any $a\in D$ and $j\in\{1,\dots,n\}$ the function $f(a_1,\dots,a_{j-1},\cdot,a_{j+1},\dots,a_n)$ is holomorphic near a_j.

For the situation in (1) or (2) we write, as usual, $f\in\mathcal{O}(D)$.

The first result dealing with separately holomorphic functions was the following one (cf. [Osg 1899]).

Theorem 2 (Osgood (1899)). *Let $D\subset\mathbb{C}^n$ be a domain. If $f\in\mathcal{O}_{\mathrm{s}}(D)$ is locally bounded, then $f\in\mathcal{O}(D)$.*

Sketch of proof. Using the Schwarz Lemma coordinatewise it follows that f is continuous. Hence Theorem 1 implies that f is holomorphic. ∎

Moreover, based on Baire's theorem Osgood showed the following result (cf. [Osg 1900]).

Theorem 3 (Osgood (1900)). *Let $f\in\mathcal{O}_{\mathrm{s}}(\mathbb{X}(D_1,D_2;D_1,D_2))$, where $D_j\subset\mathbb{C}$ are arbitrary domains. Then*

$$\forall_{(a_1,a_2)\in D_1\times D_2,\, r>0,\, D_1'\Subset D_1,\, D_2'\Subset D_2}\ \exists_{U_j\subset\mathbb{B}(a_j,r)\cap D_j,\, j=1,2}:$$
$$\text{the } U_j \text{ are open and } f \text{ is bounded on } U_1\times D_2'\cup D_1'\times U_2,$$

and therefore, $f\in\mathcal{O}(U_1\times D_2'\cup D_1'\times U_2)$.

COROLLARY 4. *If f is as above, then there exists an open and dense subset Ω of $D_1\times D_2$ such that $f|_\Omega\in\mathcal{O}(\Omega)$.*

In the same paper the following Casorati–Weierstrass type result can be found.

THEOREM 5 (Osgood (1900)). *Let the set Ω in Corollary 4 be chosen maximal and assume that $\Omega\neq D_1\times D_2$. Then for every $a\in\partial\Omega\cap(D_1\times D_2)$ we have*

$$\forall_{\alpha\in\mathbb{C},\,\varepsilon>0,\,\delta>0}\ \exists_{z\in D_1\times D_2}:\quad |z-a|<\delta,\ |f(z)-\alpha|<\varepsilon.$$

REMARK. In his second note Osgood already mentioned that in order to get $\Omega=D_1\times D_2$ it suffices to prove the following statement:

(∗) if $f\in\mathcal{O}(\Delta_0(1)\times\Delta_0(1))$ ([2]) and if for some $R>1$ the function $f(a_1,\cdot)$ belongs to $\mathcal{O}(\Delta_0(0,R))$ for all $a_1\in\Delta_0(1)$, then $f\in\mathcal{O}(\Delta_0(1)\times\Delta_0(R))$.

Indeed, the next step was based on exactly the above remark by Osgood; it is done in the work of Hartogs (cf. [Har 1906]).

THEOREM 6 (Hartogs (1906)). (a) (∗) *is true, and therefore,*

$$\mathcal{O}_{\mathrm{s}}(\mathbb{X}(D_1,D_2;D_1,D_2))=\mathcal{O}(D_1\times D_2)$$

whenever $D_j\subset\mathbb{C}$ is a domain, $j=1,2$.

(b) *Let $D\subset\mathbb{C}^n$ be a domain. Then $\mathcal{O}(D)=\mathcal{O}_{\mathrm{s}}(D)$.*

Moreover, if $D_j\subset\mathbb{C}^{k_j}$ is a domain, $j=1,2$, and if $A_1=D_1$ and A_2 is an open subset of D_2, then

$$\mathcal{O}_{\mathrm{s}}(\mathbb{X}(A_1,A_2;D_1,D_2))=\mathcal{O}(D_1\times D_2).$$

Sketch of proof. The main ingredients in the proof of the first part of (a) are nowadays called Hartogs' series and Hartogs' Lemma:

Let $(u_j)_j\subset\mathcal{PSH}(D)$, D a domain in $\mathbb{C}^n$, be a sequence of plurisubharmonic functions, locally bounded from above. Assume that $\limsup_{j\to\infty}u_j\le C$. Then

$$\forall_{\varepsilon>0,\,K\Subset D}\ \exists_{j_0}\ \forall_{j\ge j_0,\,z\in K}:\quad u_j(z)\le C+\varepsilon.$$

To get the second part of (a) use locally Theorem 3 in order to fall in the situation described by (∗). Finally, the full statement in (b) is shown by induction. ∎

([2]) For $a\in\mathbb{C}^n$ and $r>0$ we put $\Delta_a(r):=\{z\in\mathbb{C}^n: |z_j-a_j|<r,\ j=1,\dots,n\}$; $\Delta_a(r)$ is the *polycylinder* with center a and radius r.

