Regularity of domains of parameterized families of closed linear operators

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To Professor Józef Siciak on the occasion of his 70th birthday

Abstract. The purpose of this paper is to provide a method of reduction of some problems concerning families \( A_t = (A(t))_{t \in T} \) of linear operators with domains \( (D_t)_{t \in T} \) to a problem in which all the operators have the same domain \( D \). To do it we propose to construct a family \( (\Psi_t)_{t \in T} \) of automorphisms of a given Banach space \( X \) having two properties: (i) the mapping \( t \mapsto \Psi_t \) is sufficiently regular and (ii) \( \Psi_t(D) = D_t \) for \( t \in T \). Three effective constructions are presented: for elliptic operators of second order with the Robin boundary condition with a parameter; for operators in a Hilbert space for which eigenspaces form a complete orthogonal system of closed linear subspaces; and for a class of closed operators having bounded inverses.

1. Introduction. Most of the results concerning differential operators with a parameter \( t \) in the coefficients have been obtained under the assumption that the operators \( (A_t = A(t))_{t \in T} \) of a given family have domains independent of \( t \) (see e.g. [2, 6, 7, 8]).

One of possible ways of handling some problems concerning operators \( (A_t)_{t \in T} \) with domains \( D_t \subset X \) depending on \( t \) is to find a sufficiently regular (with respect to \( t \in T \)) family \( \Psi_t \) of automorphisms of the Banach space \( X \) such that \( \Psi_t(D_t) = D_t \), where \( D_t \) is a fixed linear subspace of \( X \).

In general, the domain of a differential operator is determined by some boundary conditions. Thus it would be useful to find an effective construction of a family \( \Psi_t \) using the boundary conditions only. Such a construction for a family of elliptic operators of order two with the Robin boundary condition with a parameter (i.e. \( \partial u/\partial n + a(x, t)u = 0 \) on \( \partial \Omega \)) is presented in 2.1. The problem of existence and construction of a family \( (\Psi_t)_{t \in T} \) for general types of boundary conditions is more delicate and still open.
In Section 2.2 there is a construction of a continuous family $\Psi_t$ for some families $(D_t)_{t \in \mathcal{T}}$ of domains of operators in a Hilbert space $H$ for which the corresponding eigenspaces form a complete orthogonal system of closed linear subspaces of $H$.

If $D_t$ is the domain of a closed invertible operator $A_t : X \to X$ and $R_t = A_t^{-1}$, for $t \in \mathcal{T}$, then the natural candidate for $\Phi_t = \Psi_t^{-1}$ is $R_{t_0} A_t$ whenever $R_{t_0} A_t$ is closable. If it is closable then we may use some results presented in [3] concerning the topology of generalized convergence to prove that the expected family is good (for more details see Section 3). Unfortunately, if $D_t$ depends on $t$, it may happen that $R_{t_0} A_t$ is not closable.

2. Regularity of families of linear subspaces. Let $X$ be a Banach space, $\mathcal{T}$ an interval in $\mathbb{R}$, and $(D_t)_{t \in \mathcal{T}}$ a family of linear subspaces of $X$.

Definition 1. We say that the family $(D_t)_{t \in \mathcal{T}}$ is of class $C^k$ (resp. strongly of class $C^k$) if there exist a linear subspace $D$ of $X$ and a family $(\Phi_t)_{t \in \mathcal{T}}$ of automorphisms of $X$ such that

- the mapping $\mathcal{T} \ni t \mapsto \Psi_t \in \text{Aut}(X)$ is of class $C^k$ (resp. strongly of class $C^k$) and
- $\Psi_t(D) = D_t$ for $t \in \mathcal{T}$.

Considering a family $(A_t)_{t \in \mathcal{T}}$ of closed linear operators with the family of domains $(D_t = D(A_t))_{t \in \mathcal{T}}$ of class $C^1$ we may reduce some problems to a family with a constant domain. For example, suppose that $u$ is a classical solution of the evolution equation

\[ \frac{du}{dt} = A(t)u + f(t) \]

in which the family of domains $(D_t = D(A_t))_{t \in \mathcal{T}}$ is of class $C^1$. Let $(\Psi_t)_{t \in \mathcal{T}}$ be a family of automorphisms of $X$ as above and $(\Phi_t = \Psi_t^{-1})_{t \in \mathcal{T}}$ the family of inverses.

