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On the removal of subharmonic singularities of plurisubharmonic functions

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Dedicated to Prof. J. Siciak on the occasion of his 70th birthday

Abstract. It is proved that any subharmonic function in a domain $\Omega \subset \mathbb{C}^n$ which is plurisubharmonic outside of a real hypersurface of class C^1 is indeed plurisubharmonic in Ω.

1. Let E be a closed nowhere dense subset of a domain Ω in $\mathbb{C}^n$ and u be a subharmonic function in Ω, which is plurisubharmonic (psh) in $\Omega \setminus E$. The question is what conditions on E guarantee the plurisubharmonicity of u in the whole domain Ω. For the question to be nontrivial, we consider the sets E which are nonremovable for general bounded psh functions in $\Omega \setminus E$. The simplest class of such singularities is given by smooth hypersurfaces in Ω, so the following theorem can be considered as a first step towards the solution of the general problem.

Theorem. *Let Γ be a C^1-hypersurface in a domain $\Omega \subset \mathbb{C}^n$ and u be a subharmonic function in Ω which is plurisubharmonic in $\Omega \setminus \Gamma$. Then u is plurisubharmonic in Ω.*

Note that we do not assume any smoothness of u in Ω. If Γ divides Ω into two components $\Omega_\pm$ and $u \in C^2(\Omega \setminus \Gamma) \cap C^1(\overline{\Omega}_\pm)$ is continuous in Ω, the condition of subharmonicity of u in Ω means that $\Delta u \geq 0$ in $\Omega \setminus \Gamma$ and $\partial u/\partial n_+ + \partial u/\partial n_- \geq 0$ on Γ where $\partial u/\partial n_\pm$ are the (inner) normal derivatives of $u|\Omega_\pm$ at points of Γ. This case of "classical" smoothness of u was considered by P. Blanchet [1] who proved the theorem under these additional assumptions. The general case needs another technique, even for C^∞-hypersurfaces and u piecewise smooth. On the other hand, our proof

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does not work already in the case of Lipschitz graphs (the C^1-smoothness of Γ is essential). The following question remains open:

Let E be a closed subset of $\Omega \subset \mathbb{C}^n$ with locally finite Hausdorff $(2n-1)$-measure and let a function u be subharmonic in Ω and psh in $\Omega \setminus E$. Does it follow that u is plurisubharmonic in Ω?

2. The proof of the theorem is based on the notion of positive currents (see [3, 2]). Recall that $v \in \mathrm{psh}(\Omega)$ if and only if the current $dd^c v = i \sum v_{jk} dz_j \wedge d\bar{z}_k$ of bidegree $(1,1)$ in Ω is positive, that is, $(dd^c v, \Phi) \geq 0$ for each positive $(n-1, n-1)$-form Φ of class $C_0^\infty(\Omega)$. (Here $d^c = i(\overline{\partial} - \partial)$; the class of test forms Φ can be reduced to Φ of the type $\varphi \prod_{\nu=1}^{n-1}(i\, dl_\nu \wedge d\bar{l}_\nu)$ where $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ and l_ν are $\mathbb{C}$-linear functions in $\mathbb{C}^n$, see [2].) As is well known (see [3]), the coefficients v_{jk} of the current $dd^c v$ for $v \in \mathrm{psh}(\Omega)$ are (locally finite, complex-valued) measures in Ω, $v_{jj} \geq 0$ and $|v_{jk}| \leq \sum v_{ll} = \frac{1}{2}\Delta v$, $j, k = 1, \ldots, n$.

Step 1. The theorem is local, so we can assume that Γ is the zero-set of a function $\varrho \in C^1(\Omega)$ with $d\varrho \neq 0$ on Γ. By the Whitney extension theorem, we can also assume that $\varrho \in C^\infty(\Omega \setminus \Gamma)$. Set $\lambda_\varepsilon = \chi_\varepsilon \circ \varrho$, where $\chi_\varepsilon \in C^\infty(\mathbb{R})$, $0 \leq \chi_\varepsilon \leq 1$, $\chi_\varepsilon = 0$ in a neighbourhood of 0 and $\chi_\varepsilon(t) = 1$ for $|t| \geq \varepsilon > 0$. Then $dd^c u = \mu_\varepsilon + \sigma_\varepsilon$ where $\mu_\varepsilon = \lambda_\varepsilon dd^c u$ and

$$\sigma_\varepsilon = (1 - \lambda_\varepsilon) dd^c u = d((1 - \lambda_\varepsilon) d^c u) + (\chi'_\varepsilon \circ \varrho) d\varrho \wedge d^c u \tag{1}$$

$$= -(1 - \lambda_\varepsilon) d^c du = -d^c((1 - \lambda_\varepsilon) du) - (\chi'_\varepsilon \circ \varrho) d^c \varrho \wedge du. \tag{2}$$