REMARK. To be historically correct, I should mention that already in 1911 Bernstein discussed the following general 2-fold cross situation (cf. [Ber 1912]):

$$D_1 = D_2 = \text{ellipse with foci } 1, -1, \quad A_1 = A_2 = [-1, 1], \text{ and}$$
$$f \in \mathcal{O}_{\mathrm{s}}(\mathbb{X}(A_1, A_2; D_1, D_2)) \text{ bounded.}$$

It seems that this result had not been recognized for a long time until a paper by Akhiezer and Ronkin (cf. [Akh-Ron 1973], see also [Ron 1977]).

SUMMARY. *So far we have discussed the situation $A_1 = D_1$ and $A_2 \subset D_2$ open.*

The next step in the development started in 1930 with a paper by Hukuhara (in Japanese) (cf. [Huk 1930]).

THEOREM 7 (Hukuhara (1930)). *Let $D_j \subset \mathbb{C}$ be a domain, $j = 1, 2$, and let $A_1 = D_1$; assume that $A_2 \subset D_2$ has at least one accumulation point in D_2. If $f \in \mathcal{O}_{\mathrm{s}}(\mathbb{X}(A_1, A_2; D_1, D_2))$ is locally bounded, then $f \in \mathcal{O}(D_1 \times D_2)$.*

Sketch of proof. Exploiting the theorem of Montel–Vitali leads to the fact that $f \in \mathcal{C}(D_1 \times D_2)$. Then the Cauchy integral gives another holomorphic function that coincides with $f(a_1, \cdot)$ on a dense set. Hence $f \in \mathcal{O}_{\mathrm{s}}(D_1 \times D_2)$. Applying Theorem 1 finishes the proof. ∎

According to Terada (cf. [Ter 1967]), Hukuhara asked the following question:

PROBLEM OF HUKUHARA. Let $N = 2$, $D_j \subset \mathbb{C}^{k_j}$, $A_1 = D_1$, and $A_2 \subset D_2$ arbitrary. What conditions on A_2 guarantee that

$$\mathcal{O}_{\mathrm{s}}(\mathbb{X}(A_1, A_2; D_1, D_2)) = \mathcal{O}(D_1 \times D_2)\ ?$$

It took another 30 years before Shimoda came back to that problem (cf. [Shi 1957]). He proved a result analogous to the one of Osgood.

THEOREM 8 (Shimoda (1957)). *Assume that D_j, A_j are as in the theorem of Hukuhara. Let $f \in \mathcal{O}_{\mathrm{s}}(\mathbb{X}(A_1, A_2; D_1, D_2))$. Then*

$$\forall_{a_1 \in D_1,\, r>0}\ \forall_{D_2' \Subset D_2 \text{ containing accumulation points of } A_2}$$
$$\exists_{U_1 \subset \Delta_{a_1}(r) \cap D_1}: \quad U_1 \text{ is open and } f \text{ is bounded on } U_1 \times D_2',$$

and therefore (using Hukuhara), $f \in \mathcal{O}(U_1 \times D_2')$.

Sketch of proof. Use the theorems of Baire and Montel–Vitali. ∎

COROLLARY 9. *Let A_1, A_2, D_1, D_2, and f be as in Theorem 8. Then there is an open and dense subset $\Omega \subset D_1 \times D_2$ such that $f|_\Omega \in \mathcal{O}(\Omega)$.*

The next important step was done by Terada (cf. [Ter 1967] and [Ter 1972]), who was finally able to answer the question raised by Hukuhara.

THEOREM 10 (Terada (1967, 1972)). (a) *Let* $N = 2$, $D_j \subset \mathbb{C}^{k_j}$ *be domains*, $A_1 = D_1$, *and let* $A_2 \subset D_2$ *be non-pluripolar* ([3]). *Then*

$$\mathcal{O}_{\mathrm{s}}(\mathbb{X}(A_1, A_2; D_1, D_2)) = \mathcal{O}(D_1 \times D_2).$$

(b) *Let* $A_1 = D_1 = \Delta_0(1) \subset \mathbb{C}$. *Let* $D_2 \subset \mathbb{C}^{k_2}$ *be a domain of holomorphy and* $A_2 \subset D_2$ *be pluripolar with* $A_2 = \bigcup_{j=1}^{\infty} A_{2,j}$, $A_{2,j}$ *compact. Then*

$$\exists f \in \mathcal{O}_{\mathrm{s}}(\mathbb{X}(D_1, A_2; D_1, D_2)) \setminus \mathcal{O}(D_1 \times D_2).$$

The proof of Theorem 10 is based on Baire's theorem, Hukuhara's idea, the fact that negligible sets are of zero measure, and the Hartogs theorem.

REMARK. (b) shows that the condition in (a) for the set A_2 to be non-pluripolar is almost optimal.

SUMMARY. *So far we have discussed the situation* $A_1 = D_1$ *and* $A_2 \subset D_2$ *arbitrary.*

To conclude the discussion of the first period and to have some link to Kraków I wish to mention a new proof of the Hartogs theorem given by Leja based on his so-called polynomial lemma.