Since $u(t) \in D_t$, there exists $v(t) \in D$ such that $\Psi_t(v(t)) = u(t)$. We have

\[ \frac{du}{dt} = \frac{d\Psi_t}{dt} v(t) + \Psi_t \frac{dv}{dt} \]

and after a standard calculation we obtain

\[ \frac{dv}{dt} = \left( \Phi_t A(t) \psi_t - \frac{d\psi_t}{dt} \right) v(t) + \Phi_t f(t) \]

Thus $v$ is a classical solution of the evolution equation

\[ \frac{dv}{dt} = B(t)v + F(t) \]

with the family $(B_t = B(t))_{t \in \mathcal{T}}$ of operators having domains independent of $t$.

\[ (1) \quad \text{This means that for any } x \in X \text{ the mapping } t \mapsto \Psi_t x \text{ is of class } C^k. \]
2.1. Construction using boundary conditions. Now we produce an example of a family \((D_t)_{t \in \mathcal{T}}\) of class \(C^k\), \(k \geq 1\), of the domains for elliptic operators of order two which is a nonconstant family of linear subspaces of \(X = \mathcal{L}^2(\Omega)\).

Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n\) with boundary \(S = \partial \Omega\) of class \(C^{k+1}\), \(\mathcal{T} = [0, T]\), and let \(a : \overline{\Omega} \times \mathcal{T} \to \mathbb{R}\) be a function of class \(C^k\) nonvanishing on \(S\). The sets

\[
D_t = \left\{ u \in \mathcal{L}^2(\Omega) : u \in H^2(\Omega) \text{ and } \frac{\partial u}{\partial n} + a(x, t)u = 0 \text{ on } \partial \Omega \right\},
\]

\[
\mathcal{D} = \left\{ u \in \mathcal{L}^2(\Omega) : u \in H^2(\Omega) \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \right\}
\]

are dense linear subspaces of \(\mathcal{L}^2(\Omega)\), where \(n\) is the interior unit normal vector field on \(S\).

Let \(\eta : \overline{\Omega} \times \mathcal{T} \to \mathbb{R}\) be a function of class \(C^k\) such that

\[
1/2 \leq \eta_t(x) = \eta(x, t) \quad \text{for } x \in \overline{\Omega}, \ t \in \mathcal{T},
\]

\[
\eta_t(x) = 1 \quad \text{and} \quad \frac{\partial \eta_t(x)}{\partial n} = a(x, t) \quad \text{for } x \in \partial \Omega, \ t \in \mathcal{T}.
\]

The function \(\eta\) can be constructed in the following way. We consider \(S\) as the retract of class \(C^k\) (for \(\varepsilon > 0\) small enough) of the open \(\varepsilon\)-tube

\[
\text{TUB}^\varepsilon(S) = \{ x + \tau n(x) : x \in S, \ |\tau| < \varepsilon \}.
\]

Then we take a function \(h_\varepsilon\) of class \(C^\infty\) in \(\mathbb{R}^n\) satisfying the following conditions:

\[
h_\varepsilon(x) = 0 \quad \text{for } x \in \mathbb{R}^n \setminus \text{TUB}^\varepsilon(S),
\]

\[
h_\varepsilon(x) = 1 \quad \text{for } x \in \text{TUB}^\varepsilon(S),
\]

\[
h_\varepsilon(x) \in [0, 1] \quad \text{for } x \in \mathbb{R}^n.
\]

The function

\[
f_\varepsilon : \text{TUB}^\varepsilon(S) \ni x + \tau n(x) \mapsto a(t, x)\tau \in \mathbb{R}
\]

is of class \(C^k\), and for \(\varepsilon\) small enough, the function \(\eta = h_\varepsilon f_\varepsilon + 1\) is one we have been looking for.

