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Analytic continuation for some classes of separately analytic functions of real variables

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Abstract. For functions that are separately solutions of an elliptic homogeneous PDE with constant coefficients, we prove an analogue of Siciak's theorem for separately holomorphic functions.

1. Definitions and the principal result

Definition. Let Q be an elliptic homogeneous polynomial of N real variables. A complex function f defined in an open subset Ω of $\mathbb{R}^N$ is called *Q-analytic* if it is real-analytic and satisfies

$$Q\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_N}\right)f=0.$$

The set of Q-analytic functions in Ω will be denoted $AQ(\Omega)$.

Definition. Let D_j be an open subset of $\mathbb{R}^{N_j}$ and E_j a compact subset of D_j, $j=1,\ldots,p$. Put

$$\Gamma=\Gamma(D_1,\ldots,D_p;E_1,\ldots,E_p)=(D_1\times E_2\times E_p)\cup\ldots\cup(E_1\times\ldots\times E_{p-1}\times D_p).$$

Let Q_j $(1\le j\le p)$ be an elliptic homogeneous polynomial of N_j real variables. A function $f:\Gamma\to\mathbb{C}$ is called $(Q_1,\ldots,Q_p)$-*separately analytic* if for every fixed $j\in\{1,\ldots,p\}$ and every fixed $x\in\prod_{k=1}^{p}E_k$, the function $t\mapsto f(\breve{x}_j,t)$ is Q_j-analytic in D_j, where $\breve{x}_j=(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_p)$.

Problem. Under some hypothesis on E_j and D_j, find an open neighbourhood V of Γ such that every $(Q_1,\ldots,Q_p)$-separately analytic function on Γ can be analytically continued to a real-analytic function on V.

We shall need some extremal functions studied by Hécart [2]–[4], first introduced by Zahariuta [11].

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Definition. Let D be an open subset of $\mathbb{R}^N$, E a compact set in D and Q an elliptic homogeneous polynomial of N real variables. We define

$$K_{Q,\varepsilon}(D,E,x) = \sup\{\alpha \log|u(x)| : u \in AQ(D),\ 0<\alpha<\varepsilon,\ \|u\|_E \le 1,\ \|u\|_D \le \exp(1/\alpha)\} \quad (\varepsilon>0),$$

$$\chi_{Q,\varepsilon}(D,E,x) = \limsup_{x'\to x} K(D,E,x'),$$

$$\chi_{Q,0}(D,E,x) = \lim_{\varepsilon\to 0} \chi_{Q,\varepsilon}(D,E,x),$$

$$\chi_{Q}(D,E,x) = \lim_{s\to \infty} \chi_{Q,0}(D_s,E,x),$$

where $E \subset D_s \Subset D_{s+1}$ and $\bigcup D_s = D$.

Theorem 1. *If D_j is connected and E_j is L-regular in $\mathbb{C}^{N_j}$ $(1 \le j \le p)$, then every $(Q_1,\ldots,Q_p)$-separately analytic function on Γ can be analytically continued to*

$$\widehat{\Gamma} = \Big\{(x_1,\ldots,x_p) \in D_1\times\ldots\times D_p : \sum_{j=1}^{p} \chi_{Q_j}(D_j,E_j,x_j) < 1\Big\}.$$

Remarks. • We consider $\mathbb{R}^N$ as the real part of $\mathbb{C}^N$.

• A compact set E in $\mathbb{C}^N$ is called *L-regular* if its Siciak extremal function ϕ_E is continuous; for details we refer to Klimek's book [5].

• The L-regularity of E_j implies that $\widehat{\Gamma}$ is an open neighbourhood of Γ.