THEOREM 11 (Leja (1933, 1950)). (a) [Lej 1933] *Let* $K \subset \mathbb{C}$ *be a continuum and let* $(p_j)_j$ *be a sequence of polynomials* p_j, $\deg p_j \le j$, *that is pointwise bounded on* K. *Then*

$$\forall_{\varepsilon>0,\, a\in K}\ \exists_{M>0,\, \delta>0}\ \forall_{j\in\mathbb{N},\, z:\, |z-a|<\delta}:\quad \delta|p_j(z)| \le M(1+\varepsilon)^j.$$

(b) [Lej 1950] *Let* $(f_j)_j \subset \mathcal{O}(D)$, $D \subset \mathbb{C}$ *a domain. Put*

$$R_0 := \sup\Big\{R \ge 0\colon \sum_{j=1}^{\infty} f_j(z)R^j \text{ convergent for all } z \in D\Big\},$$

$$R^* := \sup\Big\{R \ge 0\colon \sum_{j=1}^{\infty} f_j(\cdot)R^j \text{ locally uniformly convergent on } D\Big\}.$$

If $R_0 > 0$, *then either* $R_0 = R^*$ *or* $R^* = 0$.

3. Second period 1969–1997 characterized by $A_j \subset D_j$ arbitrary. This period started with the interest in finding some analogue to the Hartogs theorem for real-analytic functions.

Observe that there exists $u \in \mathcal{C}^{\infty}(\mathbb{R}^2)$, separately real-analytic but not real-analytic as a function of two real variables; e.g.

$$u\colon \mathbb{R}^2 \to \mathbb{R}, \qquad u(x) = u(x_1, x_2) := \begin{cases} x_1 x_2 \exp(-1/(x_1^2 + x_2^2)) & \text{if } x \ne 0, \\ 0 & \text{elsewhere.} \end{cases}$$

There are the following qualitative results (cf. [Bro 1961] and [Lel 1961]).

([3]) Recall that a set $M \subset \mathbb{C}^n$ is called *pluripolar* if there is a function $u \in \mathcal{PSH}(\mathbb{C}^n)$, $u \not\equiv -\infty$, with $M \subset u^{-1}(-\infty)$.

Theorem 12 (Browder, Lelong (1961)). *If f is separately real-analytic and if certain uniform estimates for derivatives hold, then f is real-analytic.*

In 1969, in a series of papers, J. Siciak started to generalize the real-analytic result; even more, he discussed separately holomorphic functions in the sense of the introduction (cf. [Sic 1969a] and [Sic 1969b]).

In order to formulate his results we shall need some more notions.

Let $A \subset \mathbb{C}$ be a compact subset. ∂A is said to fulfill the *local Leja condition* if for any $a \in \partial A$ and any $r > 0$ the following property is true: if a sequence $(p_j)_j$ of polynomials with $\deg p_j \le j$ is pointwise bounded on $\partial A \cap \Delta_a(r)$, then

$$\forall_{\varepsilon>0}\ \exists_{M>0,\,\delta>0}\colon\quad |p_j(z)| \le M\exp(\varepsilon \deg p_j), \qquad j \in \mathbb{N},\ |z-a| < \delta.$$

Moreover, let $A \subset D \subset \mathbb{C}^n$, where D is a domain. Define $h^*_{A,D}$ as the upper continuous regularization of $h_{A,D} := \sup\{u \in \mathcal{PSH}(D) : u \le 1,\ u|_A \le 0\}$. Then $h^*_{A,D}$ is the so called *relative extremal function* of the pair (A, D).

For a compact set $A \subset \mathbb{C}$ denote by $\widehat{A}$ its polynomially convex envelope. Observe that if $\partial\widehat{A}$ satisfies the Leja condition then $h^*_{A,D}|_A = 0$.

Now Siciak's result is the following:

Theorem 13 (Siciak (1969)). *Let $D_1, \dots, D_N$ be domains in $\mathbb{C}$, and let $A_j \subset D_j$ be a compact subset such that $\partial\widehat{A}_j$ fulfils the local Leja condition for $j = 1, \dots, N$. Put $X := \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N)$. Then*

$$\forall_{f\in\mathcal{O}_s(X)}\ \exists!_{\widehat{f}\in\mathcal{O}(\widehat{X})}\colon\quad \widehat{f}|_X = f,$$

where

$$\widehat{X} := \Big\{(z_1, \dots, z_N) \in D_1 \times \ldots \times D_N\colon \sum_{j=1}^N h^*_{A_j,D_j}(z_j) < 1\Big\}.$$

Observe that $\widehat{X}$ is pseudoconvex and $X \subset \widehat{X}$. In particular, there is the following generalization of the results of Browder and Lelong.