Let \(\Phi_t : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)\) be given by

\[
\Phi_t(u) = \eta_t \cdot u \quad \text{for } u \in \mathcal{L}^2(\Omega), \ t \in [0, T]
\]

and let \(\Psi_t = \Phi_t^{-1}\). One can verify that

- \(\Phi_t \in \text{Aut}(\mathcal{L}^2(\Omega))\),
- \(\Phi_t(D_t) = \mathcal{D}\) and \(\Psi_t(\mathcal{D}) = D_t\),
- the mapping \(\mathcal{T} \ni t \mapsto \Phi_t \in \mathcal{B}(\mathcal{L}^2(\Omega))\) is of class \(C^k\). Thus the mapping \(\mathcal{T} \ni t \mapsto \Psi_t \in \mathcal{B}(\mathcal{L}^2(\Omega))\) is also of class \(C^k\).
Considering parametrized boundary conditions of the form
\begin{equation}
\frac{\partial u}{\partial \mu_t} + a(x, t)u = 0 \quad \text{on } \partial \Omega,
\end{equation}
where \( \mu_t \) is a vector field on \( S \) parametrized by \( t \in T \), one can look for \( \Phi_t \) of the form
\begin{equation}
\Phi_t u = \eta_t \cdot (u \circ \varphi_t) \quad \text{for } u \in L^2(\Omega),
\end{equation}
where \( \varphi_t \) is a diffeomorphism of \( \Omega \) such that \( \varphi_t'(x) \cdot n(x) = \mu_t(x), \varphi_t(x) = x \) for \( x \in S, t \in T \), and \( \eta \) is as in (8). Indeed, if \( u \) satisfies (9) then
\begin{equation}
\frac{\partial (u \circ \varphi_t)}{\partial n}(x) + a(x, t)(u \circ \varphi_t)(x) = \frac{\partial u}{\partial \mu_t} + a(x, t)u = 0 \quad \text{on } \partial \Omega.
\end{equation}
Thus, \( v = u \circ \varphi_t \in D_t, \eta v \in D \) and vice versa.

Let us remark that the boundary conditions (9) parametrized by \( t \) are natural, for example, when we consider the family \( (A_t = tA + (1-t)\Delta)_{t \in [0,1]} \) in which \( A \) is a strongly elliptic operator of the second order. For an application see the second part of the proof of Theorem 3.4 in [1].

2.2. Construction using eigenspaces. Let \( H \) be a Hilbert space, and \( H_j, j = 1, 2, \ldots, \) a complete orthogonal sequence of closed linear subspaces of \( H \). We will use the following well known facts from the theory of Fourier series.

**Lemma 1.** If \( a_j \in H_j \) for \( j = 1, 2, \ldots \) then the series \( \sum_{j=1}^{\infty} a_j \) converges to a point \( a \in H \) if and only if the series \( \sum_{j=1}^{\infty} \|a_j\|^2 \) is convergent. Moreover, if \( a = \sum_{j=1}^{\infty} a_j \) then
\begin{equation}
\|a\|^2 = \sum_{j=1}^{\infty} \|a_j\|^2 \quad \text{and} \quad a_j = p_j(a) \text{ for } j = 1, 2, \ldots,
\end{equation}
where \( p_j : H \to H_j \) is the orthogonal projection of \( H \) onto \( H_j \) for \( j = 1, 2, \ldots \).

To any sequence \( \lambda = \{\lambda_j\}_{j=1}^{\infty} \) of real (complex if \( H \) is a complex space) numbers corresponds a closed linear operator \( A = A_{\lambda}(t) : H \to H \) given by
\begin{equation}
Ax = A_\lambda x = \sum_{j=1}^{\infty} \lambda_j p_j(x).
\end{equation}
The operator \( A \) with domain
\begin{equation}
D := D(A) = \left\{ x \in H : \sum_{j=1}^{\infty} \lambda_j p_j(x) \text{ is convergent} \right\}
\end{equation}
is a closed densely defined linear operator and \( \lambda_j \) is an eigenvalue of \( A \) corresponding to the eigenspace \( H_j \).

From now on we assume that \( H_j \) and \( \lambda_j \) depend on the parameter \( t \in \mathcal{T} \). This implies that the projections \( p_j, j = 1, 2, \ldots \), also depend on \( t \). Thus, \( H_j(t), \lambda_j(t), p_j(t), j = 1, 2, \ldots \), are sequences of closed subspaces, numbers and projections, respectively, parametrized by \( t \in \mathcal{T} \).