Step 2. As $u \in \mathrm{psh}(\Omega \setminus \Gamma)$ and $\lambda_\varepsilon = 0$ in a neighbourhood of Γ, the currents

$$\mu_\varepsilon := \lambda_\varepsilon dd^c u = \lambda_\varepsilon i \sum u_{jk} dz_j \wedge d\bar{z}_k$$

are well defined and positive in Ω. As u is subharmonic in Ω, the measure Δu is nonnegative and locally bounded in Ω. As $|u_{jk}| \leq \Delta u$, $j, k \leq n$, it follows that

$$\lim_{\varepsilon \to 0} \mu_\varepsilon =: \mu = i \sum \mu_{jk} dz_j \wedge d\bar{z}_k \text{ exists and is a positive current in } \Omega,$$

and its coefficients μ_{jk} are locally finite measures in Ω. By the construction, μ is carried by $\Omega \setminus \Gamma$, i.e., $\mu_{jk}(E) = 0$, $j, k \leq n$, for any set $E \subset \Gamma$. Moreover, $dd^c u = \mu + \sigma$, where

$$\sigma = \lim_{\varepsilon \to 0} (1 - \lambda_\varepsilon) dd^c u$$

is a current in Ω supported on Γ.

Step 3. As $u \in \mathrm{sh}(\Omega)$, the 1-currents du, $d^c u$ have coefficients in $L^1_{\mathrm{loc}}(\Omega)$ (this follows obviously from the Riesz decomposition). Thus for every $(2n-3)$-form Ψ of class $C_0^1(\Omega)$ (the coefficients belong to $C^1(\Omega)$ and have compact supports) we see, according to (1), that the value

$$\begin{aligned}(dd^c u, d\varrho \wedge \Psi) &:= \lim_{\delta\to 0}(dd^c u, d\varrho_\delta \wedge \Psi)\\ &= \lim_{\delta\to 0}((\mu_\varepsilon, d\varrho_\delta \wedge \Psi) + ((\chi'_\varepsilon \circ \varrho)d\varrho \wedge d^c u, d\varrho_\delta \wedge \Psi)\\ &\quad + ((1-\lambda_\varepsilon)d^c u, d\varrho_\delta \wedge d\Psi))\\ &= (\mu_\varepsilon, d\varrho \wedge \Psi) + o_\varepsilon(1) = (\mu, d\varrho \wedge \Psi)\end{aligned}$$

is well defined. (Here the index δ means δ-regularization, that is, convolution with a nonnegative function in $C_0^\infty(\mathbb{C}^n)$ supported in the ball $|z| < \delta$ and having Lebesgue integral 1.)

In the same way, using (2) we find that the values $(dd^c u, d^c\varrho \wedge \Psi)$ are well defined and $(dd^c u, d^c\varrho \wedge \Psi) = (\mu, d^c\varrho \wedge \Psi)$, which implies that

$$(\sigma, d\varrho \wedge \Psi) := \lim_{\delta\to 0}(\sigma, d\varrho_\delta \wedge \Psi) = 0, \tag{3}$$

$$(\sigma, d^c\varrho \wedge \Psi) := \lim_{\delta\to 0}(\sigma, d^c\varrho_\delta \wedge \Psi) = 0. \tag{3'}$$