Corollary 14. *Let $D \subset \mathbb{R}^n$ open. Then*

$$\begin{aligned}\mathcal{C}^\omega(D,\mathbb{C}) = \{f\colon D \to \mathbb{C}\colon\ &\forall_{x^0\in D}\ \exists_{r>0}\colon \Delta_{x^0}(r) \subset D \text{ and } \forall_{x\in\Delta_{x^0}(r)}\forall_{j\in\{1,\dots,N\}}\colon\\ &f(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_N) \text{ extends holomorphically to } \Delta_{x^0_j}(r) \subset \mathbb{C}\}.\end{aligned}$$

Remark. (1) Although in [Sic 1969a], Siciak studied a more restrictive geometric configuration, his result contains the situation studied by Bernstein under the additional assumption that f is bounded. The main point in the proof is approximation by Chebyshev polynomials.

In [Sic 1969b], the main tool was an approximation lemma using interpolation of separately holomorphic functions with nodes which are suitably chosen extremal points of Fekete–Leja type. To be more precise:

Let $D_1 \subset \mathbb{C}$ be a k-connected domain with a nice boundary $\varGamma_0 \cup \ldots \cup \varGamma_{k-1}$. Fix points $a_1, \ldots, a_{k-1}$, a_j in the interior of $\varGamma_j$, let $D_2 \subset \mathbb{C}^n$, and let $A_j \subset D_j$ be nice compact sets. Moreover, let $f \in \mathcal{O}_{\mathrm{s}}(\mathbb{X}(A_1, A_2; D_1, D_2))$ be bounded. Put $p(z) := (z - a_1) \ldots (z - a_{k-1})$. Then there exist systems of extremal points for $A_1 \cup \partial D_1$ with a certain weight b_λ. Choose such systems $\eta^{(k\nu)}$ and denote by $\eta_0^{(k\nu)}, \ldots, \eta_{l_\nu}^{(k\nu)}$ those points on A_1. Consider the Siciak interpolation

$$f_\nu(z, w) := \sum_{j=0}^{l_\nu} f(\eta_j^{(k\nu)}, w) L^{(j)}(z, \eta^{(k\nu)}) (p(\eta_j^{(k\nu)})/p(z))^\nu,$$

where

$$L^{(j)}(z, \eta^{(k\nu)}) := \prod_{s=0,\, s \neq j}^{k\nu} (z - \eta_s^{(k\nu)})/(\eta_j^{(k\nu)} - \eta_s^{(k\nu)})$$

is the Lagrange polynomial. Put $Q_1 := f_1$ and $Q_\nu := f_\nu - f_{\nu-1}$. Then $Q_\nu \in \mathcal{O}(D_1 \times D_2)$ with $\sum Q_\nu(z, w) = f(z, w)$ on $D_1 \times A_2$.

The main work consists in proving that this series is uniformly convergent on $\widehat{X}$, which gives the stated holomorphic extension.

(2) Observe that in [Akh-Ron 1973], Akhiezer and Ronkin proved the case of an ellipse-cross with the help of the Bernstein result using some potential theory argument (see also [Ron 1977]).

(3) Siciak used his cross theorem to give a proof of an edge of the wedge type theorem (cf. [Sic 1981]).

(4) Later, Shiffman [Shi 1989] gave an improvement of Terada's theorem using methods based on [Sic 1969b].

In 1990 J. Saint Raymond initiated the study of the singularity set of separately real-analytic functions in two variables (cf. [Ray 1990]). He showed that a function of two real variables which is separately real-analytic is jointly analytic at every point off a closed set whose projections onto both axes are polar. In addition, for any such closed set F he produced a separately analytic function whose domain of analyticity is the complement of F. Later, using the above cross theorem Siciak and Błocki were able to complete the discussion of the singularity set of separately real-analytic functions (cf. [Sic 1990] and [Blo 1992]).

Let me recall some definitions.

Definition. (a) Let $\varOmega \subset \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_s}$ be open, $1 \leq p < s$, and $f \colon \varOmega \to \mathbb{C}$. We say that f is *p-separately real-analytic* in $\varOmega$ if for any $x^0 =$

$(x_1^0, \ldots, x_s^0)$ in Ω and for any p-tuple $1 \le j_1 < \ldots < j_p \le s$ the function

$$(x_{j_1}, \ldots, x_{j_p}) \mapsto f(x_1^0, \ldots, x_{j_1-1}^0, x_{j_1}, x_{j_1+1}^0, \ldots, x_{j_p-1}^0, x_{j_p}, x_{j_p+1}^0, \ldots, x_s^0)$$

is real-analytic in a neighborhood of $(x_{j_1}^0, \ldots, x_{j_p}^0)$.

(b) The set

$$S(f) := \Omega \setminus \{x \in \Omega\colon f \text{ real-analytic in a neighborhood of } x\}$$

is called the *singular part* of f.

