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Normal families and shared values of meromorphic functions

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To Professor Józef Siciak, with admiration and friendship

Abstract. Let $\mathcal{F}$ be a family of meromorphic functions on a plane domain D, all of whose zeros are of multiplicity at least $k \ge 2$. Let a, b, c, and d be complex numbers such that $d \ne b, 0$ and $c \ne a$. If, for each $f \in \mathcal{F}$, $f(z) = a \Leftrightarrow f^{(k)}(z) = b$, and $f^{(k)}(z) = d \Rightarrow f(z) = c$, then $\mathcal{F}$ is a normal family on D. The same result holds for $k = 1$ so long as $b \ne (m+1)d$, $m = 1, 2, \ldots$.

1. Introduction. Let f and g be meromorphic functions on a domain D in $\mathbb{C}$, and let a and b be complex numbers. If $g(z) = b$ whenever $f(z) = a$, we write $f(z) = a \Rightarrow g(z) = b$. If $f(z) = a \Rightarrow g(z) = b$ and $g(z) = b \Rightarrow f(z) = a$, we write $f(z) = a \Leftrightarrow g(z) = b$. If $f(z) = a \Leftrightarrow g(z) = a$, then we say that f and g share a in D.

Mues and Steinmetz [11] proved

Theorem A. *Let f be a nonconstant meromorphic function, and let a_1, a_2, and a_3 be distinct complex numbers. If f and f' share a_1, a_2, and a_3, then $f \equiv f'$.*

Schwick [15] discovered a connection between normality criteria and shared values. He proved

Theorem B. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and let a_1, a_2, and a_3 be distinct complex numbers. If, for each $f \in \mathcal{F}$, f and f' share a_1, a_2, and a_3 in D, then $\mathcal{F}$ is normal in D.*

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This result has undergone various extensions [12], [17], [18], culminating in the following result of Pang and Zalcman [13].

Theorem C. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D; and let a, b, and c be complex numbers such that $b \neq a$ and $c \neq 0$. If, for each $f \in \mathcal{F}$, $f(z) = 0 \Leftrightarrow f'(z) = a$, and $f(z) = c \Leftrightarrow f'(z) = b$, then $\mathcal{F}$ is normal in D.*

It is natural to ask what can be said if f' is replaced by $f^{(k)}$ for $k \geq 2$ in the above theorems. Frank and Schwick observed that while Theorem A extends in a natural fashion when f' is replaced by $f^{(k)}$ [6], Theorem B does not admit such an extension [7]. Chen and Fang [4] proved

Theorem D. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D; let $k \geq 2$ be an integer; and let a, b, and c be complex numbers such that $b \neq a$. If, for each $f \in \mathcal{F}$, f and $f^{(k)}$ share a and b in D, and all zeros of $f - c$ have multiplicity at least $k+1$, then $\mathcal{F}$ is normal in D.*

In this paper, we extend Theorem C as follows.

Theorem 1. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D; let k be a positive integer; and let a, b, c, and d be complex numbers such that $b \neq a, 0$ and $c \neq 0$. If, for each $f \in \mathcal{F}$, all zeros of $f - d$ have multiplicity at least k, $f(z) = 0 \Leftrightarrow f^{(k)}(z) = a$, and $f^{(k)}(z) = b \Rightarrow f(z) = c$, then $\mathcal{F}$ is normal in D for $k \geq 2$, and for $k = 1$ so long as $a \neq (m+1)b$, $m = 1, 2, \ldots$.*

As a consequence, we obtain the following sharpening of Theorem D.

Corollary. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D; let $k \geq 2$ be an integer; and let a, b, and c be complex numbers such that $b \neq a$. If, for each $f \in \mathcal{F}$, f and $f^{(k)}$ share a and b in D, and all zeros of $f - c$ have multiplicity at least k, then $\mathcal{F}$ is normal in D.*

Proof. Since $a \neq b$, we may assume that $b \neq 0$. Consider the family of functions $\mathcal{G} = \{f - a : f \in \mathcal{F}\}$. For each $g \in \mathcal{G}$, all zeros of $g - (c - a)$ have multiplicity at least k. Further, if $g \in \mathcal{G}$, then $g(z) = 0 \Leftrightarrow g^{(k)}(z) = a$, and $g(z) = b - a \Leftrightarrow g^{(k)}(z) = b$. By Theorem 1, $\mathcal{G}$ is normal on D; and hence $\mathcal{F}$ is normal on D.

