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An extension theorem for separately holomorphic functions with analytic singularities

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Dedicated to Professor Józef Siciak in honour of his 70th birthday

Abstract. Let $D_j \subset \mathbb{C}^{k_j}$ be a pseudoconvex domain and let $A_j \subset D_j$ be a locally pluriregular set, $j = 1, \ldots, N$. Put

$$X := \bigcup_{j=1}^{N} A_1 \times \ldots \times A_{j-1} \times D_j \times A_{j+1} \times \ldots \times A_N \subset \mathbb{C}^{k_1+\ldots+k_N}.$$

Let U be an open connected neighborhood of X and let $M \subsetneq U$ be an analytic subset. Then there exists an analytic subset $\widehat{M}$ of the "envelope of holomorphy" $\widehat{X}$ of X with $\widehat{M} \cap X \subset M$ such that for every function f separately holomorphic on $X \setminus M$ there exists an $\widehat{f}$ holomorphic on $\widehat{X} \setminus \widehat{M}$ with $\widehat{f}|_{X \setminus M} = f$. The result generalizes special cases which were studied in [Ökt 1998], [Ökt 1999], [Sic 2001], and [Jar-Pfl 2001].

1. Introduction. Main theorem. Let $N \in \mathbb{N}$, $N \geq 2$, and let

$$\emptyset \neq A_j \subset D_j \subset \mathbb{C}^{k_j},$$

where D_j is a domain, $j = 1, \ldots, N$. We define an *N-fold cross*

$$(1)\quad X := \mathbb{X}(A_1, \ldots, A_N; D_1, \ldots, D_N) := \bigcup_{j=1}^{N} A_1 \times \ldots \times A_{j-1} \times D_j \times A_{j+1} \times \ldots \times A_N \subset \mathbb{C}^{k_1+\ldots+k_N}.$$

Observe that X is connected.

Let $\Omega \subset \mathbb{C}^n$ be an open set and let $A \subset \Omega$. Put

$$h_{A,\Omega} := \sup\{u : u \in \mathcal{PSH}(\Omega),\ u \leq 1 \text{ on } \Omega,\ u \leq 0 \text{ on } A\},$$

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where $\mathcal{PSH}(\Omega)$ denotes the set of all functions plurisubharmonic on Ω. Define

$$\omega_{A,\Omega} := \lim_{k\to\infty} h^*_{A\cap\Omega_k,\Omega_k},$$

where $(\Omega_k)_{k=1}^\infty$ is a sequence of relatively compact open sets $\Omega_k \subset \Omega_{k+1} \subset\subset \Omega$ with $\bigcup_{k=1}^\infty \Omega_k = \Omega$, and h^* denotes the upper semicontinuous regularization of h. Observe that the definition is independent of the exhausting sequence $(\Omega_k)_{k=1}^\infty$ chosen. Moreover, $\omega_{A,\Omega} \in \mathcal{PSH}(\Omega)$.

For an N-fold cross $X = \mathbb{X}(A_1,\dots,A_N; D_1,\dots,D_N)$ put

$$\widehat{X} := \Big\{(z_1,\dots,z_N) \in D_1 \times \dots \times D_N : \sum_{j=1}^N \omega_{A_j,D_j}(z_j) < 1\Big\};$$

notice that $\widehat{X}$ may be empty. Observe that $\widehat{X}$ is pseudoconvex if $D_1,\dots,D_N$ are pseudoconvex domains.

We say that a subset $\emptyset \neq A \subset \mathbb{C}^n$ is *locally pluriregular* if $h^*_{A\cap\Omega,\Omega}(a) = 0$ for any $a \in A$ and for any open neighborhood Ω of a (in particular, $A \cap \Omega$ is non-pluripolar).

Note that if $A_1,\dots,A_N$ are locally pluriregular, then $X \subset \widehat{X}$ and, moreover, $\widehat{X}$ is connected (Lemma 4).

Let U be a connected neighborhood of X and let $M \subsetneq U$ be an analytic subset (M may be empty). We say that a function $f\colon X\setminus M \to \mathbb{C}$ is *separately holomorphic* ($f \in \mathcal{O}_s(X\setminus M)$) if for any $(a_1,\dots,a_N) \in A_1 \times \dots \times A_N$ and $j \in \{1,\dots,N\}$ the function $f(a_1,\dots,a_{j-1},\cdot,a_{j+1},\dots,a_N)$ is holomorphic in the domain $\{z_j \in D_j : (a_1,\dots,a_{j-1},z_j,a_{j+1},\dots,a_N) \notin M\}$.

The main result of our paper is the following extension theorem for separately holomorphic functions ([1]).

Main Theorem. *Let $D_j \subset \mathbb{C}^{k_j}$ be a pseudoconvex domain and let $A_j \subset D_j$ be a locally pluriregular set, $j = 1,\dots,N$. Let $M \subsetneq U$ be an analytic subset of an open connected neighborhood U of $X = \mathbb{X}(A_1,\dots,A_N; D_1,\dots,D_N)$ (M may be empty). Then there exists a pure one-codimensional analytic subset $\widehat{M} \subset \widehat{X}$ such that*:

- *$\widehat{M} \cap U_0 \subset M$ for an open neighborhood U_0 of X, $U_0 \subset U$,*
- *for every $f \in \mathcal{O}_s(X\setminus M)$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{X}\setminus\widehat{M})$ with $\widehat{f}|_{X\setminus M} = f$.*

Moreover, if $U = \widehat{X}$, then we can take $\widehat{M}$:= the union of all one-codimensional irreducible components of M.

The proof will be given in Sections 3 (the case $U = \widehat{X}$) and 4 (the general case).

(1) We thank Professor Józef Siciak for turning our attention to this problem.

REMARK. Notice that the Main Theorem may be generalized to the case where D_j is a Riemann–Stein domain over $\mathbb{C}^{k_j}$, $j = 1, \dots, N$.

Observe that in the case $M = \emptyset$, $N = 2$, the Main Theorem is nothing else than the following cross theorem.

THEOREM 1 (cf. [Ale-Zer 2001]). *Let $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ be pseudoconvex domains and let $A \subset D$, $B \subset G$ be locally pluriregular. Put $X := \mathbb{X}(A, B; D, G)$. Then for any $f \in \mathcal{O}_s(X)$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{X})$ with $\widehat{f} = f$ on X.*

REMARK. (a) Let $M = \emptyset$. There is a long list of papers discussing the case $N = 2$ (under various assumptions): [Sic 1969], [Zah 1976], [Sic 1981], [Shi 1989], [Ngu-Zer 1991], [Ngu 1997], [Ale-Zer 2001]. The case $N \geq 2$, $k_1 = \dots = k_N = 1$ can be found in [Sic 1981]. The general case $N \geq 2$, $k_1, \dots, k_N \geq 1$ was solved in [Ngu-Zer 1995] (2).

(b) Let $M \neq \emptyset$. J. Siciak [Sic 2001] solved the case: $N \geq 2$, $k_1 = \dots = k_N = 1$, $D_1 = \dots = D_N = \mathbb{C}$, $M = P^{-1}(0)$, where P is a non-zero polynomial of N complex variables. The special subcase $N = 2$, $P(z, w) := z - w$ had been studied in [Ökt 1998]. The general case for $N = 2$, $k_1 = k_2 = 1$ was solved in [Jar-Pfl 2001]; see also [Ökt 1999] for a partial discussion of the case $N = 2$, $k_1, k_2 \geq 1$.

The case where the singular set M is a pluripolar relatively closed subset of U is studied in [Jar-Pfl 2003].

2. Auxiliary results. The following lemma gathers a few standard results, which will be frequently used in what follows.

LEMMA 2 (cf. [Kli 1991], [Jar-Pfl 2000, §3.5]). (a) *Let $\Omega \subset \mathbb{C}^n$ be a bounded open set and let $A \subset \Omega$. Then*:

- *If $P \subset \mathbb{C}^n$ is pluripolar, then $h^*_{A \setminus P, \Omega} = h^*_{A, \Omega}$.*
- *$h^*_{A_k \cap \Omega_k, \Omega_k} \searrow h^*_{A, \Omega}$ (pointwise on Ω) for any sequence of open sets $\Omega_k \nearrow \Omega$ and any sequence $A_k \nearrow A$.*
- *$\omega_{A, \Omega} = h^*_{A, \Omega}$.*
- *The following conditions are equivalent*: *for any connected component S of Ω the set $A \cap S$ is non-pluripolar*, *and $h^*_{A, \Omega}(z) < 1$ for any $z \in \Omega$.*
- *If A is non-pluripolar*, *$0 < \alpha < 1$, and $\Omega_\alpha := \{z \in \Omega : h^*_{A, \Omega}(z) < \alpha\}$, then for any connected component S of Ω_α the set $A \cap S$ is non-pluripolar (in particular, $A \cap S \neq \emptyset$).*
- *If A is locally pluriregular*, *$0 < \alpha \leq 1$, and Ω_α is as above, then $h^*_{A, \Omega} = \alpha h^*_{A, \Omega_\alpha}$ on Ω_α.*

(2) We thank Professor Nguyen Thanh Van for calling our attention to that paper.