THEOREM 15 (Siciak (1990), Błocki (1992)). (a) *If f is p-separately real-analytic, then for any $1 \le j_1 < \ldots < j_q \le s$ $(q := s - p)$ the projection of $S(f)$ onto $\mathbb{R}^{n_{j_1}} \times \ldots \times \mathbb{R}^{n_{j_q}}$ is a pluripolar set in $\mathbb{C}^{n_{j_1}} \times \ldots \times \mathbb{C}^{n_{j_q}}$.*

(b) *If $S \subset \Omega$ is closed with the above property, then there is $f\colon \Omega \to \mathbb{C}$, p-separately real-analytic, such that $S = S(f)$.*

The next deep steps in developing the theory of separately holomorphic functions were initiated in 1976 by Zahariuta (cf. [Zah 1976]) when he started to use common bases of Hilbert spaces instead of applying the more ad hoc techniques of Siciak which, of course, heavily depend on the geometry of the given 2-fold cross.

Let us repeat the main idea: under certain assumptions which may be realized via approximation one has an orthogonal basis $(b_k)_{k\in\mathbb{N}} \subset H_0 := L^2_{\mathrm{h}}(D_1) := L^2(D_1) \cap \mathcal{O}(D_1)$ with $\|b_k\|_{H_0} \to \infty$ such that

$$(b|_{A_1})_k \subset \overline{L^2_{\mathrm{h}}(D_1)|_{A_1}}^{L^2(A_1,\mu_{A_1,D_1})} =: H_1$$

is an orthonormal basis of H_1, where μ_{A_1,D_1} is a certain measure defined via the Monge–Ampère operator. Therefore, if $f \in \mathcal{O}_{\mathrm{s}}(\mathbb{X}(A_1, A_2; D_1, D_2))$ then one may assume that $f(\cdot, z_2) \in L^2_{\mathrm{h}}(D_1)$ for all $z_2 \in A_2$, and therefore, $f(\cdot, z_2) = \sum_{k=1}^{\infty} c_k(z_2) b_k$. It can be shown that the functions c_k are holomorphic on D_2. Hence it remains to discuss the domain of convergence of this series of functions holomorphic on $D_1 \times D_2$.

Zahariuta's method was also used and modified in papers by Nguyen Thanh Van and Zeriahi (cf. [Ngu-Zer 1991], [Ngu-Zer 1997], [Ngu-Zer 1995]). The most general result to date is contained in a recent paper due to Alehyane and Zeriahi [Ale-Zer 2001].

Before stating this theorem let me recall a few definitions: A set $A \subset D$, D a domain in $\mathbb{C}^n$, is called *locally pluriregular* if for any $a \in A$ and any neighborhood $U = U(a)$ we have $h^*_{A\cap U,U}(a) = 0$. Observe that such a set is "thick" in the pluripotential sense; in particular, it is not pluripolar.

Moreover, if $D_j \nearrow D$, then $\omega_{A,D} := \lim h^*_{A\cap D_j,D_j}$. Note that the definition of $\omega_{A,D}$ is independent of the exhaustion sequence, and if D is bounded, then $\omega_{A,D} = h^*_{A,D}$.

Now we are able to formulate what we will quote in the future as the *classical cross theorem.*

THEOREM 16 (Alehyane & Zeriahi (2001)). *Let* $D_j \subset \mathbb{C}^{k_j}$ *be a pseudoconvex domain and* $A_j \subset D_j$ *a locally pluriregular subset*, $1 \le j \le N$. *Put* $X := \mathbb{X}(A_1,\dots,A_N; D_1,\dots,D_N)$. *Then for any* $f \in \mathcal{O}_{\mathrm{s}}(X)$ *there is exactly one* $\widehat{f} \in \mathcal{O}(\widehat{X})$ *with* $\widehat{f}|_X = f$, *where*

$$\widehat{X} := \{(z_1,\dots,z_N) \colon \omega_{A_1,D_1}(z_1) + \ldots + \omega_{A_N,D_N}(z_N) < 1\}.$$

(Observe that also here $\widehat{X}$ is a pseudoconvex domain containing X.)

REMARK. It should be mentioned that there are much more papers in this field dealing with separately holomorphic or separately meromorphic functions or with separately holomorphic mappings. The author apologizes for not having been able to cite all of them.

SUMMARY. *So far we have discussed the situation of an arbitrary* N*-fold cross* X *and separately holomorphic functions given on the whole of* X.

3. Third period 1998–2001 characterized by cross theorems with analytic singularities. This period started with a paper by Öktem investigating the range problem in *mathematical tomography* (cf. [Ökt 1998] and [Ökt 1999]) Let me describe that problem:

The *exponential Radon transform* is given by the mapping ($\mu \neq 0$)

$$\mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^2,\mathbb{R}) \ni h \mapsto R_\mu(h) \colon S^1 \times \mathbb{R} \to \mathbb{R},$$
$$R_\mu(h)(\omega,p) := \int\limits_{x\cdot\omega = p} h(x)\exp(\mu x \cdot \omega^{\perp})\, d\Lambda_1(x),$$

where S^1 denotes the unit circle in $\mathbb{R}^2$, $\omega^{\perp} := (-\sin\alpha, \cos\alpha) \in S^1$ the vector orthogonal to $\omega = (\cos\alpha, \sin\alpha)$, Λ_1 the one-dimensional Lebesgue measure, and where "$\cdot$" means the scalar product in $\mathbb{R}^2$.

