ANNALES
POLONICI MATHEMATICI
80 (2003)

On the Hartogs extension theorem

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Abstract. This paper contains a new approach to a proof of the Hartogs extension theorem and its generalisation. The proof bases only on one complex variable methods.

1. Introduction. In this paper we consider the well known Hartogs extension theorem:

Theorem 1. *Let G be a domain in $\mathbb{C}^n$, $n \geq 2$, and let K be a compact subset of G such that $G \setminus K$ is connected. Then every holomorphic function $f : G \setminus K \to \mathbb{C}$ has a unique holomorphic extension $f^* : G \to \mathbb{C}$.*

There are several methods of proving this theorem; one is based on the theorem on existence of compactly supported solutions to the inhomogeneous Cauchy–Riemann equations (see e.g. [H, Th. 2.3.2]), another one makes use of the Bochner–Martinelli formula (see e.g. [S, 6.17, Th. 2]). In still another method one defines local "extensions" of f by the formula

$$\widetilde{f}(z) := \frac{1}{2\pi i} \int_{\gamma} \frac{f('z, \xi)}{\xi - z_n}\, d\xi, \qquad \gamma \in \Gamma('z),\ z = ('z, z_n) \in G \subset \mathbb{C}^{n-1} \times C,$$

where $\Gamma('z)$ is some admissible system of curves in the open set $\{\xi \in \mathbb{C} : ('z, \xi) \in G \setminus K\}$.

In the last, and as we think the most elementary method, the possible difficulties lie in proving that the locally defined functions $\widetilde{f}$ coincide in their common domains. In this paper we tackle these difficulties by applying an elementary lemma on the choice of a suitable cycle in an open set in the plane. The method allows proving a generalisation of Theorem 1 (presented in [JJ, Th. 2.1.1]).

For all definitions we refer to [R1] and [R2].

2. Hartogs theorem. For the sake of completeness, we recall the lemma mentioned in the introduction.

2000 *Mathematics Subject Classification*: Primary 32D15.
Key words and phrases: Hartogs theorem, holomorphic extension.

Lemma 1. *Let $G \subset \mathbb{C}$ be a nonempty open set and K a compact subset of G. Then there is a cycle C in $\mathbb{C}$ (or equivalently in $G \setminus K$) such that*

(i) $\mathrm{ind}_C(z) = 0$ *for* $z \in C \setminus G$,
(ii) $\mathrm{ind}_C(z) = 1$ *for* $z \in K$.

(For the proof see e.g. [R2, 12.4, formula (3) in "Cauchy integral formula for compact sets"]).

We shall need a version of this in $\mathbb{C}^n$.

Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $'z = (z_1, \ldots, z_{n-1})$. Define $\pi : \mathbb{C}^n \ni z \mapsto {'z} \in \mathbb{C}^{n-1}$. For $E \subset \mathbb{C}^n$ and $'z \in \mathbb{C}^{n-1}$ set $E('z) := \{\xi \in \mathbb{C} \mid ('z, \xi) \in E\}$.

Lemma 2. *Let G be an open set in $\mathbb{C}^n$ $(n \geq 2)$. Then for every $'a \in \pi(G)$ and every compact subset K of G there is a cycle C in $\mathbb{C}$ and a polydisk $'P \subset \pi(G)$ with centre at $'a$ such that for every $'z \in {'P}$ we have*

(i) $\mathrm{ind}_C(\xi) = 0$ *for* $\xi \in \mathbb{C} \setminus G('z)$,
(ii) $\mathrm{ind}_C(\xi) = 1$ *for* $\xi \in K('z)$

(*and so also* $|C| \subset G('z) \setminus K('z)$).

Proof. By Lemma 1 there is a cycle C in $\mathbb{C}$ such that

$$\mathrm{ind}_C(\xi) = 0 \quad \text{for } \xi \in \mathbb{C} \setminus G('a), \tag{1}$$

$$\mathrm{ind}_C(\xi) = 1 \quad \text{for } \xi \in K('a). \tag{2}$$

We shall show that the cycle C is good for (i); the case of (ii) is similar and left to the reader. Indeed, suppose to the contrary that one can find a sequence $(\overset{\nu}{z})_{\nu \geq 1}$, $\overset{\nu}{z} \in \mathbb{C}^n \setminus G$, such that $'\overset{\nu}{z} \to {'a}$ and for each $\nu \geq 1$ either $\mathrm{ind}_C(\overset{\nu}{z}_n) \neq 0$ or $\overset{\nu}{z}_n \in |C|$. In any case the sequence $(\overset{\nu}{z}_n)_{\nu \geq 1}$ is bounded. By choosing a subsequence we may assume that $\overset{\nu}{z}_n \to a_n \in \mathbb{C} \setminus G('a)$. By (1), $\mathrm{ind}_C(a_n) = 0$, which is impossible. ∎