**Proposition 2.** Suppose that for given \( t, t_0 \in \mathcal{T}, \Phi_j(t) : H \to H, j = 1, 2, \ldots \), are bounded linear mappings satisfying the following conditions:

(i) \( \Phi_j(t)(H_j(t)) = H_j(t_0) \) and \( \Phi_j(t)|_{H_j(t)} : H_j(t) \to H_j(t_0) \) is an isomorphism of Banach spaces for \( j = 1, 2, \ldots \),

(ii) there exist positive constants \( M(t), m(t) > 0 \) such that

\[
m(t)\|x\| \leq \|\Phi_j(t)x\| \leq M(t)\|x\| \quad \text{for} \ x \in H_j(t), \ j = 1, 2, \ldots ,
\]

(iii) there exist positive constants \( \delta(t), \Delta(t) > 0 \) such that

\[
\delta(t) \leq \left| \frac{\lambda_j(t_0)}{\lambda_j(t)} \right| \leq \Delta(t) \quad \text{for} \ j = 1, 2, \ldots .
\]

Then

\[
(11) \quad \Phi_t := \Phi(t) = \sum_{j=1}^{\infty} \Phi_j(t) \circ p_j(t)
\]

is an automorphism of \( H \) such that \( \Phi_t(\mathcal{D}_t) = \mathcal{D}_{t_0} \).

**Proof.** We begin by proving that \( \Phi_t \) is well defined. Since

\[
\|\Phi_j(t)(p_j(t)x)\|^2 \leq \|\Phi_j(t)\|^2\|p_j(t)x\|^2 \leq M^2(t)\|p_j(t)x\|^2
\]

and the series \( \sum_{j=1}^{\infty} \|p_j(t)x\|^2 \) is convergent (because \( \sum_{j=1}^{\infty} p_j(t)x \) is convergent), the series defining \( \Phi_t x \) is convergent for any \( (t, x) \in I \times H \).

Since

\[
\|\Phi_t x\|^2 = \sum_{j=1}^{\infty} \|\Phi_j(t)p_j(t)x\|^2 \leq M^2(t)\|x\|^2,
\]

the operator \( \Phi_t \) is bounded.

Injectivity of \( \Phi_t \) follows from Lemma 1. Indeed,

\[
\ker \Phi_t = \{ x \in H : \Phi_j(t)p_j(t)x = 0, j = 1, 2, \ldots \} = \{0\}.
\]

Let \( y \in H \) and

\[
(12) \quad x = \sum_{j=1}^{\infty} (\Phi_j(t))^{-1} p_j(t_0)y.
\]

To prove surjectivity we must prove that the series (12) defining \( x \) is convergent and that \( \Phi_t x = y \).
Assuming the convergence for the moment, we have
\[ \Phi_t x = \sum_{j=1}^{\infty} \Phi_j(t) p_j(t) x = \sum_{j=1}^{\infty} \Phi_j(t) (\Phi_j(t))^{-1} p_j(t_0) y = \sum_{j=1}^{\infty} p_j(t_0) y = y. \]

The convergence of \( \sum_{j=1}^{\infty} (\Phi_j(t))^{-1} p_j(t_0) y \) follows from Lemma 1, because of the estimates
\[ \| (\Phi_j(t))^{-1} p_j(t_0) y \|^2 \leq \frac{1}{m(t)} \| p_j(t) y \|^2 \quad \text{for} \quad j = 1, 2, \ldots, y \in X. \]

For \( x \in D_t \) the series \( \sum_{j=1}^{\infty} \lambda_j(t) p_j(t) x \) is convergent and we have \( \Phi_t x = y = \sum_{j=1}^{\infty} p_j(t_0) y \). Thus
\[ \sum_{j=1}^{\infty} \Phi_j(t) p_j(t) x = \sum_{j=1}^{\infty} p_j(t_0) y, \]
which implies that
\[ p_j(t_0) y = \Phi_j(t) p_j(t) x. \]

Since
\[ \left\| \frac{\lambda_j(t_0)}{\lambda_j(t)} \Phi_j(t) (\lambda_j(t) p_j(t)) x \right\|^2 \leq \left| \frac{\lambda_j(t_0)}{\lambda_j(t)} \right|^2 M^2(t) \| \lambda_j(t) p_j(t) \|^2 \leq \Delta^2(t) M^2(t) \| \lambda_j(t) p_j(t) \|^2, \]
the series
\[ \sum_{j=1}^{\infty} \frac{\lambda_j(t_0)}{\lambda_j(t)} \Phi_j(t) (\lambda_j(t) p_j(t)) x \]
is convergent and hence, because of (13), so is \( \sum_{j=1}^{\infty} \lambda_j(t_0) p_j(t_0) y \). This means that \( \Phi_t(D_t) \subset D_{t_0} \). The proof of the inverse inclusion is similar. \( \blacksquare \)

Remark 1. If, in Proposition 2, \( \Phi_j(t) : H_j(t) \to H_j(t_0) \) is an isometry for all \( j \) and \( t \), then \( \Phi_t \) is also an isometry.