(b) *Let $\Omega \subset \mathbb{C}^n$ be an open set and let $A \subset \Omega$. Then*:

- $\omega_{A,\Omega} \in \mathcal{PSH}(\Omega)$.
- *If A is locally pluriregular*, *then $\omega_{A,\Omega}(a) = 0$ for any $a \in A$.*
- *If $P \subset \mathbb{C}^n$ is pluripolar*, *then $\omega_{A\setminus P,\Omega} = \omega_{A,\Omega}$.*
- *If A is locally pluriregular and $P \subset \mathbb{C}^n$ is pluripolar*, *then $A \setminus P$ is locally pluriregular.*

Moreover, we get:

LEMMA 3. (a) *Let $A_j \subset \mathbb{C}^{k_j}$ be locally pluriregular*, *$j = 1, \ldots, N$. Then $A_1 \times \ldots \times A_N$ is locally pluriregular.*

(b) *Let $A_j \subset \Omega_j \Subset \mathbb{C}^{k_j}$*, *$\Omega_j$ a domain*, *A_j locally pluriregular*, *$j = 1, \ldots, N$*, *$N \geq 2$. Put*

$$\Omega := \Big\{(z_1, \ldots, z_N) \in \Omega_1 \times \ldots \times \Omega_N : \sum_{j=1}^N h^*_{A_j,\Omega_j}(z_j) < 1\Big\}$$

(*observe that $A_1 \times \ldots \times A_N \subset \Omega$*). *Then*

$$h^*_{A_1\times\ldots\times A_N,\Omega} = \sum_{j=1}^N h^*_{A_j,\Omega_j} \quad \textit{on } \Omega.$$

Proof. (a) is an immediate consequence of the product property for the relatively extremal function

$$h^*_{B_1\times\ldots\times B_N,\Omega_1\times\ldots\times\Omega_N} = \max\{h^*_{B_j,\Omega_j} : j = 1, \ldots, N\};$$

cf. [Ngu-Sic 1991].

(b) First observe that

$$\sum_{j=1}^N h^*_{A_j,\Omega_j} \leq h^*_{A_1\times\ldots\times A_N,\Omega} \quad \text{on } \Omega.$$

To get the opposite inequality we proceed by induction on $N \geq 2$.

Let $N = 2$. The proof of this step is taken from [Sic 1981]. For the reader's convenience we repeat the details.

Put $u := h^*_{A_1\times A_2,\Omega} \in \mathcal{PSH}(\Omega)$ and fix a point $(a_1, a_2) \in \Omega$. If $a_1 \in A_1$ (thus $h^*_{A_1,\Omega_1}(a_1) = 0$), then $u(a_1, \cdot) \in \mathcal{PSH}(\Omega_2)$ with $u(a_1, \cdot) \leq 1$ and $u(a_1, \cdot) \leq 0$ on A_2. Therefore,

$$u(a_1, \cdot) \leq h^*_{A_2,\Omega_2} = h^*_{A_1,\Omega_1}(a_1) + h^*_{A_2,\Omega_2} \quad \text{on } \Omega_2.$$

In particular, $u(a_1, a_2) \leq h^*_{A_1,\Omega_1}(a_1) + h^*_{A_2,\Omega_2}(a_2)$.

Observe that the same argument shows that if $a_2 \in A_2$, then $u(\cdot, a_2) \leq h^*_{A_1,\Omega_1}$ on Ω_1.

If $a_1 \notin A_1$, then $h^*_{A_1,\Omega_1}(a_1) + h^*_{A_2,\Omega_2}(a_2) < 1$ and therefore $\alpha := 1 - h^*_{A_1,\Omega_1}(a_1) \in (0,1]$. Put

$$(\Omega_2)_\alpha := \{z_2 \in \Omega_2 : h^*_{A_2,\Omega_2}(z_2) < \alpha\}.$$

It is clear that $A_2 \subset (\Omega_2)_\alpha \ni a_2$. Put

$$v := \frac{1}{\alpha}\,(u(a_1,\cdot) - h^*_{A_1,\Omega_1}(a_1)) \in \mathcal{PSH}((\Omega_2)_\alpha).$$

Then $v \le 1$ and $v \le 0$ on A_2. Therefore, by Lemma 2(a),

$$v \le h^*_{A_2,(\Omega_2)_\alpha}(a_2) = \frac{1}{\alpha}\,h^*_{A_2,\Omega_2}(a_2) \quad \text{on } (\Omega_2)_\alpha.$$

Consequently, $u(a_1,a_2) \le h^*_{A_1,\Omega_1}(a_1) + h^*_{A_2,\Omega_2}(a_2)$, which finishes the proof for $N = 2$.

Now, assume that the formula is true for $N-1 \ge 2$. Put

$$\widetilde{\Omega} := \Big\{(z_1,\ldots,z_{N-1}) \in \Omega_1 \times \ldots \times \Omega_{N-1} : \sum_{j=1}^{N-1} h^*_{A_j,\Omega_j}(z_j) < 1\Big\}$$

and fix an arbitrary $z = (\widetilde{z}, z_N) \in \Omega$. Obviously, $\widetilde{z} \in \widetilde{\Omega}$. By the inductive hypothesis, we conclude that

$$h^*_{A_1\times\ldots\times A_{N-1},\widetilde{\Omega}}(\widetilde{z}) = \sum_{j=1}^{N-1} h^*_{A_j,\Omega_j}(z_j). \tag{2}$$

Now we apply the case $N = 2$ to the following situation:

$$\Omega' := \{(\widetilde{w}, w_N) \in \widetilde{\Omega} \times \Omega_N : h^*_{A_1\times\ldots\times A_{N-1},\widetilde{\Omega}}(\widetilde{w}) + h^*_{A_N,\Omega_N}(w_N) < 1\}.$$

So

$$\begin{aligned} h^*_{A_1\times\ldots\times A_{N-1},\widetilde{\Omega}}(\widetilde{w}) &+ h^*_{A_N,\Omega_N}(w_N) \\ &= h^*_{A_1\times\ldots\times A_N,\Omega'}(\widetilde{w}, w_N), \quad (\widetilde{w}, w_N) \in \Omega'. \end{aligned}$$

Note that $\Omega' = \Omega$. Hence

$$h^*_{A_1\times\ldots\times A_N,\Omega}(\widetilde{z}, z_N) = h^*_{A_1\times\ldots\times A_{N-1},\widetilde{\Omega}}(\widetilde{z}) + h_{A_N,\Omega_N}(z_N) \overset{(2)}{=} \sum_{j=1}^{N} h^*_{A_j,\Omega_j}(z_j). \quad \blacksquare$$

LEMMA 4. *Let* $X = \mathbb{X}(A_1,\ldots,A_N; D_1,\ldots,D_N)$ *be an* N*-fold cross as in* (1). *If* $A_1,\ldots,A_N$ *are locally pluriregular*, *then* $\widehat{X}$ *is a domain.*

Proof. Using exhaustion by bounded domains we may assume that the D_j's are bounded.

We know that $X \subset \widehat{X}$. Let $z^0 = (z_1^0,\ldots,z_N^0) \in \widehat{X}$ be an arbitrary point.

If $\sum_{j=2}^N h^*_{A_j,D_j}(z_j^0) = 0$, then $D_1 \times \{(z_2^0,\ldots,z_N^0)\} \subset \widehat{X}$. Therefore, z^0 can be joined inside $D_1 \times \{(z_2^0,\ldots,z_N^0)\}$ to $(a_1, z_2^0,\ldots,z_N^0)$ for some $a_1 \in A_1$.

If $\sum_{j=2}^{N} h^*_{A_j,D_j}(z_j^0) =: \varepsilon > 0$, put

$$(D_1)_{1-\varepsilon} := \{z_1 \in D_1 : h^*_{A_1,D_1}(z_1) < 1-\varepsilon\}.$$

Then, by Lemma 2(a), the connected component S of $(D_1)_{1-\varepsilon}$ that contains z_1^0 intersects A_1. Therefore, z^0 can be joined inside $S \times \{(z_2^0, \dots, z_N^0)\} \subset \widehat{X}$ to $(a_1, z_2^0, \dots, z_N^0)$ for some $a_1 \in A_1$.

Now we repeat the above argument for the second component of the point $(a_1, z_2^0, \dots, z_N^0)$. Finally, the point z^0 can be joined inside $\widehat{X}$ to $(a_1, \dots, a_N) \in A_1 \times \dots \times A_N \subset X$. Since X is connected, the proof is complete. ∎

LEMMA 5 (Identity theorems). (a) *Let $\Omega \subset \mathbb{C}^n$ be a domain and let $A \subset \Omega$ be non-pluripolar. Then any $f \in \mathcal{O}(\Omega)$ with $f|_A = 0$ vanishes identically on Ω.*

(b) *Let $D \subset \mathbb{C}^p, G \subset \mathbb{C}^q$ be domains, let $A \subset D$, $B \subset G$ be locally pluriregular sets, and let $X := \mathbb{X}(A, B; D, G)$. Let $M \subsetneq U$ be an analytic subset of an open connected neighborhood U of X. Assume that $A' \subset A$, $B' \subset B$ are such that*:

- *$A \setminus A'$ and $B \setminus B'$ are pluripolar (in particular, A', B' are also locally pluriregular),*
- *$M_z := \{w \in G : (z, w) \in M\} \neq G$ for any $z \in A'$,*
- *$M^w := \{z \in D : (z, w) \in M\} \neq D$ for any $w \in B'$.*

Then:

(b$_1$) *If $f \in \mathcal{O}_{\mathrm{s}}(X \setminus M)$ and $f = 0$ on $A' \times B' \setminus M$ ([3]), then $f = 0$ on $X \setminus M$.*

(b$_2$) *If $g \in \mathcal{O}(U \setminus M)$ and $g = 0$ on $A' \times B' \setminus M$, then $g = 0$ on $U \setminus M$.*

Proof. (a) is obvious.