Theorem 2. *Let G be a domain in $\mathbb{C}^n$, $n \geq 2$, and let K be a subset of G such that*

(a) *$G \setminus K$ is a domain,*
(b) *for any $'a \in \pi(G)$ there is a neighbourhood $'N \subset \pi(G)$ of $'a$ such that $\pi^{-1}('N) \cap K \Subset G$,*
(c) $\pi(K) \subsetneq \pi(G)$.

Then every holomorphic function $f : G \setminus K \to \mathbb{C}$ has a unique holomorphic extension $f^* : G \to \mathbb{C}$.

Proof. The uniqueness follows from the identity theorem. Fix $a = ('a, a_n) \in G$, a polydisk $'Q \subset \mathbb{C}^{n-1}$ with centre at $'a$ and a disk $D_a \subset \mathbb{C}$ with centre at a_n such that $\pi^{-1}('Q) \cap K \Subset G$ and $'Q \times D_a \Subset G$. Then the open set G, the point $'a$ and the compact set $[\overline{\pi^{-1}('Q)} \cap K] \cup \overline{'Q \times D_a}$ satisfy the assumptions of Lemma 2. Therefore there exists a cycle C_a and a polydisk $'P \subset {'Q}$

with centre at $'a$ such that for every $'z \in {'P}$,

$$\operatorname{ind}_{C_a}(\xi) = 0 \quad \text{for } \xi \in \mathbb{C} \setminus G('z), \tag{3}$$

$$\operatorname{ind}_{C_a}(\xi) = 1 \quad \text{for } \xi \in K('z) \cup \overline{D}_a, \tag{4}$$

$$|C_a| \subset G('z) \setminus [K('z) \cup \overline{D}_a]. \tag{5}$$

Write $P_a := {'P} \times D_a$ and define $f^a : P_a \to \mathbb{C}$ by

$$f^a(z) := \frac{1}{2\pi i} \int\limits_{C_a} \frac{f('z, \xi)}{\xi - z_n}\, d\xi, \qquad z \in P_a,$$

which is holomorphic by (5) and n-fold differentiation under the integral sign (see e.g. [R1, 82.2]).

Now, we claim that for any other point $b = ('b, b_n) \in G$,

$$f^a|_{P_a \cap P_b} \;=\; f^b|_{P_a \cap P_b}. \tag{6}$$

Indeed, fix $z \in P_a \cap P_b$. Put $\Omega := G('z) \setminus [K('z) \cup (\overline{D}_a \cap \overline{D}_b)]$ and consider a holomorphic function

$$g(\xi) := \frac{f('z, \xi)}{\xi - z_n}, \qquad \xi \in \Omega.$$

Observe that the properties (3) and (4) lead to

$$\operatorname{ind}_{C_a - C_b}(\xi) = 0 \quad \text{for } \xi \in [\mathbb{C} \setminus G('z)] \cup [K('z) \cup (\overline{D}_a \cap \overline{D}_b)] = \mathbb{C} \setminus \Omega.$$

The Cauchy theorem (see e.g. [R2, 13.1.3]) yields $\int_{C_a - C_b} g(\xi)\, d\xi = 0$, i.e. (6).

Now, (6) allows us to define a global holomorphic function $f^* : G \to \mathbb{C}$ by

$$f^*(z) := f^a(z) \quad \text{for } z \in P_a.$$

It remains to show that $f = f^*|_{G \setminus K}$. In order to do it, observe that for $a \in G \cap \pi^{-1}[\pi(G) \setminus \pi(K)]$, a nonempty open set, we have $K('a) = \emptyset$ and therefore we can apply the Cauchy integral formula (see e.g. [R2, 13.1.3]),

$$f^*(a) = \frac{1}{2\pi i} \int\limits_{C_a} \frac{f('a, \xi)}{\xi - a_n}\, d\xi = f(a) \operatorname{ind}_{C_a}(a) = f(a).$$

The identity theorem finishes the proof. ■

REMARK 1. Theorem 1 follows immediately from Theorem 2.

Acknowledgements. The author is grateful to his teacher Jacek Chądzyński, who proposed the problem and helped “organise” the proof.

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Reçu par la Rédaction le 7.9.2001
Révisé le 27.3.2002 (1377)