Theorem 3. If the mappings
\[ I \ni t \mapsto p_j(t) \quad \text{and} \quad I \ni t \mapsto \Phi_j(t) \quad \text{for} \quad j = 1, 2, \ldots \]
are continuous and there exist \( M, m > 0 \) such that
\[ m \| x \| \leq \| \Phi_j(t) x \| \leq M \| x \| \quad \text{for} \quad j = 1, 2, \ldots, x \in X, \]
then for any compact set \( K \subset I \times H, \) the mapping \( K \ni (t, x) \mapsto \Phi_t x \) is continuous.

Proof. By Dini's theorem, the sequence
\[ S_\nu(t, x) = \sum_{j=1}^{\nu} \| p_j(t) x \|^2, \quad \nu = 1, 2, \ldots, \]
converges uniformly to \( \|x\|^2 \) on compact subsets of \( I \times H \). Since
\[
\left\| \sum_{j=p}^{p+s} \Phi_j(t)(p_j(t)x) \right\|^2 \leq M^2 \sum_{j=p}^{p+s} \|p_j(t)x\|^2,
\]
the Cauchy condition of uniform convergence is satisfied for the series \( \sum_{j=1}^{\infty} \Phi_j(t)(p_j(t)x) \) and so \( \Phi \) is continuous on \( K \). \( \blacksquare \)

To obtain a higher regularity for the family \( (\Phi_t) \), we must assume a higher regularity for \( p_j(t) \) and some assumptions that guarantee differentiability of series term by term.

**Example 1.** The mapping
\[\Phi_j(t) := p_j(t_0) \circ p_j(t) : H \to H_j(t_0) \subset H\]
is a bounded linear map. Assuming, for example, that
\[H_j^+(t_0) \cap H_j(t) = \{0\}\]
we see that \( \Phi_j(t) \) is injective.

If additionally we assume that \( \dim H_j(t) = k_j < \infty \) is independent of \( t \) then
\[\Phi_j(t)|_{H_j(t)} : H_j(t) \to H_j(t_0)\]
is an isomorphism. Moreover
\[\|\Phi_j(t)x\|^2 \leq \|p_j(t)x\|^2 \quad \text{and} \quad \Phi_j(t_0) = p_j(t_0).\]
Therefore, using the same method as in the proof of Theorem 3, we may prove that the continuity of the mapping \( I \times H \ni (t, x) \mapsto p_j(t)x \in H \) for \( j = 1, 2, \ldots \) implies the continuity of \( \Phi \) on compact subsets of \( I \times H \).

**3. Families of closed operators with bounded inverses.** Let \( X, Y \) be Banach spaces over the field \( \mathbb{K} \) of real or complex numbers. We endow the space \( C(X, Y) \) of closed linear operators \( A : X \to Y \) with the topology of generalized convergence [3, Ch. IV]. The domain of a given operator \( A : X \to Y \) is denoted by \( D(A) \). The space of bounded linear operators \( A : X \to Y \) is denoted by \( B(X, Y) \), and \( \text{Isom}(X, Y) \) is the subspace of \( B(X, Y) \) of bijective bounded linear operators with bounded inverses. The subspace of \( C(X, Y) \) consisting of the invertible densely defined operators \( A \) such that \( A^{-1} \in B(Y, X) \) will be denoted by \( \mathcal{R}(X, Y) \). If \( X = Y \) we will write \( C(X), B(X), \text{Aut}(X), \mathcal{R}(X) \) instead of \( C(X, X), B(X, X), \text{Isom}(X, X), \mathcal{R}(X, X) \), respectively. Since \( B(X, Y) \subset C(X, Y) \), we may consider \( B(X, Y) \) with the induced topology, which by [3, Ch. IV, Theorem 2.23] is equivalent to the norm topology in \( B(X, Y) \). Let us also recall that by the same theorem, the convergence of \( A_n \) to \( A \) in \( \mathcal{R}(X, Y) \) is equivalent to the convergence of \( A_n^{-1} \) to \( A^{-1} \) in \( B(Y, X) \), \( \text{Isom}(X, Y) \) is open in \( C(X, Y) \) and \( \text{Aut}(X) \) is open in \( C(X) \).
\textbf{Lemma 4.} Let $\mathcal{H}$ be a metric space, $A_h \in \mathcal{R}(X,Y)$ and $\Phi_h \in \text{Aut}(X)$ for $h \in \mathcal{H}$. If the mappings
\begin{equation}
H \ni h \mapsto A_h \in C(X,Y) \quad \text{and} \quad H \ni h \mapsto \Phi_h \in \mathcal{B}(X)
\end{equation}
are continuous then the mapping
\begin{equation}
H \ni h \mapsto A_h \circ \Phi_h \in \mathcal{R}(X,Y)
\end{equation}
is also continuous.