(b$_1$) Take a point $(a_0, b_0) \in X \setminus M$. We may assume that $a_0 \in A$. Since $A \setminus A'$ is pluripolar, there exists a sequence $(a_k)_{k=1}^{\infty} \subset A'$ such that $a_k \to a_0$. The set $Q := \bigcup_{k=0}^{\infty} M_{a_k}$ is pluripolar. Consequently, the set $B'' := B' \setminus Q$ is non-pluripolar. We have $f(a_k, w) = 0$, $w \in B''$, $k = 1, 2, \dots$ Hence $f(a_0, w) = 0$ for any $w \in B''$. Finally, $f(a_0, w) = 0$ on $G \setminus M_{a_0} \ni b_0$.

(b$_2$) Take an $a_0 \in A'$. Since $M_{a_0} \neq G$, there exists a $b_0 \in B' \setminus M_{a_0}$. Let $P = \Delta_{a_0}(r) \times \Delta_{b_0}(r) \subset U \setminus M$ ($\Delta_{z_0}(r) \subset \mathbb{C}^p$ denotes the polydisc with center $z_0 \in \mathbb{C}^p$ and radius $r > 0$). Then $g(\cdot, w) = 0$ on $A' \cap \Delta_{a_0}(r)$ for any $w \in B' \cap \Delta_{b_0}(r)$. The set $A' \cap \Delta_{a_0}(r)$ is non-pluripolar. Hence $g(\cdot, w) = 0$ on $\Delta_{a_0}(r)$ for any $w \in B' \cap \Delta_{b_0}(r)$. By the same argument for the second variable we get $g = 0$ on P and, consequently, on $U \setminus M$. ∎

([3]) Here and below, to simplify notation we write $P \times Q \setminus M$ instead of $(P \times Q) \setminus M$.

3. Proof of the Main Theorem in the case where $U = \widehat{X}$. We proceed by several reduction steps. First observe that, by Lemma 5(a), the function $\widehat{f}$ is uniquely determined (if it exists).

STEP 1. *To prove the Main Theorem for $M \neq \emptyset$ it suffices to consider the case where M is pure one-codimensional.*

Proof. Since $\widehat{X}$ is pseudoconvex, the arbitrary analytic set $M \subset \widehat{X}$ can be written as

$$M = \{z \in \widehat{X} : g_1(z) = \ldots = g_k(z) = 0\},$$

where $g_j \in \mathcal{O}(\widehat{X})$, $g_j \not\equiv 0$, $j = 1, \ldots, k$. Then $M_j := g_j^{-1}(0)$ is pure one-codimensional.

Take an $f \in \mathcal{O}_s(X \setminus M)$. Observe that $f_j := f|_{X \setminus M_j} \in \mathcal{O}_s(X \setminus M_j)$. By the reduction assumption there exists an $\widehat{f}_j \in \mathcal{O}(\widehat{X} \setminus M_j)$ such that $\widehat{f}_j = f$ on $X \setminus M_j$. In view of Lemma 5(a), gluing the functions $(\widehat{f}_j)_{j=1}^k$ leads to an $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus M)$ with $\widehat{f} = \widehat{f}_j$ on $\widehat{X} \setminus M_j$, $j = 1, \ldots, k$. Therefore, $\widehat{f} = f$ on $X \setminus M$.

Finally, since $\operatorname{codim}(M \setminus \widehat{M}) \geq 2$, the function $\widehat{f}$ extends holomorphically across $M \setminus \widehat{M}$. ∎

From now on we assume that M is empty or pure one-codimensional.

STEP 2. *To prove the Main Theorem it suffices to consider the case where M is empty or pure one-codimensional and $D_1, \ldots, D_N$ are bounded pseudoconvex.*

Proof. Let $D_1, \ldots, D_N$ be arbitrary pseudoconvex domains, and let $D_{j,k} \nearrow D_j$, $D_{j,k} \Subset D_j$, where $D_{j,k}$ are pseudoconvex domains with $A_{j,k} := A_j \cap D_{j,k} \neq \emptyset$. Observe that all the $A_{j,k}$'s are locally pluriregular. Put

$$X_k := \mathbb{X}(A_{1,k}, \ldots, A_{N,k}; D_{1,k}, \ldots, D_{N,k}) \subset X;$$

note that $\widehat{X}_k \nearrow \widehat{X}$.

Let $f \in \mathcal{O}_s(X \setminus M)$ be given. By the reduction assumption, for each k there exists an $\widehat{f}_k \in \mathcal{O}(\widehat{X}_k \setminus M)$ with $\widehat{f}_k = f$ on $X_k \setminus M$. By Lemma 5(a), $\widehat{f}_{k+1} = \widehat{f}_k$ on $\widehat{X}_k \setminus M$. Therefore, gluing the $\widehat{f}_k$'s, we obtain an $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus M)$ with $\widehat{f} = f$ on $X \setminus M$. ∎

From now on we assume that M is empty or pure one-codimensional and $D_1, \ldots, D_N$ are bounded pseudoconvex.

STEP 3. *To prove the Main Theorem it suffices to consider the case $N = 2$.*

REMARK. By Theorem 1, Step 3 finishes the proof of the Main Theorem for $M = \emptyset$.

Proof of Step 3. We proceed by induction on $N \geq 2$. Suppose that the Main Theorem is true for $N-1 \geq 2$. We have to discuss the case of an N-fold cross $X = \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N)$, where $D_1, \dots, D_N$ are bounded pseudoconvex. Let $M \subset \widehat{X}$ be empty or pure one-codimensional.

Let $f \in \mathcal{O}_{\mathrm{s}}(X \setminus M)$. Observe that

$$X = (Y \times A_N) \cup (\widehat{A} \times D_N),$$

where

$$Y := \mathbb{X}(A_1, \dots, A_{N-1}; D_1, \dots, D_{N-1}), \qquad \widehat{A} := A_1 \times \dots \times A_{N-1}.$$

We also mention that for any $a_N \in A_N$ we have

$$\{(z_1, \dots, z_{N-1}) \in \mathbb{C}^{k_1} \times \dots \times \mathbb{C}^{k_{N-1}} : (z_1, \dots, z_{N-1}, a_N) \in \widehat{X}\} = \widehat{Y}.$$

Now fix an $a_N \in A_N$ such that

$$M_{a_N} := \{(z_1, \dots, z_{N-1}) \in \widehat{Y} : (z_1, \dots, z_{N-1}, a_N) \in M\} \subsetneq \widehat{Y};$$

in particular, M_{a_N} is empty or one-codimensional (in $\widehat{Y}$). Recall that A_1, $\dots$, A_{N-1} are locally pluriregular. By the inductive assumption there exists an $\widehat{f}_{a_N} \in \mathcal{O}(\widehat{Y} \setminus M_{a_N})$ with $\widehat{f}_{a_N} = f(\cdot, a_N)$ on $Y \setminus M_{a_N}$.

To continue define the following 2-fold cross:

$$Z := \mathbb{X}(\widehat{A}, A_N; \widehat{Y}, D_N).$$

Notice that Z satisfies all the properties for the case $N = 2$: $\widehat{Y}, D_N$ are bounded pseudoconvex domains, $\widehat{A} \subset \widehat{Y}$, $A_N \subset D_N$ are locally pluriregular.

By Lemma 3, we have

$$\widehat{Z} = \{(\widehat{z}, z_N) \in \widehat{Y} \times D_N : h^*_{\widehat{A},\widehat{Y}}(\widehat{z}) + h^*_{A_N,D_N}(z_N) < 1\} = \widehat{X}.$$

Define $\widetilde{f} \colon Z \setminus M \to \mathbb{C}$ by

$$\widetilde{f}(z) = \widetilde{f}(\widehat{z}, z_N) := \begin{cases} \widehat{f}_{z_N}(\widehat{z}) & \text{if } z \in \widehat{Y} \times A_N, \\ f(z) & \text{if } z \in \widehat{A} \times D_N. \end{cases}$$

Obviously, $\widetilde{f}$ is well defined and therefore $\widetilde{f} \in \mathcal{O}_{\mathrm{s}}(Z \setminus M)$.

Using the case $N = 2$, we find another function $\widehat{f} \in \mathcal{O}(\widehat{Z} \setminus M)$ with $\widehat{f} = \widetilde{f}$ on $Z \setminus M$. Recall that $\widehat{Z} = \widehat{X}$. Hence $\widehat{f} = f$ on $X \setminus M$. ∎

What remains is to prove the case $N = 2$ and $M \neq \emptyset$. *From now on we simplify our notation and consider the following configuration*:

Let $A \subset D \Subset \mathbb{C}^p$, $B \subset G \Subset \mathbb{C}^q$, where D, G are bounded pseudoconvex domains, A, B are locally pluriregular. Put, as always,

$$X := \mathbb{X}(A, B; D, G), \qquad \widehat{X} := \{(z, w) \in D \times G : h^*_{A,D}(z) + h^*_{B,G}(w) < 1\}.$$

Moreover, let M be a pure one-codimensional analytic subset of $\widehat{X}$.