Example 1. Consider the family $\mathcal{F} = \{a(e^{nz} - 1)/n : n = 1, 2, \ldots\}$ on $D = \{z : |z| < 1\}$. Then, for every $f \in \mathcal{F}$, $f(z) = 0 \Leftrightarrow f'(z) = a$, and $f'(z) \neq 0$ (and hence $f'(z) = 0 \Rightarrow f(z) = c$ for any c). But $\mathcal{F}$ is not normal in D. This shows that $b \neq 0$ is necessary in Theorem 1 when $k = 1$. For $k \geq 2$, Theorem 1 actually holds even when $b = 0$. However, we shall not prove that here.

Example 2. Let a and b be two nonzero numbers such that $a = (m+1)b$, where m is a positive integer. Set

$$f_n(z) = b\left(z - \frac{1}{n}\right) + \frac{1}{m(nz-1)^m}, \qquad n = 1, 2, \ldots,$$

and let $\mathcal{F} = \{f_n\}$, $D = \{z : |z| < 1\}$. Then

$$f_n'(z) = b - \frac{n}{(nz-1)^{m+1}}.$$

Clearly, for every $f \in \mathcal{F}$, $f(z) = 0 \Leftrightarrow f'(z) = a$, and $f'(z) \neq b$ (hence $f'(z) = b \Rightarrow f(z) = c$). But $\mathcal{F}$ is not normal in D. This means that $a \neq (m+1)b$ ($m = 1, 2, \ldots$) is necessary in Theorem 1 when $k = 1$.

Example 3. Fix k and let $\{\omega_1, \ldots, \omega_k\}$ be the kth roots of unity (with $\omega_k = 1$). Any function of the form

$$F(z) = \sum_{j=1}^{k} c_j e^{\omega_j z}$$

clearly satisfies $F^{(k)} \equiv F$. The $k \times k$ Vandermonde determinant defined by ω_j, $1 \leq j \leq k$, does not vanish. Hence, solving k linear equations in k unknowns, we may choose the c_j so that the first $k-1$ Taylor coefficients of F vanish at the origin, i.e., so that F has a zero of exact order $k-1$ at 0. Let $D = \{z : |z| < 1\}$, and set $f_n(z) = nF(z)$, $n = 1, 2, \ldots$. Let $\mathcal{F} = \{f_n\}$; then $\mathcal{F}$ is a family of holomorphic functions on D. Obviously, for each $f \in \mathcal{F}$, $f^{(k)} \equiv f$, so f and $f^{(k)}$ share *every* complex value in D. But $\mathcal{F}$ is not normal in D. This shows that the requirement of multiplicity k in Theorem 1 cannot be dropped in general.

Example 4. Theorem 1 does not hold if the requirement that $f^{(k)}(z) = b \Rightarrow f(z) = c$ is replaced by $f(z) = c \Rightarrow f^{(k)}(z) = b$. Indeed, set

$$f_n(z) = \frac{(nz)^2}{(nz)^2 - 1}, \qquad n = 1, 2, \ldots,$$

and let $\mathcal{F} = \{f_n\}$, $D = \{z : |z| < 1\}$. Then

$$f_n'(z) = \frac{-2n^2 z}{[(nz)^2 - 1]^2}.$$

Obviously, if $f \in \mathcal{F}$, then f and f' vanish only at 0; also, $f(z) \neq 1$. Thus, if we choose $k = 1$, $a = 0$, and $c = 1$, we have $f(z) = 0 \Leftrightarrow f'(z) = 0$, and $f(z) = 1 \Rightarrow f'(z) = b$ for *any* b (since $f(z) \neq 1$). However, $\mathcal{F}$ is not normal on D.