The main problem is to recover h from $R_\mu(h)$ which is measured. So it is important to know the shape of the range of R_μ.

THEOREM 17 (Öktem (1998)). *Let* $g \colon S^1 \times \mathbb{R} \to \mathbb{C}$ *and* $\mu \neq 0$. *Then the following statements are equivalent*:

(a) *there is an* $h \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^2,\mathbb{C})$ *with* $g = R_\mu(h)$;

(b) $g \in \mathcal{C}_{\mathrm{c}}^{\infty}(S^1 \times \mathbb{R},\mathbb{C})$ *and* $\widehat{g}(\omega, it) = \widehat{g}(\sigma, -it)$ *whenever* $t\omega + \mu\omega^{\perp} = -t\sigma + \mu\sigma^{\perp}$.

To prove Theorem 17, Öktem used the following theorem, whose proof is based on the classical cross theorem.

THEOREM 18 (Öktem (1998–1999)). *Let* $D_1 = D_2 = \mathbb{C}$, $A_1 = A_2 = \mathbb{R}$, *and* $M := \{z \in \mathbb{C}^2 \colon z_1 = z_2\}$. *Put* $X := \mathbb{X}(A_1, A_2; D_1, D_2)$. *Then*

$$\forall_{f \in \mathcal{O}_s(X \setminus M)}\ \exists!_{\widehat{f} \in \mathcal{O}(\mathbb{C}^2 \setminus M)} \colon \quad \widehat{f}|_{X \setminus M} = f.$$

Observe that Theorem 18 is the first result dealing with a cross theorem with singularities.

This result was generalized by Siciak [Sic 2000].

THEOREM 19 (Siciak (2000)). *Let* $D_1 = \ldots = D_N = \mathbb{C}$, $A_j \subset D_j$ *with* $\operatorname{Cap} A_j > 0$ ([4]), *and let* $M := \{z \in \mathbb{C}^N \colon P(z) = 0\}$, P *a polynomial. Define* X *as above. Then*

$$\forall_{f \in \mathcal{O}_s(X \setminus M)}\ \exists!_{\widehat{f} \in \mathcal{O}(\mathbb{C}^N \setminus M)} \colon \quad \widehat{f}|_{X \setminus M} = f.$$

Observe that the following *general principle of analytic continuation across thin subsets* (cf. [Gr-Re 1956/57]) was used in the proof of Theorem 19.

THEOREM (Grauert & Remmert (1956/57)). *Let* $G \subset \mathbb{C}^n$ *be a domain and* $\widehat{G}$ *its envelope of holomorphy. Moreover*, *let* $A \subset \widehat{G}$ *be a pure* 1-*codimensional analytic subset of* $\widehat{G}$. *Then the envelope of holomorphy* $\widehat{G \setminus A}$ *of* $G \setminus A$ *is* $\widehat{G} \setminus A$. (Here $\widehat{G}$ may be thought of as a Riemann domain over $\mathbb{C}^n$.)

This general principle was generalized by Dloussky (cf. [Dlo 1977]); whereas above the analytic singularity set is already given in the whole envelope of holomorphy, it could also be the case that A is only assumed to exist in G.

THEOREM (Dloussky (1977)). *Let* $G \subset \mathbb{C}^n$ *be a domain and assume that* $A \subset G$ *is a proper analytic subset. Then there exists an analytic subset* $\widehat{A}$ *of* $\widehat{G}$ *with*

$$\widehat{G \setminus A} = \widehat{G} \setminus \widehat{A} \quad \text{and} \quad \widehat{A} \cap G \subset A.$$

REMARK. Recently a nice proof of the theorem of Dloussky was given by Porten (2001) (cf. [Por 2000]).

Based on this extension result of Dloussky the following general cross theorem with analytic singularities is true (cf. [Jar-Pfl 2001a], [Jar-Pfl 2001b], [Jar-Pfl 2001c] ([5])).

THEOREM 20 (Jarnicki & Pflug (2000–2001)). (a) *Let* $D_j \subset \mathbb{C}^{k_j}$ *be a pseudoconvex domain*, *and let* $A_j \subset D_j$ *be a locally pluriregular subset* $(j = 1, \ldots, N)$. *Put*, *as usual*, $X := \mathbb{X}(A_1, \ldots, A_N; D_1, \ldots, D_N)$. *Moreover*, *let* U

([4]) Here Cap means the logarithmic capacity.