We want to show that any $f \in \mathcal{O}_{\mathrm{s}}(X \setminus M)$ extends holomorphically to $\widehat{X} \setminus M$.

STEP 4. *Let X, M, and f be as above. Let $(D_j)_{j=1}^{\infty}$, $(G_j)_{j=1}^{\infty}$ be sequences of pseudoconvex domains, $D_j \Subset D$, $G_j \Subset G$, with $D_j \nearrow D$, $G_j \nearrow G$. Moreover, let $A' \subset A$, $B' \subset B$ be such that $A \setminus A'$, $B \setminus B'$ are pluripolar, and $A' \cap D_j \neq \emptyset$, $B' \cap G_j \neq \emptyset$, $j \in \mathbb{N}$. For each $j \in \mathbb{N}$ assume that for any $(a,b) \in (A' \cap D_j) \times (B' \cap G_j)$ there exist polydiscs $\Delta_a(r_{a,j}) \subset D_j$, $\Delta_b(s_{b,j}) \subset G_j$ with $(\Delta_a(r_{a,j}) \times G_j) \cup (D_j \times \Delta_b(s_{b,j})) \subset \widehat{X}$, and functions $f_{a,j} \in \mathcal{O}(\Delta_a(r_{a,j}) \times G_j \setminus M)$, $f^{b,j} \in \mathcal{O}(D_j \times \Delta_b(s_{b,j}) \setminus M)$ such that*

- $f_{a,j} = f$ *on* $(A' \cap \Delta_a(r_{a,j})) \times G_j \setminus M$,
- $f^{b,j} = f$ *on* $D_j \times (B' \cap \Delta_b(s_{b,j})) \setminus M$.

Then there exists an $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus M)$ with $\widehat{f} = f$ on $X \setminus M$.

Proof. Fix a $j \in \mathbb{N}$. Put

$$\widetilde{U}_j := \bigcup_{\substack{a \in A' \cap D_j \\ b \in B' \cap G_j}} (\Delta_a(r_{a,j}) \times G_j) \cup (D_j \times \Delta_b(s_{b,j})),$$

$$X_j := ((A \cap D_j) \times G_j) \cup (D_j \times (B \cap G_j)).$$

Note that

$$X_j' := ((A' \cap D_j) \times G_j) \cup (D_j \times (B' \cap G_j)) \subset \widetilde{U}_j.$$

We wish to glue the functions $(f_{a,j})_{a \in A' \cap D_j}$ and $(f^{b,j})_{b \in B' \cap G_j}$ to obtain a global holomorphic function f_j on $\widetilde{U}_j \setminus M$. Let $a \in A' \cap D_j$, $b \in B' \cap G_j$. Observe that

$$f_{a,j} = f \quad \text{on } (A' \cap \Delta_a(r_{a,j})) \times G_j \setminus M,$$
$$f^{b,j} = f \quad \text{on } D_j \times (B' \cap \Delta_b(s_{b,j})) \setminus M.$$

Thus $f_{a,j} = f^{b,j}$ on $(A' \cap \Delta_a(r_{a,j})) \times (B' \cap \Delta_b(s_{b,j})) \setminus M$. Applying Lemma 5(a), we conclude that

$$f_{a,j} = f^{b,j} \quad \text{on } (\Delta_a(r_{a,j}) \times \Delta_b(s_{b,j})) \setminus M.$$

Now let $a', a'' \in A' \cap D_j$ be such that $\Delta_{a'}(r_{a',j}) \cap \Delta_{a''}(r_{a'',j}) \neq \emptyset$. Fix a $b \in B' \cap G_j$. We know that $f_{a',j} = f^{b,j} = f_{a'',j}$ on $(\Delta_{a'}(r_{a',j}) \cap \Delta_{a''}(r_{a'',j})) \times \Delta_b(r_{b,j}) \setminus M$. Hence, by the identity principle, we conclude that $f_{a',j} = f_{a'',j}$ on $(\Delta_{a'}(r_{a',j}) \cap \Delta_{a''}(r_{a'',j})) \times G_j \setminus M$. The same argument works for $b', b'' \in B' \cap G_j$. Consequently, we obtain a function $f_j \in \mathcal{O}(\widetilde{U}_j \setminus M)$ with $f_j = f$ on $X_j' \setminus M$.

Let U_j be the connected component of $\widetilde{U}_j \cap \widehat{X}_j'$ with $X_j' \subset U_j$. Thus we have $f_j \in \mathcal{O}(U_j \setminus M)$ with $f_j = f$ on $X_j' \setminus M$.

Recall that $X'_j \subset U_j \subset \widehat{X}'_j$. We claim that the envelope of holomorphy of U_j coincides with $\widehat{X}'_j$. In fact, let $h \in \mathcal{O}(U_j)$; then $h|_{X'_j} \in \mathcal{O}_{\mathrm{s}}(X'_j)$. So, by Theorem 1, there exists an $\widehat{h} \in \mathcal{O}(\widehat{X}'_j)$ with $\widehat{h} = h$ on X'_j. Lemma 5(b_2) implies that $\widehat{h} = h$ on U_j.

Applying the Grauert–Remmert theorem (cf. [Jar-Pfl 2000, Th. 3.4.7]), we find a function $\widehat{f}_j \in \mathcal{O}(\widehat{X}'_j \setminus M)$ with $\widehat{f}_j = f_j$ on $U_j \setminus M$. In particular, $\widehat{f}_j = f$ on $X'_j \setminus M$.

Since $A \setminus A'$, $B \setminus B'$ are pluripolar, we get

$$\begin{aligned}\widehat{X}'_j &= \{(z,w) \in D_j \times G_j : h^*_{A' \cap D_j, D_j}(z) + h^*_{B' \cap G_j, G_j}(w) < 1\} \\ &= \{(z,w) \in D_j \times G_j : h^*_{A \cap D_j, D_j}(z) + h^*_{B \cap G_j, G_j}(w) < 1\} = \widehat{X}_j.\end{aligned}$$

So, in fact, $\widehat{f}_j \in \mathcal{O}(\widehat{X}_j \setminus M)$. Using Lemma 5($\mathrm{b}_1$), we even see that $\widehat{f}_j = f$ on $X_j \setminus M$.

Observe that $\bigcup_{j=1}^{\infty} X_j = X$, $\widehat{X}_j \subset \widehat{X}_{j+1}$, and $\bigcup_{j=1}^{\infty} \widehat{X}_j = \widehat{X}$. Using again Lemma 5(a), by gluing the $\widehat{f}_j$'s, we get a function $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus M)$ with $\widehat{f} = f$ on $X \setminus M$. ∎

To apply Step 4 we introduce the following condition (∗). Let $\varrho > 0$, $0 < r < R$. Put

$$\Omega := \Delta_{a_0}(\varrho) \times \Delta_{b_0}(R) \subset \mathbb{C}^p \times \mathbb{C}^q, \qquad \widetilde{\Omega} := \Delta_{a_0}(\varrho) \times \Delta_{b_0}(r) \subset \mathbb{C}^p \times \mathbb{C}^q.$$

Let $A \subset \Delta_{a_0}(\varrho) \subset \mathbb{C}^p$ be locally pluriregular, $a_0 \in A$, and let M be a pure one-codimensional analytic subset of Ω with $M \cap \widetilde{\Omega} = \emptyset$. Put $M_a := \{w \in \Delta_{b_0}(R) : (a,w) \in M\}$, $a \in A$. Condition (∗) reads:

(∗) For any $R' \in (r, R)$ there exists $\varrho' \in (0, \varrho)$ such that for any function $f \in \mathcal{O}(\widetilde{\Omega})$ with $f(a,\cdot) \in \mathcal{O}(\Delta_{b_0}(R) \setminus M_a)$, $a \in A$, there exists an extension $\widehat{f} \in \mathcal{O}(\Delta_{a_0}(\varrho') \times \Delta_{b_0}(R') \setminus M)$ with $\widehat{f} = f$ on $\Delta_{a_0}(\varrho') \times \Delta_{b_0}(r)$.

Step 5. *If condition* (∗) *holds*, *then the assumptions of Step* 4 *are satisfied.*

Proof. Take X, M, $f \in \mathcal{O}_{\mathrm{s}}(X \setminus M)$ as is in Step 4. Define

$$A' := \{a \in A : M_a \neq G\}, \qquad B' := \{a \in B : M^b \neq D\},$$

where $M_a := \{w \in G : (a,w) \in M\}$, $M^b := \{z \in D : (z,b) \in M\}$. It is clear that $A \setminus A'$, $B \setminus B'$ are pluripolar.

Let $(D_j)_{j=1}^{\infty}$, $(G_j)_{j=1}^{\infty}$ be approximation sequences: $D_j \Subset D_{j+1} \Subset D$, $G_j \Subset G_{j+1} \Subset G$, $D_j \nearrow D$, $G_j \nearrow G$, $A' \cap D_j \neq \emptyset$, and $B' \cap G_j \neq \emptyset$, $j \in \mathbb{N}$.