2. Preliminary results

Proposition 1. *Let Q be an elliptic homogeneous polynomial in $\mathbb{R}^N$. For every $r>0$, there exist $\varrho=\varrho(r,Q)>0$ and $C=C(r,Q)$ such that every Q-analytic function f in the real ball $B(0,r)$ is analytically continuable to the complex ball $\widehat{B}(0,\varrho)$, and the continuation $\widehat{f}$ satisfies $\|\widehat{f}\|_{\widehat{B}} \le C\|f\|_B$.*

Proposition 2. *Let Ω_j be a bounded pseudoconvex open set in $\mathbb{C}^{N_j}$ and E_j a Borel set in Ω_j $(1\le j\le p)$. If f is separately holomorphic on the crossed set*

$$X = (\Omega_1\times E_2\times\ldots\times E_p)\cup\ldots\cup(E_1\times\ldots\times E_{p-1}\times\Omega_p),$$

then f is holomorphically continuable to

$$\widehat{X} = \Big\{(z_1,\ldots,z_p) \in \Omega_1\times\ldots\times\Omega_p : \sum_{j=1}^{p} \omega^*(\Omega_j,E_j,z_j) < 1\Big\}$$

where $\omega^*(\Omega_j,E_j,\cdot)$ is the $(0,1)$-psh extremal function associated to E_j and Ω_j.

Proposition 3. *Let D be a bounded domain in $\mathbb{R}^N$, E an L-regular compact subset of D and $\mu=(dd^c\log\phi_E)^N$ the Monge–Ampère measure* *on E. Let Q be an elliptic homogeneous polynomial in $\mathbb{R}^N$, and put*

$$L^2_Q(D) = AQ(D) \cap L^2(D, \lambda),$$

where λ is the Lebesgue N-dimensional measure. Then there exists an orthogonal basis (B_k) of the Hilbert space $L^2_Q(D)$ such that

(i) $\sum \mu_k^{-\delta} < \infty$ *for all* $\delta > 0$, *with* $\mu_k = (\int_D |B_k|^2 \, d\lambda)^{1/2}$ *increasing to* ∞,

(ii) $\int\limits_E B_k \overline{B}_l \, d\mu = \begin{cases} 0 & \text{if } k \neq l, \\ 1 & \text{if } k = l, \end{cases}$

(iii) *for every* $\varepsilon > 0$ *there exists* $C = C(\varepsilon)$ *such that* $\|B_k\|_E := \sup_E |B_k| \leq C\mu_k^{\varepsilon}$.

Proposition 1 is taken from Armitage, Bagby and Gauthier [1].

Proposition 2, a general version of the Siciak–Zahariuta theorem, is from [10].

Proof of Proposition 3. It is easy to verify that

$$L^2_Q(D) \hookrightarrow AQ(D) \hookrightarrow L^2(E, \mu),$$

where $\hookrightarrow$ means a compact imbedding defined by the restriction operator. Because $AQ(D)$ is a nuclear Fréchet space with the compact convergence topology [3], a theorem of Mityagin [7] gives the existence of an orthogonal basis (B_k) of $L^2_Q(D)$ satisfying (i) and (ii).

We now deduce (iii). Put

$$\widehat{D} = \bigcup_{z \in D} \widehat{B}(z, \varrho_z) \quad \text{with} \quad \varrho_z = \varrho(\operatorname{dist}(z, \partial D), Q).$$

Following Proposition 1, we can continue B_k to a holomorphic function $\widehat{B}_k$ in $\widehat{D}$ such that for every K compact $\subset \widehat{D}$,

$$\|\widehat{B}_k\|_K \leq M(K)\mu_k, \qquad \forall k.$$

From this inequality we see that the family $(\log \left|B_k\right| / \log \mu_k)$ is locally upper bounded in $\widehat{B}$. We put

$$W = \limsup \frac{\log |B_k|}{\log \mu_k}, \qquad W^* = \operatorname{reg} \sup W.$$

A known argument [10] yields

$$W^*(z) \leq 0, \qquad \forall z \in E \setminus E',$$

for some E' with $\mu(E') = 0$. By Levenberg [6], $E \setminus E'$ satisfies the (L_0) condition of Leja at every point of E. Because of the polynomial convexity of E in $\mathbb{C}^N$ and the fact that $W^* \in \operatorname{PSH}(\widehat{D})$ with $\widehat{D} \supset E$, we have $W^* \leq 0$ on E, by [8, Th. 2]. This last inequality and the classical Hartogs Lemma imply (iii).