Theorem 2. *Let f be a transcendental meromorphic function, $k \geq 2$ an integer, and $a \in \mathbb{C}$. If all zeros of f have multiplicity at least k and $f(z) = 0 \Leftrightarrow f^{(k)}(z) = a$, then $f^{(k)}$ takes on each nonzero finite value b infinitely many times.*

2. Some lemmas. For the proofs of our theorems, we need the following lemmas.

LEMMA 1 ([14, Lemma 2]). *Let $\mathcal{F}$ be a family of functions meromorphic on the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal, there exist, for each $0 \leq \alpha \leq k$,*

(a) *a number $0 < r < 1$;*
(b) *points z_n, $|z_n| < r$;*
(c) *functions $f_n \in \mathcal{F}$; and*
(d) *positive numbers $\varrho_n \to 0$*

such that $\varrho_n^{-\alpha} f_n(z_n + \varrho_n \zeta) = g_n(\zeta) \to g(\zeta)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$. In particular, g has order at most 2.

Here, as usual, $g^{\#}(\zeta) = |g'(\zeta)|/(1 + |g(\zeta)|^2)$ is the spherical derivative.

REMARK. That all zeros of g have multiplicity at least k is immediate from the argument principle. That g has order at most 2 follows easily from the fact that $g^{\#}$ is bounded; cf. [19, p. 217]. For $0 \leq \alpha < k$, the hypothesis on $f^{(k)}(z)$ can be dropped, and $kA + 1$ can be replaced by an arbitrary positive constant.

LEMMA 2 ([3, Corollary 3]). *Let g be a meromorphic function with finite order. If g has only finitely many critical values, then it has only finitely many asymptotic values.*

LEMMA 3 ([1, Lemma 2]; cf. [2, Lemma 3]). *Let g be a transcendental meromorphic function such that $g(0) \neq \infty$ and the set of finite critical and asymptotic values of g is bounded. Then there exists $R > 0$ such that*

$$|g'(z)| \geq \frac{|g(z)|}{2\pi|z|} \log \frac{|g(z)|}{R}$$

for all $z \in \mathbb{C} \setminus \{0\}$ which are not poles of g.

LEMMA 4 ([8, Theorem 3], [9, Corollary to Theorem 3.5]). *Let f be a transcendental meromorphic function, and let b be a nonzero value. Then, for each positive integer k, either f or $f^{(k)} - b$ has infinitely many zeros.*

LEMMA 5. *Let f be a transcendental meromorphic function of finite order in the complex plane, k a positive integer, and a and $b \neq 0$ complex numbers. If all zeros of f have multiplicity at least k and $f(z) = 0 \Leftrightarrow f^{(k)}(z) = a$, then $f^{(k)} - b$ has infinitely many zeros.*

Proof. We consider two cases.

CASE 1: f has only finitely many zeros. In this case, $f^{(k)} - b$ has infinitely many zeros by Lemma 4.

CASE 2: f has infinitely many zeros $z_1, z_2, \ldots$. We define $g(z) = f^{(k-1)}(z) - bz$; then $g'(z) = f^{(k)}(z) - b$. We have to show that g' has infinitely many zeros. Suppose that g' has only finitely many zeros; then g has finitely many critical values. Hence, by Lemma 2, g has only finitely many asymptotic values. Without loss of generality, we may assume that $f(0) \neq \infty$ (and hence $g(0) \neq \infty$). Then by Lemma 3 we have

$$\frac{|z_j g'(z_j)|}{|g(z_j)|} \geq \frac{1}{2\pi} \log \frac{|g(z_j)|}{R} = \frac{1}{2\pi} \log \frac{b|z_j|}{R}.$$

In particular,

$$\frac{|z_j g'(z_j)|}{|g(z_j)|} \to \infty \quad \text{as } j \to \infty.$$

On the other hand,

$$\frac{|z_j g'(z_j)|}{|g(z_j)|} = \left|\frac{a-b}{b}\right|,$$

a contradiction. It follows that $g'(z) = f^{(k)}(z) - b$ has infinitely many zeros. This completes the proof of Lemma 5.