Fix $j \in \mathbb{N}$, $a \in A' \cap D_j$ and let Ω_j be the set of all $b \in G_{j+1}$ such that there exist a polydisc $\varDelta_{(a,b)}(r_b) \subset D_j \times G_{j+1}$ and a function $\widetilde{f}_b \in \mathcal{O}(\varDelta_{(a,b)}(r_b) \setminus M)$ with $\widetilde{f}_b = f$ on $(A \cap \varDelta_a(r_b)) \times \varDelta_b(r_b) \setminus M$.

It is clear that Ω_j is open. Observe that $\Omega_j \neq \emptyset$. Indeed, as $B \cap G_j \setminus M_a \neq \emptyset$, we can choose a point $b \in B \cap G_j \setminus M_a$. Therefore there is a polydisc $\varDelta_{(a,b)}(r) \subset D_j \times G_j \setminus M$. Put

$$Y := \mathbb{X}(A \cap \varDelta_a(r), B \cap \varDelta_b(r); \varDelta_a(r), \varDelta_b(r)).$$

By Theorem 1, we find $r_b \in (0, r)$ and $\widetilde{f}_b \in \mathcal{O}(\varDelta_{(a,b)}(r_b))$ with $\widetilde{f}_b = f$ on $\varDelta_{(a,b)}(r_b) \cap Y \supset (A \cap \varDelta_a(r_b)) \times \varDelta_b(r_b)$. Consequently, $b \in \Omega_j$.

Moreover, Ω_j is relatively closed in G_{j+1}. Indeed, let c be an accumulation point of Ω_j in G_{j+1} and let $\varDelta_c(3R) \subset G_{j+1}$. Take a point $b \in \Omega_j \cap \varDelta_c(R) \setminus M_a$ and let $r \in (0, r_b]$, $r < 2R$, be such that $\varDelta_{(a,b)}(r) \cap M = \emptyset$. Observe that $\widetilde{f}_b \in \mathcal{O}(\varDelta_{(a,b)}(r))$ and $\widetilde{f}_b(z, \cdot) = f(z, \cdot) \in \mathcal{O}(\varDelta_b(2R) \setminus M_z)$ for any $z \in A \cap \varDelta_a(r)$. Hence, by $(*)$ (with $R' := R$), there exists an extension $\widehat{\widetilde{f}}_b \in \mathcal{O}(\varDelta_a(\varrho') \times \varDelta_b(R) \setminus M)$ ($\varrho' \in (0, r)$) such that $\widehat{\widetilde{f}}_b = \widetilde{f}_b$ on $\varDelta_{(a,b)}(r)$. Take an $r_c > 0$ so small that $\varDelta_{(a,c)}(r_c) \subset \varDelta_a(\varrho') \times \varDelta_b(R)$ and put $\widetilde{f}_c := \widehat{\widetilde{f}}_b$ on $\varDelta_{(a,c)}(r_c) \setminus M$. Obviously $\widetilde{f}_c = \widehat{\widetilde{f}}_b = f$ on $(A \cap \varDelta_a(r_c)) \times \varDelta_c(r_c) \setminus M$. Hence $c \in \Omega_j$.

Thus $\Omega_j = G_{j+1}$. There exists a finite set $T \subset \overline{G}_j$ such that

$$\overline{G}_j \subset \bigcup_{b \in T} \varDelta_b(r_b).$$

Define $r_{a,j} := \min\{r_b : b \in T\}$. Take $b', b'' \in T$ with $\varDelta_{b'}(r_{b'}) \cap \varDelta_{b''}(r_{b''}) \neq \emptyset$. Then $\widetilde{f}_{b'} = f = \widetilde{f}_{b''}$ on $(A' \cap \varDelta_a(r_{a,j})) \times (\varDelta_{b'}(r_{b'}) \cap \varDelta_{b''}(r_{b''})) \setminus M$. Consequently, by Lemma 5(a), $\widetilde{f}_{b'} = \widetilde{f}_{b''}$ on $\varDelta_a(r_{a,j}) \times (\varDelta_{b'}(r_{b'}) \cap \varDelta_{b''}(r_{b''})) \setminus M$. In particular, by gluing the functions $(\widetilde{f}_b)_{b \in T}$, we get a function $f_{a,j} \in \mathcal{O}(\varDelta_a(r_{a,j}) \times G_j \setminus M)$ such that $f_{a,j} = f$ on $(A' \cap \varDelta_a(r_{a,j})) \times G_j \setminus M$.

Changing the roles of z and w, we get $f^{b,j}$, $b \in B' \cap G_j$.

Thus the assumptions of Step 4 are satisfied. ∎

It remains to check $(*)$.

STEP 6. *Condition* $(*)$ *is always satisfied*, *i.e. the Main Theorem is true.*

Proof. Fix a function $f \in \mathcal{O}(\widetilde{\Omega})$ such that $f(a, \cdot) \in \mathcal{O}(\varDelta_{b_0}(R) \setminus M_a)$ for any $a \in A$ with $M_a \neq \varDelta_{b_0}(R)$. Define

$$(3) \qquad R_0^* := \sup\{R' \in [r, R) : \exists_{\varrho' \in (0,\varrho]} \exists_{\widehat{f} \in \mathcal{O}(\varDelta_{a_0}(\varrho') \times \varDelta_{b_0}(R') \setminus M)} : \widehat{f} = f \text{ on } \varDelta_{a_0}(\varrho') \times \varDelta_{b_0}(r)\}.$$

It suffices to show that $R_0^* = R$.

Suppose that $R_0^* < R$. Fix $R_0^* < R_0' < R_0 < R$ and choose $R', \varrho', \widehat{f}$ as in (3) with $R' \in [r, R_0^*)$, $\sqrt[q]{R'^{q-1}R_0'} > R_0^*$. Write $w = (w', w_q) \in \mathbb{C}^q = \mathbb{C}^{q-1} \times \mathbb{C}$. Put $\widetilde{A} := A \cap \Delta_{a_0}(\varrho')$.

Let A' denote the set of all $(a, b') \in \widetilde{A} \times \Delta_{b_0'}(R')$ which satisfy the following condition:

$\binom{*}{*}$ There exist $R'' \in (R_0, R)$, $\delta > 0$, $m \in \mathbb{N}$, $c_1, \dots, c_m \in \Delta_{b_{0,q}}(R'')$, $\varepsilon > 0$, and holomorphic functions $\phi_\mu\colon \Delta_{(a,b')}(\delta) \to \Delta_{c_\mu}(\varepsilon)$, $\mu = 1, \dots, m$, such that:

- $\Delta_{(a,b')}(\delta) \subset \Delta_{a_0}(\varrho') \times \Delta_{b_0'}(R')$,
- $\Delta_{c_\mu}(\varepsilon) \Subset \Delta_{b_{0,q}}(R'')$, $\mu = 1, \dots, m$,
- $\bar{\Delta}_{c_\mu}(\varepsilon) \cap \bar{\Delta}_{c_\nu}(\varepsilon) = \emptyset$ for $\mu \neq \nu$, $\mu, \nu = 1, \dots, m$,
- $\widetilde{H} := \Delta_{b_{0,q}}(R') \cap H \neq \emptyset$, where $H := \Delta_{b_{0,q}}(R'') \setminus \bigcup_{\mu=1}^m \bar{\Delta}_{c_\mu}(\varepsilon)$,
- $(\Delta_{(a,b')}(\delta) \times \Delta_{b_{0,q}}(R'')) \cap M = \bigcup_{\mu=1}^m \{(z, w', \phi_\mu(z, w')) : (z, w') \in \Delta_{(a,b')}(\delta)\}$.

For any $(a, b') \in A'$ define a new cross

$$Y := \mathbb{X}((A \cap \Delta_a(\delta)) \times \Delta_{b'}(\delta), \widetilde{H}; \Delta_{(a,b')}(\delta), H).$$

Notice that Y does not intersect M. In particular, $\widehat{f}|_Y \in \mathcal{O}_{\mathrm{s}}(Y)$. Hence, by Theorem 1, there exists an $\widehat{f}_1 \in \mathcal{O}(\widehat{Y})$ with $\widehat{f}_1 = \widehat{f}$ on Y. Take $R''' \in (R_0, R'')$ and $\varepsilon'' > \varepsilon' > \varepsilon$ ($\varepsilon'' \approx \varepsilon$) such that

- $\Delta_{c_\mu}(\varepsilon'') \Subset \Delta_{b_{0,q}}(R''')$, $\mu = 1, \dots, m$,
- $\bar{\Delta}_{c_\mu}(\varepsilon'') \cap \bar{\Delta}_{c_\nu}(\varepsilon'') = \emptyset$ for $\mu \neq \nu$, $\mu, \nu = 1, \dots, m$.

Then there exists $\delta' \in (0, \delta]$ such that

- $\Delta_{(a,b')}(\delta') \times H' \subset \widehat{Y}$, where $H' := \Delta_{b_{0,q}}(R''') \setminus \bigcup_{\mu=1}^m \bar{\Delta}_{c_\mu}(\varepsilon')$.

In particular, $\widehat{f}_1 \in \mathcal{O}(\Delta_{(a,b')}(\delta') \times H')$.