3. Proof of Theorem 1

3.1. For every $j = 1, \dots, p$, we choose a domain D'_j such that $E_j \subset D'_j \Subset D_j$. Because $\overline{D}_j$ is a polynomially convex subset of $\widehat{D}_j$, it has a bounded pseudoconvex open neighbourhood Ω_j in $\widehat{D}_j$ (for the definition of $\widehat{D}_j$, see the beginning of the proof of Proposition 3).

For fixed $(x_1, \dots, x_p) \in E_1 \times \dots \times E_p$, let $\widehat{f}(\breve{x}_j, \cdot)$ be the analytic continuation of $f(\breve{x}_j, \cdot)$ to $\widehat{D}_j$; gluing up these functions, we obtain a separately holomorphic function $\widehat{f}$ on the crossed set

$$X = (\Omega_1 \times E_2 \times \dots \times E_p) \cup \dots \cup (E_1 \times \dots \times E_{p-1} \times \Omega_p).$$

By Proposition 2, there exists a holomorphic function g on

$$\widehat{X} = \Big\{(z_1, \dots, z_p) \in \Omega_1 \times \dots \times \Omega_p : \sum_{j=1}^{p} \omega^*(\Omega_j, E_j, x_j) < 1\Big\},$$

with $g = \widehat{f}$ on X.

In the next paragraphs, it is important to note that $\widehat{X}$ is an open neighbourhood in $\mathbb{C}^{N_1+\dots+N_p}$ of $(\overline{D'_1} \times E_2 \times \dots \times E_p) \cup \dots \cup (E_1 \times \dots \times E_{p-1} \times \overline{D'_p})$.

3.2. For every $j = 1, \dots, p$, let $(B^j_k)_{k=0,1,\dots}$ be the doubly orthogonal basis of $L^2_{Q_j}(D'_j)$ given by Proposition 3, and μ^j the measure $(dd^c \log \phi_{E_j})^{N_j}$. For $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{N}^p$, put

$$C_\alpha = \int\limits_{E_1 \times \dots \times E_p} f(x_1, \dots, x_p) \overline{B^1_{\alpha_1}(x_1)} \dots \overline{B^p_{\alpha_p}(x_p)}\, d\mu^1(x_1) \wedge \dots \wedge d\mu^p(x_p).$$

We have $|C_\alpha| \le M/\mu^j_{\alpha_j}$, with M independent of j and α_j, and

$$\mu^j_{\alpha_j} = \Big(\int\limits_{D'_j} |B^j_{\alpha_j}|^2\, d\lambda^j\Big)^{1/2} \qquad (\lambda^j = \text{Lebesgue measure in } \mathbb{R}^{N_j}).$$

In fact, this inequality results, via obvious estimates, from the following identities:

$$C_\alpha = \int\limits_{\breve{E}_j} \Big(\int\limits_{E_j} f(x_1, \dots, x_p)\, d\mu^j(x_j)\Big) \prod_{k \ne j} B^k_{\alpha_k}(x_k) \bigwedge_{k \ne j} d\mu^k(x_k),$$

where $\breve{E}_j = E_1 \times \dots \times E_{j-1} \times E_{j+1} \times \dots \times E_p$, and

$$\int\limits_{E_j} f(x_1, \dots, x_p) \overline{B^j_{\alpha_j}(x_j)}\, d\mu^j(x_j) = (\mu^j_{\alpha_j})^{-2} \int\limits_{D'_j} f(x_1, \dots, x_p) \overline{B^j_{\alpha_j}(x_j)}\, d\lambda^j(x_j).$$