LEMMA 6 ([16, Lemma 8]). *Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 + q(z)/p(z)$ where $a_0, a_1, \ldots, a_n$ are constants with $a_n \neq 0$ and p and q are two coprime polynomials, neither of which vanishes identically, with $\deg q < \deg p$; and let k be a positive integer. If $f^{(k)}(z) \neq 1$, then*

$$f(z) = \frac{z^k}{k!} + \ldots + a_0 + \frac{1}{(\alpha z + \beta)^m}.$$

Here $\alpha \neq 0$ and β are constants and m is a positive integer.

LEMMA 7. *Let f be a meromorphic function of finite order, a and $b \neq 0$ distinct complex numbers, and $k \geq 2$ a positive integer. If all zeros of f have multiplicity at least k, $f(z) = 0 \Leftrightarrow f^{(k)}(z) = a$, and $f^{(k)}(z) \neq b$, then f is constant.*

Proof. By Lemma 5, f is a rational function. We assume $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 + q(z)/p(z)$, where $a_0, a_1, \ldots, a_n$ are constants with $a_n \neq 0$, q and p are two coprime polynomials with $\deg q < \deg p$, and n is a positive integer. Without loss of generality, we assume that $b = 1$. Suppose that q does not vanish identically. Then by Lemma 6,

$$f(z) = \frac{1}{k!} z^k + \ldots + a_0 + \frac{1}{(\alpha z + \beta)^m}, \qquad f^{(k)}(z) = 1 + \frac{A}{(\alpha z + \beta)^{k+m}},$$

where $A \neq 0$, $\alpha \neq 0$ and β are constants. Since the zeros of f all have multiplicity at least k, the set $\{z \in \mathbb{C} : f(z) = 0\}$ has at most $(k+m)/k$

distinct elements, while the set $\{z \in \mathbb{C} : f^{(k)}(z) = a\}$ has $k+m$ distinct elements. This contradicts the assumptions that $f(z) = 0 \Leftrightarrow f^{(k)}(z) = a$ and $k \geq 2$.

It follows that f is a polynomial. In this case, one checks easily that f is constant. The lemma is proved.

Using Lemmas 5 and 6, we obtain, after a simple calculation, the following result.

Lemma 8 (cf. [13, Lemma 6]). *Let f be a nonconstant meromorphic function of finite order, and let a and $b \neq 0$ be complex numbers. If $f(z) = 0 \Leftrightarrow f'(z) = a$, and $f'(z) \neq b$ in $\mathbb{C}$, then*

$$f(z) = b(z-d) + \frac{A}{m(z-d)^m}, \qquad a = (m+1)b,$$

for some $d \in \mathbb{C}$ and some positive integer m.

Lemma 9 ([5], [10]; cf. [2]). *Let f be a nonconstant meromorphic function on the plane and $k \geq 2$ a positive integer. Suppose that $f(z) \neq 0$ and $f^{(k)}(z) \neq 0$ for all $z \in \mathbb{C}$. Then either $f(z) = e^{Az+B}$ or $f(z) = \frac{1}{(Az+B)^m}$, where $A \neq 0$ and B are constants and m is a positive integer.*

3. Proof of Theorem 1. We may assume that $D = \Delta$, the unit disc. Suppose that $\mathcal{F}$ is not normal on Δ. We consider separately the cases $d = 0$ and $d \neq 0$.

Case I. Suppose $d = 0$. Then by Lemma 1, we can find $f_n \in \mathcal{F}$, $z_n \in \Delta$, and $\varrho_n \to 0^+$ such that $g_n(\zeta) = \varrho_n^{-k} f_n(z_n + \varrho_n\zeta)$ converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function g on $\mathbb{C}$, all of whose zeros have multiplicity at least k, which satisfies $g^{\#}(\zeta) \leq g^{\#}(0) = k(|a|+1)+1$. In particular, g is of order at most 2.

We claim that

(i) $g(\zeta) = 0 \Leftrightarrow g^{(k)}(\zeta) = a$, and
(ii) $g^{(k)}(\zeta) \neq b$ on $\mathbb{C}$.