Fix $\mu \in \{1, \dots, m\}$. Then $\widehat{f}_1 \in \mathcal{O}(\Delta_{(a,b')}(\delta') \times (\Delta_{c_\mu}(\varepsilon'') \setminus \bar{\Delta}_{c_\mu}(\varepsilon')))$ and $\widehat{f}_1(z, w', \cdot) \in \mathcal{O}(\Delta_{c_\mu}(\varepsilon'') \setminus \{\phi_\mu(z, w')\})$ for any $(z, w') \in (A \cap \Delta_a(\delta')) \times \Delta_{b'}(\delta')$. Using the biholomorphic mapping

$$\Phi_\mu\colon \Delta_{(a,b')}(\delta') \times \mathbb{C} \to \Delta_{(a,b')}(\delta') \times \mathbb{C},$$
$$\Phi_\mu(z, w', w_q) := (z, w', w_q - \phi_\mu(z, w')),$$

we see that the function $g := \widehat{f}_1 \circ \Phi_\mu^{-1}$ is holomorphic in $\Delta_{(a,b')}(\delta'') \times (\Delta_0(\eta'') \setminus \bar{\Delta}_0(\eta'))$ for some $\delta'' \in (0, \delta']$ and $\varepsilon' < \eta' < \eta'' < \varepsilon''$. Moreover, $g(z, w', \cdot) \in \mathcal{O}(\Delta_0(\eta'') \setminus \{0\})$ for any $(z, w') \in (A \cap \Delta_a(\delta'')) \times \Delta_{b'}(\delta'')$. Using Theorem 1 for the cross

$$\mathbb{X}((A \cap \Delta_a(\delta'')) \times \Delta_{b'}(\delta''), \Delta_0(\eta'') \setminus \bar{\Delta}_0(\eta'); \Delta_{(a,b')}(\delta''), \Delta_0(\eta'') \setminus \{0\})$$

shows that g extends holomorphically to $\Delta_{(a,b')}(\delta'')\times(\Delta_0(\eta'')\setminus\{0\})$ (because $h^*_{\Delta_0(\eta'')\setminus\{0\},\Delta_0(\eta'')\setminus\bar{\Delta}_0(\eta')}\equiv 0$).

Translating the above information back via Φ_μ for all μ, we conclude that the function $\widehat{f}_1$ extends holomorphically to $\Delta_{(a,b')}(\delta''')\times\Delta_{b_{0,q}}(R''')\setminus M$ for some $\delta'''\in(0,\delta'']$; in particular, $\widehat{f}_1$ extends holomorphically to $\Delta_{(a,b')}(\delta''')\times\Delta_{b_{0,q}}(R_0)\setminus M$.

Now we prove that $(\widetilde{A}\times\Delta_{b_0'}(R'))\setminus A'$ is pluripolar. Write

$$M\cap(\Delta_{a_0}(\varrho')\times\Delta_{b_0'}(R')\times\Delta_{b_{0,q}}(R))=\bigcup_{\nu=1}^{\infty}\{\zeta\in P_\nu:g_\nu(\zeta)=0\},$$

where $P_\nu\Subset\Delta_{a_0}(\varrho')\times\Delta_{b_0'}(R')\times\Delta_{b_{0,q}}(R)$ is a polydisc and $g_j\in\mathcal{O}(P_j)$ is a defining function for $M\cap P_j$; cf. [Chi 1989, §2.9]. Define

$$S_\nu:=\left\{\zeta=(\widetilde{\zeta},\zeta_{p+q})\in P_\nu:g_\nu(\zeta)=\frac{\partial g_\nu}{\partial\zeta_{p+q}}(\zeta)=0\right\}$$

and observe that, by the implicit function theorem, any point from

$$(\widetilde{A}\times\Delta_{b_0'}(R'))\setminus\bigcup_{\nu=1}^{\infty}\mathrm{pr}_{\widetilde{\zeta}}(S_\nu)$$

satisfies $(^*_*)$. It is enough to show that each set $\mathrm{pr}_{\widetilde{\zeta}}(S_\nu)$ is pluripolar. Fix ν. Let S be an irreducible component of S_ν. We have to show that $\mathrm{pr}_{\widetilde{\zeta}}(S)$ is pluripolar. If S has codimension ≥ 2, then $\mathrm{pr}_{\widetilde{\zeta}}(S)$ is contained in a countable union of proper analytic sets (cf. [Chi 1989, §3.8]). Consequently, $\mathrm{pr}_{\widetilde{\zeta}}(S)$ is pluripolar. Thus we may assume that S is pure one-codimensional. The same argument as above shows that $\mathrm{pr}_{\widetilde{\zeta}}(\mathrm{Sing}(S))$ is pluripolar. It remains to prove that $\mathrm{pr}_{\widetilde{\zeta}}(\mathrm{Reg}(S))$ is pluripolar. Since g_ν is a defining function, for any $\zeta\in\mathrm{Reg}(S)$ there exists a $k\in\{1,\dots,p+q-1\}$ such that

$$\frac{\partial g_\nu}{\partial\zeta_k}(\zeta)\neq 0.$$

Thus

$$\mathrm{Reg}(S)=\bigcup_{k=1}^{p+q-1}T_k,$$

where

$$T_k:=\left\{\zeta\in\mathrm{Reg}(S):\frac{\partial g_\nu}{\partial\zeta_k}(\zeta)\neq 0\right\}.$$

We only need to prove that each set $\mathrm{pr}_{\widetilde{\zeta}}(T_k)$ is pluripolar, $k=1,\dots,p+q-1$. Fix k. To simplify notation, assume that $k=1$. Observe that, by the implicit function theorem, we can write

$$T_1 = \bigcup_{l=1}^{\infty}\{\zeta \in Q_l : \zeta_1 = \psi_l(\zeta_2, \dots, \zeta_{p+q})\},$$

where $Q_l \subset P_\nu$ is a polydisc, $Q_l = Q_l' \times Q_l'' \subset \mathbb{C} \times \mathbb{C}^{p+q-1}$, and $\psi_l\colon Q_l'' \to Q_l'$ is holomorphic, $l \in \mathbb{N}$. It suffices to prove that the projection of each set $T_{1,l} := \{\zeta \in Q_l : \zeta_1 = \psi_l(\zeta_2, \dots, \zeta_{p+q})\}$ is pluripolar. Fix l. Since

$$g_\nu(\psi_l(\zeta_2, \dots, \zeta_{p+q}), \zeta_2, \dots, \zeta_{p+q}) = 0, \qquad (\zeta_2, \dots, \zeta_{p+q}) \in Q_l'',$$

we conclude that $\partial\psi_l/\partial\zeta_{p+q} \equiv 0$ and consequently ψ_l is independent of ζ_{p+q}. Thus $\mathrm{pr}_{\widetilde{\zeta}}(T_{1,l}) = \{\zeta_1 = \psi_l(\zeta_2, \dots, \zeta_{p+q-1})\}$ and therefore the projection is pluripolar. The proof that $(\widetilde{A} \times \varDelta_{b_0'}(R')) \setminus A'$ is pluripolar is complete.

Using Step 4, we conclude that $\widehat{f}$ extends holomorphically to the domain $\widehat{Y} \setminus M$, where

$$\begin{aligned}\widehat{Y} &:= \{(z, w', w_q) \in \varDelta_{a_0}(\varrho') \times \varDelta_{b_0'}(R') \times \varDelta_{b_{0,q}}(R_0) : \\ &\qquad h^*_{A', \varDelta_{a_0}(\varrho') \times \varDelta_{b_0'}(R')}(z, w') + h^*_{\varDelta_{b_{0,q}}(R'), \varDelta_{b_{0,q}}(R_0)}(w_q) < 1\} \\ &= \{(z, w', w_q) \in \varDelta_{a_0}(\varrho') \times \varDelta_{b_0'}(R') \times \varDelta_{b_{0,q}}(R_0) : \\ &\qquad h^*_{\widetilde{A}, \varDelta_{a_0}(\varrho')}(z) + h^*_{\varDelta_{b_{0,q}}(R'), \varDelta_{b_{0,q}}(R_0)}(w_q) < 1\}\end{aligned}$$

(here we have used the product property of the relative extremal function). Since $R_0' < R_0$, we find a $\varrho_q \in (0, \varrho']$ and a function $\widetilde{f}_q \in \mathcal{O}(\varDelta_{a_0}(\varrho_q) \times \varDelta_{b_0'}(R') \times \varDelta_{b_{0,q}}(R_0') \setminus M)$ such that

$$\widetilde{f}_q = \widehat{f} \quad \text{on } \varDelta_{a_0}(\varrho_q) \times \varDelta_{b_0}(R') \setminus M.$$

If $q = 1$ we get a contradiction (because $R_0' > R_0^*$).