3.3. Now we prove the local absolute uniform convergence of the series $\sum_{\alpha\in\mathbb{N}^p} C_\alpha B_\alpha(z)$ with $B_\alpha(z) = B^1_{\alpha_1}(z_1)\ldots B^p_{\alpha_p}(z_p)$ in the open set

$$\widehat{\Gamma}' = \Big\{(x_1,\ldots,x_p) \in D'_1 \times \ldots \times D'_p : \sum_{j=1}^{p} \chi_{Q_j}(D'_j, E_j, x_j) < 1\Big\}.$$

Let $a = (a_1,\ldots,a_p) \in \widehat{\Gamma}'$. We can choose domains D''_j and $\theta_j > 0$ ($j = 1,\ldots,p$) such that

- $\sum \theta_j < 1$,
- $E_j \subset D''_j \Subset D'_j$,
- $a_j \in D''_j(\theta_j) := \{x \in D''_j : \chi_{Q_j}(D''_j, E_j, x) < \theta_j\}$.

We now choose $\theta'_j > \theta_j$ with $\sum \theta'_j = 1$. From the estimate of $|C_\alpha|$ given in 3.2, we deduce that

$$|C_\alpha| \le M(\mu^1_{\alpha_1})^{-\theta'_1}\ldots(\mu^p_{\alpha_p})^{-\theta'_p}.$$

Let V_j be a compact neighbourhood of a_j in $D''_j(\theta_j)$. By the two-constants theorem [2], [3], for every $\varepsilon \in {]0, 1-\theta_j[}$ there exists $C = C(V_j, \varepsilon)$ such that

$$\|u\|_{V_j} \le C\|u\|_{E_j}^{1-\theta_j-\varepsilon}\|u\|_{D''_j}^{\theta_j+\varepsilon}, \quad \forall u \in AQ(D''_j).$$

From Proposition 3(iii) and the obvious fact

$$\|B^j_{\alpha_j}\|_{D''_j} \le C'\mu^j_{\alpha_j}$$

we have, for every $\varepsilon \in {]0, \inf\theta_j[}$,

$$\|B_\alpha\|_V \le C''\prod_{j=1}^{p}(\mu^j_{\alpha_j})^{\varepsilon(1-\theta_j-\varepsilon)}(\mu^j_{\alpha_j})^{\theta_j+\varepsilon},$$

where $V = V_1 \times \ldots \times V_p$, so

$$|C_\alpha|\,\|B_\alpha\|_V \le C''\prod_{j=1}^{p}(\mu^j_{\alpha_j})^{\varepsilon-\varepsilon^2-\varepsilon\theta_j+\theta_j-\theta'_j}.$$

Since $\theta'_j > \theta_j$, for ε small enough we have $\varepsilon - \varepsilon^2 - \varepsilon\theta_j + \theta_j - \theta'_j =: -\delta_j < 0$, for all j. Hence $\sum_\alpha |C_\alpha|\,\|B_\alpha\|_V$ is majorized by the series

$$\sum_\alpha C''(\mu^1_{\alpha_1})^{-\delta_1}\ldots(\mu^p_{\alpha_p})^{-\delta_p} = C''\prod_{j=1}^{p}\sum_{k=0}^{\infty}(\mu^j_k)^{-\delta_j} < \infty$$

(here we use Proposition 3(i)).