STEP 4. Let $\varphi \in C_0(\Omega)$, $\varphi \ge 0$, and let $\omega = \frac{1}{4}dd^c|z|^2$ be the fundamental form in $\mathbb{C}^n$. Then

$$(dd^c u, \varphi\omega^{n-1}) := \lim_{\delta\to 0}(dd^c u, \varphi_\delta\omega^{n-1}) = 4(n-1)!(\Delta u, \varphi) \ge 0,$$

hence,

$$\lim_{\delta\to 0}(\mu+\sigma, (1-\lambda_\varepsilon)\varphi_\delta\omega^{n-1}) = (\mu, (1-\lambda_\varepsilon)\varphi\omega^{n-1}) + \lim_{\delta\to 0}(\sigma, \varphi_\delta\omega^{n-1})$$

is nonnegative. As μ is carried by $\Omega \setminus \Gamma$ and $1-\lambda_\varepsilon \to 0$ there, we can pass to the limit as $\varepsilon \to 0$ and obtain

$$(\sigma, \varphi\omega^{n-1}) := \lim_{\delta\to 0}(\sigma, \varphi_\delta\omega^{n-1}) \ge 0. \tag{4}$$

STEP 5. Let Φ be an arbitrary $(n-1,n-1)$-form of class $C_0^\infty(\Omega)$ and Φ_τ be its projection onto the complex tangent planes T_z^c to the levels $\{\varrho(z) = \text{const}\}$. (Representing $\Phi = \sum_{j,k=1}^n \Phi_{jk}\prod_{\alpha\ne j} dz_\alpha \wedge \prod_{\beta\ne k} d\bar z_\beta$ and assuming that $T_a^c = \{z_n = 0\}$, which can be done by a unitary transform, we see that $\Phi_\tau|_a$ is the same sum from 1 to $n-1$ at the point a.) It is obvious that Φ_τ is a $(2n-2)$-form of class $C_0(\Omega)$. Moreover, $\Phi = \Phi_\tau + \Phi_\nu$, where Φ_ν is in the same class and $\Phi_\nu(\tau_1 \wedge \ldots \wedge \tau_{n-1}) = 0$ for any vector fields $\tau_1, \ldots, \tau_{n-1}$ which are complex orthogonal to $\nabla\varrho$.

By the Wirtinger theorem, the restrictions $\omega^{n-1}/(n-1)!|T_z^c$ coincide with the usual volume form on the planes T_z^c, hence

$$\Phi|T_z^c = \Phi_\tau|T_z^c = \varphi\omega^{n-1}|T_z^c$$

for some $\varphi \in C_0(\Omega)$. Moreover, $\varphi \ge 0$ if the form Φ is positive (because T_z^c are complex planes).

Decomposing $\omega^{n-1} = (\omega^{n-1})_\tau + (\omega^{n-1})_\nu$, we obtain, for arbitrary Φ, the decomposition $\Phi = \varphi\omega^{n-1} + \Phi_0$, where Φ_0 is orthogonal to T_z^c in the sense that $\Phi_0(\tau_1 \wedge \ldots \wedge \tau_{n-1}) = 0$ for any fields τ_j as above. As $d\varrho, d^c\varrho$ constitute a basis of 1-covectors annihilating all such τ_j, the form Φ_0 is represented as a sum $d\varrho \wedge \Psi_1 + d^c\varrho \wedge \Psi_2$ with continuous Ψ_1, Ψ_2 in Ω. Finally, we have the decomposition

$$\Phi = \varphi\omega^{n-1} + \Phi_0 = \varphi\omega^{n-1} + d\varrho \wedge \Psi_1 + d^c\varrho \wedge \Psi_2, \tag{5}$$

where all the terms belong to the class $C_0(\Omega)$, and $\varphi \geq 0$ if Φ is a positive form of bidegree $(n-1, n-1)$.

Step 6. Let now

$$\Phi^\delta := \varphi_\delta\omega^{n-1} + \Phi_0^\delta := \varphi_\delta\omega^{n-1} + d\varrho \wedge (\Psi_1)_\delta + d^c\varrho \wedge (\Psi_2)_\delta$$

for a positive Φ of class $C_0^\infty(\Omega)$ and bidegree $(n-1, n-1)$. Then

$$(dd^c u, \Phi^\delta) = (\mu, \Phi^\delta) + (\sigma, \varphi_\delta\omega^{n-1}),$$

according to (3) and (3′). As $\varphi \geq 0$, we have $(\sigma, \varphi_\delta\omega^{n-1}) \geq 0$ by (4). Thus

$$(dd^c u, \Phi) = \lim_{\delta\to 0}(dd^c u, \Phi^\delta) \geq \lim_{\delta\to 0}(\mu, \Phi^\delta) = (\mu, \Phi) \geq 0,$$

as Φ has bidegree $(n-1, n-1)$ and μ, Φ are positive.

The theorem is proved.

References

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