([5]) M. Jarnicki and the author learnt about this extension problem at the Complex Analysis Seminar in Kraków when Siciak was lecturing on his theorem.

be a domain with $X \subset U \subset \widehat{X}$, and let $M \subset U$ be a proper analytic subset. Then there are an analytic subset $\widehat{M} \subset \widehat{X}$ and an open set U_0 such that

- *$X \subset U_0 \subset U$ and $\widehat{M} \cap U_0 \subset M$,*
- *for any $f \in \mathcal{O}_s(X \setminus M)$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M})$ with $\widehat{f}|_{X\setminus M} = f$.*

(b) *Let the situation be as in* (a) *with $U = \widehat{X}$. Define $\widehat{M}$ to be the union of all irreducible* 1*-codimensional components of M. Then for any $f \in \mathcal{O}_s(X \setminus M)$ there is exactly one $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M})$ such that $\widehat{f}|_{X\setminus M} = f$.*

Sketch of proof. Part (a) is obtained by applying part (b), the classical cross theorem, and the theorem of Dloussky. Using the classical cross theorem, the description of pure 1-codimensional analytic sets, and the Grauert–Remmert theorem finally leads to (b). ∎

Summarizing we have the same general extension principle for separately holomorphic function with analytic singularities on an N-fold cross as described for holomorphic functions on domains by Dloussky's theorem.

Summary. *So far we have discussed the situation of an arbitrary N-fold cross X and separately holomorphic functions given on X off a set* (*perhaps empty*) *which is analytic in a neighborhood of X.*

4. Fourth period (2001–????) characterized by cross theorems with more general singularities. Let me first recall what could happen in Theorem 20 with the analytic singularities.

Fix $a \in A_1 \times \ldots \times A_N$ and $j \in \{1, \ldots, N\}$. Then the fiber of M over $(a_1, \ldots, \widehat{a}_j, \ldots, a_N)$ has only two possibilities, namely either $M_{(a_1,\ldots,\widehat{a}_j,\ldots,a_N)} = D_j$ or $M_{(a_1,\ldots,\widehat{a}_j,\ldots,a_N)}$ is a proper analytic subset of D_j. Moreover, the set $\{(a_1,\ldots,\widehat{a}_j,\ldots,a_N)\in(A_1\times\ldots\times A_{j-1})\times(A_{j+1}\times\ldots\times A_N): M_{(a_1,\ldots,\widehat{a}_j,\ldots,a_N)}=D_j\}$ is analytic in D_j; in particular, it is pluripolar.

Concerning the notion of pluripolarity, we recall another principle of analytic extension through thin sets of singularities (cf. [Chi 1993]).

Theorem (Chirka (1993)). *Let $G \subset \mathbb{C}^n$ be a domain and assume that $A \subset G$ is a pluripolar subset, closed in G. Then there exists a pluripolar subset $\widehat{A}$ of $\widehat{G}$, closed in $\widehat{G}$, with*

$$\widehat{G \setminus A} = \widehat{G} \setminus \widehat{A} \quad \textit{and} \quad \widehat{A} \cap G \subset A.$$

So it seems reasonable to consider the following situation of separately holomorphic functions on N-fold crosses with singularities:

General assumptions. Let $M \subset X$. Put

$$\Sigma_j := \{(a', a'') \in (A_1 \times \ldots \times A_{j-1}) \times (A_{j+1} \times \ldots \times A_N): M_{(a_1,\ldots,\widehat{a}_j,\ldots,a_N)} \text{ not pluripolar}\}.$$

In the future we will always assume that all the sets Σ_j are pluripolar. *So thick fibers of the singularity set M are only allowed over a thin set.*

A surprise. Examples:

(a) Let $D_1 = D_2 = \mathbb{C}$ and $A_1 = A_2 = E$, where E denotes the unit disc in the plane. Put $M := \{0\} \times \overline{E}$. Obviously, M is pluripolar in $\mathbb{C}^2$. Put $X := \mathbb{X}(A_1, A_2; D_1, D_2)$. Then $\Sigma_1 = \emptyset$ and $\Sigma_2 = \{0\}$ are pluripolar. Observe that $\widehat{X} = \mathbb{C}^2$.

Now consider the following function $f_0 \in \mathcal{O}_s(X \setminus M)$:

$$f_0(z, w) := \begin{cases} 1/z & \text{if } z \neq 0, \\ 0 & \text{if } z = 0,\ |w| > 1. \end{cases}$$

The best one can get via continuation is the function $\widehat{f_0} \in \mathcal{O}(\mathbb{C}_* \times \mathbb{C})$ defined by $\widehat{f_0}(z, w) := 1/z$.

Therefore, the "old" singularities propagate inside X to $\widehat{M} := \{0\} \times \mathbb{C}$, which is strictly larger than M.

(b) Let $D_1 = D_2 = \mathbb{C}$, $A_1 := E$, $A_2 := \{w \in \mathbb{C}\colon r < |w| < 1\}$, where $0 < r < 1$, and X as usual. Put $M := \{0\} \times \{w \in \mathbb{C}\colon |w| = r\}$. Then M is pluripolar in $\mathbb{C}^2$ and $\widehat{X} = \mathbb{C}^2$. Again, the sets $\Sigma_1 = \emptyset$ and $\Sigma_2 = \{0\}$ are pluripolar.