Let $q \geq 2$. Repeating the above argument for the coordinates w_ν, $\nu = 1, \dots, q-1$, we find a $\varrho_0 \in (0, \varrho']$ and a function $\widetilde{f}$ holomorphic in

$$\varDelta_{a_0}(\varrho_0) \times \Big(\bigcup_{\nu=1}^{q} \varDelta_{(b_{0,1}, \dots, b_{0,\nu-1})}(R') \times \varDelta_{b_{0,\nu}}(R_0') \times \varDelta_{(b_{0,\nu+1}, \dots, b_{0,q})}(R')\Big) \setminus M$$

such that $\widetilde{f} = \widehat{f}$ on $\varDelta_{a_0}(\varrho_0) \times \varDelta_{b_0}(R') \setminus M$. Let $\mathcal{H}$ denote the envelope of holomorphy of the domain

$$\bigcup_{\nu=1}^{q} \varDelta_{(b_{0,1}, \dots, b_{0,\nu-1})}(R') \times \varDelta_{b_{0,\nu}}(R_0') \times \varDelta_{(b_{0,\nu+1}, \dots, b_{0,q})}(R').$$

Applying the Grauert–Remmert theorem, we can extend $\widetilde{f}$ holomorphically to $\varDelta_{a_0}(\varrho_0) \times \mathcal{H} \setminus M$, i.e. there exists an $\widehat{\widetilde{f}} \in \mathcal{O}(\varDelta_{a_0}(\varrho_0) \times \mathcal{H} \setminus M)$ with $\widehat{\widetilde{f}} = f$ on $\varDelta_{a_0}(\varrho_0) \times \varDelta_{b_0}(r)$. Observe that $\varDelta_{b_0}(\sqrt[q]{R'^{q-1} R_0'}) \subset \mathcal{H}$. Recall that $\sqrt[q]{R'^{q-1} R_0'} > R_0^*$; a contradiction. ■

REMARK. Notice that the proof of Step 6 shows that the following stronger version of $(*)$ is true: Let $\varrho > 0$, $0 < r < R$, Ω, $\widetilde{\Omega}$, A, and a be as in $(*)$. Let M be a pure one-codimensional analytic subset of Ω (we do not assume that $M \cap \widetilde{\Omega} = \emptyset$). Then:

For any $R' \in (r, R)$ there exists $\varrho' \in (0, \varrho)$ such that for any function $f \in \mathcal{O}(\widetilde{\Omega} \setminus M)$ with $f(a, \cdot) \in \mathcal{O}(\Delta_{b_0}(R) \setminus M_a)$, $a \in A$, there exists an extension $\widehat{f} \in \mathcal{O}(\Delta_{a_0}(\varrho') \times \Delta_{b_0}(R') \setminus M)$ with $\widehat{f} = f$ on $\Delta_{a_0}(\varrho') \times \Delta_{b_0}(r) \setminus M$.

4. Proof of the Main Theorem in the general case. First observe that the function $\widehat{f}$ is uniquely determined (cf. §3).

We proceed by induction on N. Let $D_{j,k} \nearrow D_j$, $D_{j,k} \Subset D_{j,k+1} \Subset D_j$, where $D_{j,k}$ are pseudoconvex domains with $A_{j,k} := A_j \cap D_{j,k} \neq \emptyset$. Put

$$X_k := \mathbb{X}(A_{1,k}, \ldots, A_{N,k}; D_{1,k}, \ldots, D_{N,k}) \subset X.$$

It suffices to show that for each $k \in \mathbb{N}$ the following condition $({}^{**}_{\ *})$ holds:

$({}^{**}_{\ *})$ There exists a domain U_k, $X_k \subset U_k \subset U \cap \widehat{X}_k$, such that for any $f \in \mathcal{O}_{\mathrm{s}}(X \setminus M)$ there exists an $\widetilde{f}_k \in \mathcal{O}(U_k \setminus M)$ with $\widetilde{f}_k|_{X_k \setminus M} = f|_{X_k \setminus M}$.

Indeed, fix $k \in \mathbb{N}$ and observe that $\widehat{X}_k$ is the envelope of holomorphy of U_k (cf. the proof of Step 4). Hence, by the Dloussky theorem (cf. [Jar-Pfl 2000, Th. 3.4.8], see also [Por 2002]), there exists an analytic subset $\widetilde{M}_k$ of $\widehat{X}_k$, $\widetilde{M}_k \cap U_k \subset M$, such that $\widehat{X}_k \setminus \widetilde{M}_k$ is the envelope of holomorphy of $U_k \setminus M$. In particular, for each $f \in \mathcal{O}_{\mathrm{s}}(X \setminus M)$ there exists an $\widetilde{\widetilde{f}}_k \in \mathcal{O}(\widehat{X}_k \setminus \widetilde{M}_k)$ with $\widetilde{\widetilde{f}}_k|_{U_k \setminus M} = \widetilde{f}_k$. Let $\mathcal{F}_k := \{\widetilde{\widetilde{f}}_k : f \in \mathcal{O}_{\mathrm{s}}(X \setminus M)\} \subset \mathcal{O}(\widehat{X}_k \setminus \widetilde{M}_k)$. It is known (cf. [Jar-Pfl 2000, Prop. 3.4.5]) that there exists a pure one-codimensional analytic subset $\widehat{M}_k \subset \widehat{X}_k$, $\widehat{M}_k \subset \widetilde{M}_k$, such that any point of $\widehat{M}_k$ is singular with respect to $\mathcal{F}_k$, i.e.

- any function $\widetilde{\widetilde{f}}_k$ extends to a function $\widehat{f}_k \in \mathcal{O}(\widehat{X}_k \setminus \widehat{M}_k)$, and
- for any $a \in \widehat{M}_k$ and an open neighborhood V of a, $V \subset \widehat{X}_k$, there exists an $f \in \mathcal{O}_{\mathrm{s}}(X \setminus M)$ such that $\widehat{f}_k|_{V \setminus \widehat{M}_k}$ cannot be holomorphically extended to the whole V.

In particular, $\widehat{M}_{k+1} \cap \widehat{X}_k = \widehat{M}_k$. Consequently, $\widehat{M} := \bigcup_{k=1}^{\infty} \widehat{M}_k$ is a pure one-codimensional analytic subset of $\widehat{X}$, $\widehat{M} \cap \bigcup_{k=1}^{\infty} U_k \subset M$, and for each $f \in \mathcal{O}_{\mathrm{s}}(X \setminus M)$, the function $\widehat{f} := \bigcup_{k=1}^{\infty} \widehat{f}_k$ is holomorphic on $\widehat{X} \setminus \widehat{M}$ with $\widehat{f}|_{X \setminus M} = f$.

It remains to prove $({}^{**}_{\ *})$. Fix $k \in \mathbb{N}$. For any $a = (a_1, \ldots, a_N) \in A_{1,k} \times \ldots \times A_{N,k}$ let $\varrho = \varrho_k(a)$ be such that $\Delta_a(\varrho) \subset D_{1,k} \times \ldots \times D_{N,k}$. If $N \geq 4$, then we additionally define $(N-2)$-fold crosses

$$Y_{k,\mu,\nu} := \mathbb{X}(A_{1,k},\dots,A_{\mu-1,k},A_{\mu+1,k},\dots,A_{\nu-1,k},A_{\nu+1,k},\dots,A_{N,k}; D_{1,k},\dots,D_{\mu-1,k},D_{\mu+1,k},\dots,D_{\nu-1,k},D_{\nu+1,k},\dots,D_{N,k}), \quad 1\le\mu<\nu\le N,$$

and we assume that ϱ is so small that

$$\Delta_{(a_1,\dots,a_{\mu-1},a_{\mu+1},\dots,a_{\nu-1},a_{\nu+1},\dots,a_N)}(\varrho)\subset \widehat{Y}_{k,\mu,\nu}, \quad 1\le\mu<\nu\le N.$$

Since $\{(a_1,\dots,a_{j-1})\}\times\overline{D}_{j,k+1}\times\{(a_{j+1},\dots,a_N)\}\Subset U$, we may assume that

$$\Delta_{(a_1,\dots,a_{j-1})}(\varrho)\times D_{j,k+1}\times\Delta_{(a_{j+1},\dots,a_N)}(\varrho)\subset U, \quad j=1,\dots,N. \tag{4}$$

We define N-fold crosses

$$\begin{aligned}Z_{k,a,j} := \mathbb{X}(&A_1\cap\Delta_{a_1}(\varrho),\dots,A_{j-1}\cap\Delta_{a_{j-1}}(\varrho),A_{j,k+1},\\ &A_{j+1}\cap\Delta_{a_{j+1}}(\varrho),\dots,A_N\cap\Delta_{a_N}(\varrho);\\ &\Delta_{a_1}(\varrho),\dots,\Delta_{a_{j-1}}(\varrho),D_{j,k+1},\Delta_{a_{j+1}}(\varrho),\dots,\Delta_{a_N}(\varrho))\end{aligned}$$

for $j=1,\dots,N$. Note that $\widehat{Z}_{k,a,j}\subset U$. Since $\{(a_1,\dots,a_{j-1})\}\times\overline{D}_{j,k}\times\{(a_{j+1},\dots,a_N)\}\Subset\widehat{Z}_{k,a,j}$, there exists an $r=r_k(a)$, $0<r\le\varrho$, so small that

$$V_{k,a,j} := \Delta_{(a_1,\dots,a_{j-1})}(r)\times D_{j,k}\times\Delta_{(a_{j+1},\dots,a_N)}(r)\subset\widehat{Z}_{k,a,j}, \quad j=1,\dots,N.$$

Put

$$V_k := \bigcup_{\substack{a\in A_{1,k}\times\ldots\times A_{N,k}\\ j\in\{1,\dots,N\}}} V_{k,a,j}.$$

Note that $X_k\subset V_k$. Let U_k be the connected component of $V_k\cap\widehat{X}_k$ that contains X_k.