Suppose that $g(\zeta_0) = 0$. Then by the Hurwitz Theorem, there exist ζ_n, $\zeta_n \to \zeta_0$, such that (for n sufficiently large) $0 = g_n(\zeta_n) = f_n(z_n + \varrho_n\zeta_n)/\varrho_n^k$. Thus $f_n(z_n + \varrho_n\zeta_n) = 0$. Hence $f_n^{(k)}(z_n + \varrho_n\zeta_n) = a$, so that $g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \varrho_n\zeta_n) = a$. Since $g^{(k)}(\zeta_0) = \lim_{n\to\infty} g_n^{(k)}(\zeta_n) = a$, we have shown that $g(\zeta) = 0 \Rightarrow g^{(k)}(\zeta) = a$.

Suppose now that $g^{(k)}(\zeta_0) = a$. We claim that $g^{(k)} \not\equiv a$. Indeed, if $a = 0$, g would be a polynomial of degree less than k and so could not have zeros of multiplicity at least k. If $a \neq 0$, g must be a polynomial of exact degree k. Since each zero of g has multiplicity at least k, g must have a single zero

ζ_1 of multiplicity k, so that $g(\zeta) = a(\zeta - \zeta_1)^k/k!$. A simple calculation then shows that

$$g^{\#}(0) \le \begin{cases} k/2 & \text{if } |\zeta_1| \ge 1, \\ |a| & \text{if } |\zeta_1| < 1, \end{cases}$$

so that $g^{\#}(0) < k(|a| + 1) + 1$, a contradiction. Since $g^{(k)}(\zeta_0) = a$ but $g^{(k)} \not\equiv a$, there exist ζ_n, $\zeta_n \to \zeta_0$, such that $f_n^{(k)}(z_n + \varrho_n\zeta_n) = g_n^{(k)}(\zeta_n) = a$ for n sufficiently large. It follows that $f_n(z_n + \varrho_n\zeta_n) = 0$, so that $g_n(\zeta_n) = f_n(z_n + \varrho_n\zeta_n)/\varrho_n^k = 0$. Since $g(\zeta_0) = \lim_{n\to\infty} g_n(\zeta_n) = 0$, we have shown that $g^{(k)}(\zeta) = a \Rightarrow g(\zeta) = 0$.

This proves (i).

Next we prove (ii). Suppose $g^{(k)}(\zeta_0) = b$. Then $g(\zeta_0) \neq \infty$. Further $g^{(k)} \not\equiv b$, since that would imply $g(\zeta) = b(\zeta - \zeta_1)^k/k!$, which is inconsistent with (i). Thus, by the Hurwitz Theorem, there exist ζ_n, $\zeta_n \to \zeta_0$, such that (for large n) $g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \varrho_n\zeta_n) = b$. Since $f_n^{(k)}(z) = b \Rightarrow f_n(z) = c$, we have $f_n(z_n + \varrho_n\zeta_n) = c$ and $g_n(\zeta_n) = f_n(z_n + \varrho_n\zeta_n)/\varrho_n^k = c/\varrho_n^k \to \infty$, which contradicts $\lim_{n\to\infty} g_n(\zeta_n) = g(\zeta_0) \neq \infty$. That proves (ii).

If $k \ge 2$, g is constant by Lemma 7, a contradiction. If $k = 1$, then by Lemma 8,

$$g(\zeta) = b(\zeta - d) + \frac{A}{m(\zeta - d)^m}, \qquad a = (m + 1)b,$$

for some positive integer m, a possibility that is ruled out explicitly in the hypothesis of the theorem. Thus $\mathcal{F}$ is normal on D.

Case II. Suppose now that $d \neq 0$. We may assume that $k \ge 2$. By Lemma 1, we can find $f_n \in \mathcal{F}$, $z_n \in \Delta$, and $\varrho_n \to 0^+$ such that $g_n(\zeta) = f_n(z_n + \varrho_n\zeta) - d$ converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function g on $\mathbb{C}$, all of whose zeros have multiplicity at least k.

We claim that

(iii) $g^{(k)}(\zeta) \neq 0$ on $\mathbb{C}$, and
(iv) $g(\zeta) \neq -d$ on $\mathbb{C}$.