Now we look at the following function $f_0 \in \mathcal{O}_s(X \setminus M)$:

$$f_0(z, w) := \begin{cases} w & \text{if } z \neq 0 \text{ or } z = 0,\ |w| > r, \\ 0 & \text{if } z = 0,\ |w| < r. \end{cases}$$

Obviously, $\widehat{f_0} \in \mathcal{O}(\mathbb{C}^2)$ with $\widehat{f_0}(z, w) = w$ is the maximal extension of f_0. But now

$$f_0(0, w) \neq \widehat{f_0}(0, w), \quad 0 < |w| < 1.$$

Therefore, the maximal extension $\widehat{f_0}$ may not coincide with f_0 on $X \setminus M$.

Conclusion. From the examples it follows that we can only hope to get the following result when dealing with pluripolar singularities:

- $\widehat{M} \cap X' \subset M$, where $X' := \mathbb{X}(A_1 \setminus \Sigma_2, A_2 \setminus \Sigma_1; D_1, D_2) \subset X$,
- $\widehat{f} = f$ only on $X' \setminus M$.

According to the experiences with the examples above we introduce the following modified N-fold cross:

$$X' := \mathbb{T}(A_1, \dots, A_N; D_1, \dots, D_N; \Sigma_1, \dots, \Sigma_N)$$
$$:= \bigcup_{j=1}^{N} \{(z', z_j, z'') \in (A_1 \times \ldots \times A_{j-1}) \times D_j \times (A_{j+1} \times \ldots \times A_N)\colon (z', z'') \notin \Sigma_j\}.$$

Observe that

1) $\mathbb{T}(A_1, \dots, A_N; D_1, \dots, D_N; \emptyset, \dots, \emptyset) = X$;

2) if $N = 2$, then $\mathbb{T}(A_1, A_2; D_1, D_2; \Sigma_1, \Sigma_2) = \mathbb{X}(A_1 \setminus \Sigma_2, A_2 \setminus \Sigma_1; D_1, D_2)$, i.e. the modified 2-fold cross is always a 2-fold cross in the usual sense.

With these notions we have the following final result (cf. [Jar-Pfl 2001d]):

Theorem 21 (Jarnicki & Pflug (December 2001)). *Let N, A_j, D_j be as before. Put $X := \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N)$. Let U be an open neighborhood of X, and let $M \subset U$ be relatively closed such that Σ_j is pluripolar, $j = 1, \dots, N$. Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{X}$ with the following properties*:

- $\widehat{M} \cap X' \subset M$,
- *for any function $f \in \mathcal{O}_{\mathrm{s}}(X \setminus M)$ there exists a unique extension $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M})$ such that $\widehat{f} = f$ on $X' \setminus M$,*
- *$\widehat{X} \setminus \widehat{M}$ is pseudoconvex.*

Remark. (a) If M is pluripolar, then the assumptions are fulfilled.

(b) In the case that all fibers $M_{(a_1, \dots, \widehat{a}_j, \dots, a_N)}$ are pluripolar we obviously have $X' = X$, i.e. we are back in the good situation we have discussed in the analytic case.

The proof of the last theorem is based on the Chirka theorem, the structure of polar sets, the classical cross theorem, and the following modification of a result due to Chirka and Sadullaev [Chi-Sad 1988] (see also [Jar-Pfl 2001d]):

Theorem (Sadullaev & Chirka (1988), Jarnicki & Pflug (2001)). *Let $D_1 = E^{n-1}$, $D_2 := \mathbb{C}$, $A_1 = A$ and $A_2 := E$. Put $X := \mathbb{X}(A_1, A_2; D_1, D_2)$. Assume $U \subset E^{n-1} \times \mathbb{C}$ is an open neighborhood of X and let $M \subset U$, relatively closed, with $M \cap E^n = \emptyset$ and $M_{(a, \widehat{b})}$ polar for all $(a, b) \in A_1 \times A_2$. Then there exists a relatively closed pluripolar set $S \subset E^{n-1} \times \mathbb{C}$ such that*

- $S \cap X \subset M$,
- *$\widehat{X} \setminus S = E^{n-1} \times \mathbb{C} \setminus S$ is pseudoconvex,*
- *for any function $f \in \mathcal{O}_{\mathrm{s}}(X \setminus M)$ there is an extension $\widehat{f} \in \mathcal{O}(E^{n-1} \times \mathbb{C} \setminus S)$ with $f = \widehat{f}$ on E^n.*

Notice that this result may be viewed as a special case of Theorem 21.

Open problems. (a) Observe that Theorem 20 is a special case of Theorem 21, except that we do not see how to prove directly that the exceptional set $\widehat{M}$ is analytic.

(b) It is not clear what will happen with the statement in Theorem 21 when the singularity set M is not assumed to be closed in X.

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