In view of (4), the Main Theorem with $U=\widehat{X}$ (which is already proved in §3) implies that for any $f\in\mathcal{O}_{\mathrm{s}}(X\setminus M)$ there exists an extension $\widehat{f}_{k,a,j}\in\mathcal{O}(\widehat{Z}_{k,a,j}\setminus M)$ of $f|_{Z_{k,a,j}\setminus M}$. It remains to glue the functions

$$\widetilde{f}_{k,a,j} := \widehat{f}_{k,a,j}|_{V_{k,a,j}\setminus M}, \quad a\in A_{1,k}\times\ldots\times A_{N,k}, \quad j=1,\dots,N;$$

then the function

$$\widetilde{f}_k := \Big(\bigcup_{\substack{a\in A_{1,k}\times\ldots\times A_{N,k}\\ j\in\{1,\dots,N\}}}\widetilde{f}_{k,a,j}\Big)\Big|_{U_k\setminus M}$$

gives the required extension of $f|_{X_k\setminus M}$.

To check that the gluing process is possible, let $a,b\in A_{1,k}\times\ldots\times A_{N,k}$ and $i,j\in\{1,\dots,N\}$ be such that $V_{k,a,i}\cap V_{k,b,j}\ne\emptyset$. We have the following two cases:

(a) $i \neq j$: We may assume that $i = N-1$, $j = N$. Write $w = (w', w'') \in \mathbb{C}^{k_1+\ldots+k_{N-2}} \times \mathbb{C}^{k_{N-1}+k_N}$. Observe that

$$V_{k,a,N-1} \cap V_{k,b,N} = (\varDelta_{a'}(r_k(a)) \cap \varDelta_{b'}(r_k(b))) \times \varDelta_{b_{N-1}}(r_k(b)) \times \varDelta_{a_N}(r_k(a)).$$

For $c = (c', c'')$, let

$$M_{c'} := \{w'' \in \mathbb{C}^{k_{N-1}+k_N} : (c', w'') \in M\},$$
$$M^{c''} := \{w' \in \mathbb{C}^{k_1+\ldots+k_{N-2}} : (w', c'') \in M\};$$

$M_{c'}$ and $M^{c''}$ are analytic subsets of

$$U_{c'} := \{w'' \in \mathbb{C}^{k_{N-1}+k_N} : (c', w'') \in U\},$$
$$U^{c''} := \{w' \in \mathbb{C}^{k_1+\ldots+k_{N-2}} : (w', c'') \in U\},$$

respectively.

We consider the following three subcases:

(a_1) $N = 2$: Then $V_{k,a,1} \cap V_{k,b,2} = \varDelta_{b_1}(r_k(b)) \times \varDelta_{a_2}(r_k(a))$. Since $\widetilde{f}_{k,a,1} = \widetilde{f}_{k,b,2}$ on the non-pluripolar set $(A_1 \cap \varDelta_{b_1}(r_k(b))) \times (A_2 \cap \varDelta_{a_2}(r_k(a))) \setminus M$, by the identity principle, $\widetilde{f}_{k,a,1} = \widetilde{f}_{k,b,2}$ on $V_{k,a,1} \cap V_{k,b,2} \setminus M$.

(a_2) $N = 3$: Then $V_{k,a,2} \cap V_{k,b,3} = (\varDelta_{a_1}(r_k(a)) \cap \varDelta_{b_1}(r_k(b))) \times \varDelta_{b_2}(r_k(b)) \times \varDelta_{a_3}(r_k(a))$. Let C'' denote the set of all points $c'' \in (A_2 \cap \varDelta_{b_2}(r_k(b))) \times (A_3 \cap \varDelta_{a_3}(r_k(a)))$ such that the set $M^{c''}$ has codimension ≥ 1 (i.e. for any $w' \in M^{c''}$ the codimension of $M^{c''}$ at w' is ≥ 1). Note that C'' is non-pluripolar. We have $\widetilde{f}_{k,a,2}(\cdot, c'') = f(\cdot, c'') = \widetilde{f}_{k,b,3}(\cdot, c'')$ on $\varDelta_{a_1}(r_k(a)) \cap \varDelta_{b_1}(r_k(b)) \setminus M^{c''}$.

Now, let $c' \in \varDelta_{a_1}(r_k(a)) \cap \varDelta_{b_1}(r_k(b))$ be such that the set $M_{c'}$ has codimension ≥ 1. Then $\widetilde{f}_{k,a,2}(c', \cdot) = \widetilde{f}_{k,b,3}(c', \cdot)$ on $C'' \setminus M_{c'}$. Hence, by the identity principle, $\widetilde{f}_{k,a,2}(c', \cdot) = \widetilde{f}_{k,b,3}(c', \cdot)$ on $\varDelta_{b_2}(r_k(b)) \times \varDelta_{a_3}(r_k(a)) \setminus M_{c'}$. Finally, $\widetilde{f}_{k,a,2} = \widetilde{f}_{k,b,3}$ on $V_{k,a,2} \cap V_{k,b,3} \setminus M$.

If $N \in \{2, 3\}$, then we jump directly to (b) and we conclude that the Main Theorem is true for $N \in \{2, 3\}$.

(a_3) $N \geq 4$: Here is the only place where the induction over N is used. We assume that the Main Theorem is true for $N - 1 \geq 3$.

Similarly to the case $N = 3$, let C'' denote the set of all points $c'' \in (A_{N-1} \cap \varDelta_{b_{N-1}}(r_k(b))) \times (A_N \cap \varDelta_{a_N}(r_k(a)))$ such that the set $M^{c''}$ has codimension ≥ 1; C'' is non-pluripolar. The function $f_{c''} := f(\cdot, c'')$ is separately holomorphic on $Y_{k,N-1,N} \setminus M^{c''}$. By the inductive assumption, $f_{c''}$ extends to a function $\widehat{f}_{c''} \in \mathcal{O}(\widehat{Y}_{k,N-1,N} \setminus \widehat{M}(c''))$, where $\widehat{M}(c'')$ is an analytic subset of $\widehat{Y}_{k,N-1,N}$ with $\widehat{M}(c'') \subset M^{c''}$ in an open neighborhood of

$Y_{k,N-1,N}$. Recall that

$$\varDelta_{a'}(r_k(a)) \cup \varDelta_{b'}(r_k(b)) \subset \widehat{Y}_{k,N-1,N}.$$

Since $\widetilde{f}_{k,a,N-1}(\cdot, c'') = f_{c''}$ on $\varDelta_{a'}(r_k(a)) \cap Y_{k,N-1,N} \setminus M^{c''}$ and $\widetilde{f}_{k,b,N}(\cdot, c'') = f_{c''}$ on $\varDelta_{b'}(r_k(b)) \cap Y_{k,N-1,N} \setminus M^{c''}$, we conclude that $\widetilde{f}_{k,a,N-1}(\cdot, c'') = \widehat{f}_{c''} = \widetilde{f}_{k,b,N}(\cdot, c'')$ on $\varDelta_{a'}(r_k(a)) \cap \varDelta_{b'}(r_k(b)) \setminus M^{c''}$.

Let $c' \in \varDelta_{a'}(r_k(a)) \cap \varDelta_{b'}(r_k(b))$ be such that the set $M_{c'}$ has codimension ≥ 1. Then $\widetilde{f}_{k,a,N-1}(c', \cdot) = \widetilde{f}_{k,b,N}(c', \cdot)$ on $C'' \setminus M_{c'}$. Consequently, by the identity principle, $\widetilde{f}_{k,a,N-1}(c', \cdot) = \widetilde{f}_{k,b,N}(c', \cdot)$ on $\varDelta_{b_{N-1}}(r_k(b)) \times \varDelta_{a_N}(r_k(a)) \setminus M_{c'}$ and, finally, $\widetilde{f}_{k,a,N-1} = \widetilde{f}_{k,b,N}$ on $V_{k,a,N-1} \cap V_{k,b,N} \setminus M$.

(b) $i = j$: We may assume that $i = j = N$. Observe that

$$V_{k,a,N} \cap V_{k,b,N} = (\varDelta_{(a_1,\ldots,a_{N-1})}(r_k(a)) \cap \varDelta_{(b_1,\ldots,b_{N-1})}(r_k(b))) \times D_{N,k}.$$

By (a) we know that

$$\begin{aligned} \widetilde{f}_{k,a,N} &= \widetilde{f}_{k,a,N-1} \quad && \text{on } V_{k,a,N} \cap V_{k,a,N-1} \setminus M, \\ \widetilde{f}_{k,a,N-1} &= \widetilde{f}_{k,b,N} && \text{on } V_{k,a,N-1} \cap V_{k,b,N} \setminus M. \end{aligned}$$

Hence $\widetilde{f}_{k,a,N} = \widetilde{f}_{k,b,N}$ on

$$\begin{aligned} &V_{k,a,N} \cap V_{k,a,N-1} \cap V_{k,b,N} \setminus M \\ &\quad = (\varDelta_{(a_1,\ldots,a_{N-1})}(r_k(a)) \cap \varDelta_{(b_1,\ldots,b_{N-1})}(r_k(b))) \times \varDelta_{a_N}(r_k(a)) \setminus M, \end{aligned}$$

and finally, by the identity principle,

$$\widetilde{f}_{k,a,N} = \widetilde{f}_{k,b,N} \quad \text{on } V_{k,a,N} \cap V_{k,b,N} \setminus M.$$

The proof of the Main Theorem is complete.

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