Suppose that $g^{(k)}(\zeta_0) = 0$. Clearly $g^{(k)} \not\equiv 0$, for otherwise g would be a polynomial of degree less than k, and so could not have zeros of multiplicity at least k. Hence, since $g_n^{(k)}(\zeta) - \varrho_n^k a \to g^{(k)}(\zeta)$ on a neighborhood of ζ_0, there exist ζ_n, $\zeta_n \to \zeta_0$, such that (for n sufficiently large)

$$0 = g^{(k)}(\zeta_0) = g_n^{(k)}(\zeta_n) - \varrho_n^k a = \varrho_n^k[f_n^{(k)}(z_n + \varrho_n\zeta_n) - a].$$

Thus $f_n^{(k)}(z_n + \varrho_n\zeta_n) = a$, so that $f_n(z_n + \varrho_n\zeta_n) = 0$. It follows that $g_n(\zeta_n) = f_n(z_n + \varrho_n\zeta_n) - d = -d$, and so $g(\zeta_0) = \lim_{n\to\infty} g_n(\zeta_n) = -d$.

In a similar fashion, considering $g_n^{(k)}(\zeta)-\varrho_n^k b$ instead of $g_n^{(k)}(\zeta)-\varrho_n^k a$, we obtain $g(\zeta_0)=c-d$. Thus $c=0$, contrary to assumption. This completes the proof of (iii).

Finally, we prove (iv). Suppose that $g(\zeta_0)=-d$. Then there exist ζ_n, $\zeta_n\to\zeta_0$, such that (for n sufficiently large) $-d=g(\zeta_0)=g_n(\zeta_n)=f_n(z_n+\varrho_n\zeta_n)-d$. Thus $f_n(z_n+\varrho_n\zeta_n)=0$, and hence $f_n^{(k)}(z_n+\varrho_n\zeta_n)=a$. It follows that $g_n^{(k)}(\zeta_n)=\varrho_n^k f_n^{(k)}(z_n+\varrho_n\zeta_n)\to 0$. Therefore, $g^{(k)}(\zeta_0)=\lim_{n\to\infty} g_n^{(k)}(\zeta_n)=0$. But this contradicts (iii). That proves (iv).

Now by Lemma 9, either $g(\zeta)=-d+e^{A\zeta+B}$ or $g(\zeta)=-d+1/(Az+B)^m$, where $A\neq 0$ and B are constants and m is a positive integer. In either case, g has a nonempty set of zeros (it is here that we use the assumption $d\neq 0$), all of which are obviously simple. This contradicts the fact that all zeros of g have multiplicity at least $k\geq 2$. Thus, in Case II also, $\mathcal{F}$ is normal. This completes the proof of Theorem 1.

4. Proof of Theorem 2. From Theorem 1, we obtain the following result, which will be used in the proof of Theorem 2.

LEMMA 10. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D; let $k\geq 2$ be an integer; and let a and $b\neq 0$ be distinct complex numbers. If, for each $f\in\mathcal{F}$, all zeros of f have multiplicity at least k, $f(z)=0\Leftrightarrow f^{(k)}(z)=a$, and $f^{(k)}(z)\neq b$, then $\mathcal{F}$ is normal in D.*

Now we prove Theorem 2.

In case $b=a\neq 0$, the theorem follows at once from Lemma 4. Suppose then that $b\neq a,0$. If f has finite order, the theorem then follows from Lemma 5. So suppose that f has infinite order. Then $f^{\#}$ is unbounded on $\mathbb{C}$, so there exist $w_n\to\infty$ such that $f^{\#}(w_n)\to\infty$. Let $f_n(z)=f(z+w_n)$ and consider the family $\mathcal{F}=\{f_n\}$ on the unit disc Δ. Clearly, for each n, all zeros of f_n have multiplicity at least k and $f_n(z)=0\Leftrightarrow f_n^{(k)}(z)=a$. Since $f_n^{\#}(0)=f^{\#}(w_n)\to\infty$, no infinite subfamily of $\mathcal{F}$ is normal on Δ. Suppose now that $f^{(k)}(z)=b$ has only finitely many solutions. Then, since $w_n\to\infty$, there exists N such that no function in $\mathcal{F}_N=\{f_n : n\geq N\}$ takes on the value b in Δ. By Lemma 10, $\mathcal{F}_N$ is normal on Δ, a contradiction.

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