3.4. Put $g(x) = \sum_\alpha C_\alpha B_\alpha(x)$, $x \in \widehat{\Gamma}'$. The real analyticity of g in $\widehat{\Gamma}'$ can be proved easily by using the local uniform convergence and Proposition 1; we also have the $(Q_1,\ldots,Q_p)$-analyticity of g in $\widehat{\Gamma}'$, i.e. for all $x \in \widehat{\Gamma}'$ and

$j = 1, \dots, p$, $g(\breve{x}_j, \cdot)$ satisfies

$$Q_j\left(\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_{N_j}}\right) g(\breve{x}_j, \cdot) = 0 \quad \text{in } \{t \in \mathbb{C} : (\breve{x}_j, t) \in \widehat{\Gamma}'\}.$$

We now prove that $g = f$ on $\breve{E}_j \times D_j$ for all $j = 1, \dots, p$. We can suppose $j = p$ and that the property to be proved true for $p - 1$ (induction hypothesis). For $x' = (x_1, \dots, x_{p-1}) \in E_1 \times \ldots \times E_{p-1}$, the function $t \mapsto f(x', t)$ is Q_p-analytic in $D_p \Supset D'_p$, so it belongs to $L^2_{Q_p}(D'_j)$ and we can write

$$f(x', t) = \sum_k A_k(x') B^p_k(t), \tag{1}$$

where the series converges uniformly on any compact set in D'_j and

$$A_k(x') = \int_{E_p} f(x', v) \overline{B^p_k}(v)\, d\mu^p(v).$$

Put $x = (x', t)$ with $x' = (x_1, \dots, x_{p-1})$ and $\alpha = (\alpha', k)$, $\alpha' = (\alpha_1, \dots, \alpha_{p-1})$. By the absolute convergence of $\sum_\alpha C_\alpha B_\alpha$ on $E_1 \times \ldots \times E_p$, we can write

$$g(x', t) = \sum_{k=0}^{\infty} \Big(\sum C_{\alpha', k} B^1_{\alpha_1}(x_1) \dots B^{p-1}_{\alpha_{p-1}}(x_{p-1}) \Big) B^p_k(t) \tag{2}$$

for $(x', t) \in E_1 \times \ldots \times E_p$. We claim that

$$\sum_{\alpha'} C_{\alpha', k} B^1_{\alpha_1}(x_1) \dots B^{p-1}_{\alpha_{p-1}}(x_{p-1}) = \int_{E_p} f(x', v) \overline{B^p_k}(v)\, d\mu^p(v). \tag{3}$$

In fact with the notation $B_{\alpha'}(u) = B^1_{\alpha_1}(u_1) \dots B^{p-1}_{\alpha_{p-1}}(u_{p-1})$, we see that the left hand side is equal to

$$\int_{E_p} \Big(\sum_{\alpha'} \Big(\int_{\breve{E}_p} f(\cdot, v) \overline{B}_{\alpha'}\, d(\mu^1 \otimes \ldots \otimes \mu^{p-1}) \Big) B_{\alpha'}(x') \Big) \overline{B^p_k}(v)\, d\mu^p(v). \tag{4}$$

By the induction hypothesis and paragraph 3.3, we have

$$f(\cdot, v) = \sum_{\alpha'} \Big(\int_{\breve{E}_p} f(\cdot, v) \overline{B}_{\alpha'}\, d(\mu^1 \otimes \ldots \otimes \mu^{p-1}) \Big) B_{\alpha'}. \tag{5}$$

with absolute uniform convergence on $\breve{E}_p$.

Now (5) and (4) give (3).

Finally (3), (2) and (1) give

$$f(x', t) = g(x', t), \quad \forall t \in E_p,\ \forall x' \in \breve{E}_p.$$

Because $f(x', t)$ and $g(x', t)$ are Q_p-analytic functions of t in the domain D'_1, and E_p is L-regular in $\mathbb{C}^{N_p}$, we conclude that $f(x', t) = g(x', t)$ on $\breve{E}_p \times D'_p$.

Because of the arbitrariness of $D'_1, \dots, D'_p$, the theorem is true.

REMARKS. We conjecture that the result is true if instead of the L-regularity of E_j, we suppose that E_j satisfies the Leja condition for Q_j-analytic polynomials. This is proved in the following cases:

- $N_j = 2,\ Q_j(D) =$ Laplacian (Zeriahi [12]).
- $p = 2$ (Hécart [3], [4]).

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