\textit{Proof.} Since $A_h \in \mathcal{R}(X,Y)$, the continuity of $H \ni h \mapsto A_h \in C(X,Y)$ is equivalent to the continuity of $H \ni h \mapsto A_h^{-1} \in \mathcal{B}(Y,X)$. Thus the mapping $H \ni h \mapsto (A_h \circ \Phi_h)^{-1} = \Phi_h^{-1} \circ A_h^{-1} \in \mathcal{B}(Y,X)$ is continuous, and hence so is the mapping (15). \hfill \blacksquare

\textbf{Lemma 5.} Let $A_i \in \mathcal{R}(X,Y)$ and $R_j = A_j^{-1}$ for $j = 1,2$. If $R_j \circ A_i$ is bounded for $i,j = 1,2$, then $\mathcal{D}(A_i^*) = \mathcal{D}(A_2^*)$.

\textit{Proof.} By symmetry, it is enough to prove that $\mathcal{D}(A_1^*) \subset \mathcal{D}(A_2^*)$. Take $y^* \in \mathcal{D}(A_1^*)$. Since
\begin{align*}
|\langle A_2 x, y^* \rangle| &= |\langle A_1 \circ (R_1 \circ A_2) x, y^* \rangle| = |\langle (R_1 \circ A_2) x, A_1^* y^* \rangle| \\
&\leq \|A_1^* y^*\| \cdot \|R_1 \circ A_2\| \cdot \|x\|
\end{align*}
we have $y^* \in \mathcal{D}(A_2^*)$ and so $\mathcal{D}(A_1^*) \subset \mathcal{D}(A_2^*)$. \hfill \blacksquare

\textbf{Lemma 6.} If $A \in C(X,Y)$ and $\Phi \in \mathcal{B}(X)$ is such that $\mathcal{D}(A \circ \Phi)$ is dense in $X$ then $\mathcal{D}(A^*) \subset \mathcal{D}((A \circ \Phi)^*)$. Moreover, if $\Phi \in \text{Aut}(X)$ then $\mathcal{D}(A^*) = \mathcal{D}((A \circ \Phi)^*)$.

\textit{Proof.} Let $y^* \in \mathcal{D}(A^*)$. Since $\Phi$ is continuous, for $x \in \mathcal{D}(A \circ \Phi)$ we have
\begin{align*}
|\langle (A \circ \Phi) x, y^* \rangle| &= |\langle \Phi x, A^* y^* \rangle| \leq \|A^* y^*\| \cdot \|\Phi x\| \leq \|A^* y^*\| \cdot \|\Phi\| \cdot \|x\|
\end{align*}
and so $y^* \in \mathcal{D}((A \circ \Phi)^*)$.

If $\Phi$ is invertible then by the above $\mathcal{D}((A \circ \Phi)^*) \subset \mathcal{D}((A \circ \Phi \circ \Phi^{-1})^*) = \mathcal{D}(A^*)$. \hfill \blacksquare

\textbf{Theorem 7.} Let $(\mathcal{H}, \rho)$ be a connected metric space and $(A_h)_{h \in \mathcal{H}}$ a family of linear operators $A_h \in \mathcal{R}(X,Y)$. If $R_k \circ A_h$ is closable for each $h, k \in \mathcal{H}$, and for each $k \in \mathcal{H}$ the mapping
\begin{equation}
H \ni h \mapsto R_k \circ A_h \in C(X)
\end{equation}
is continuous, then:

(i) $R_k \circ A_h \in \text{Aut}(X)$ for each $h, k \in \mathcal{H}$,

(ii) for any $h,k \in \mathcal{H}$ there exist $m,M > 0$ such that
\begin{equation*}
\|R_h y\| \leq \|R_k y\| \leq M \|R_h y\| \quad \text{for} \quad y \in Y,
\end{equation*}

(iii) $\mathcal{D}(A_h^*) = \mathcal{D}^* = \text{const}$,

(iv) $\mathcal{D}(A_h^* \circ R_k^*) = X^*$ for all $h,k \in \mathcal{H}$.
Proof. Since Aut(X) is open in C(X) and \( R_k \circ A_k = \text{Id}_X \in \text{Aut}(X) \), there exists \( \delta = \delta(k) > 0 \) such that \( R_k \circ A_h \in \text{Aut}(X) \) for any \( h \in \mathcal{H} \) such that \( \rho(h, k) < \delta \). Thus, for a given \( k \in \mathcal{H} \),

\[
\mathcal{M} = \{ h \in \mathcal{H} : R_k \circ A_h \in \text{Aut}(X) \} \neq \emptyset.
\]

To prove that \( \mathcal{M} = \mathcal{H} \) it is enough to prove that \( \mathcal{M} \) is open and closed. For given \( h_0 \in \mathcal{M} \), \( h \in \mathcal{H} \) we have

\[
R_k \circ A_h = R_k \circ A_{h_0} \circ R_{h_0} \circ A_h.
\]

Since \( R_k \circ A_{h_0} \in \text{Aut}(X) \), and by the same argument as before there exists \( \delta = \delta(h_0) > 0 \) such that \( R_{h_0} \circ A_h \in \text{Aut}(X) \) for any \( h \in \mathcal{H} \) satisfying \( \rho(h, h_0) < \delta \), the set \( \mathcal{M} \) is open. Suppose now that \( h_n \in \mathcal{M} \) for \( n = 1, 2, \ldots \) and \( h_n \to h_0 \in \mathcal{H} \) as \( n \to \infty \). Then there exists \( n \in \mathbb{N} \) such that \( R_{h_0} \circ A_{h_n} \in \text{Aut}(X) \), by the previous part of the proof. Since \( R_k \circ A_{h_n} = R_k \circ A_{h_0} \circ R_{h_0} \circ A_{h_n} \) and \( R_k \circ A_{h_n}, R_{h_0} \circ A_{h_n} \) are automorphisms of \( X \), it follows that \( h_0 \in \mathcal{M} \) and so \( \mathcal{M} \) is closed.

To prove (ii) fix \( h, k \in \mathcal{H} \). Since \( R_k \circ A_h \in \text{Aut}(X) \), there exist \( m, M > 0 \) such that

\[
m \lVert x \rVert \leq \lVert (R_k \circ A_h)x \rVert \leq M \lVert x \rVert \quad \text{for } x \in \mathcal{D}(A_h).
\]

Since \( A_h \) is onto, taking \( y = A_h x \) we get

\[
m \lVert R_k y \rVert \leq \lVert R_k y \rVert \leq M \lVert R_k y \rVert \quad \text{for } y \in Y.
\]

To prove (iii) observe that for \( k, h \in \mathcal{H} \) we have \( A_h = A_k \circ R_k \circ A_h \). Thus, by Lemma 6, \( \mathcal{D}(A_h^*) = \mathcal{D}(A_k^*) \), because \( R_k \circ A_h \in \text{Aut}(X) \).

(iv) is a consequence of the fact that \( R_k^* \in \mathcal{B}(X^*, Y^*) \) is the inverse to \( A_k^* \) (see e.g. [3, Ch. III, Theorem 5.30]), which has the same domain as \( A_h^* \), because of (3).

Remark 2. Observe that for \( k, h \in \mathcal{H} \), if \( R_k \circ A_h \) and \( R_h \circ A_k \) are closable then conditions (i)–(iv) of Theorem 7 are equivalent. If \( \mathcal{D}(A_h^* \circ R_k^*) \) is dense in \( X^* \) in the weak* topology on \( X^* \) then \( R_k \circ A_h \) is closable. If condition (iii) of Theorem 7 is satisfied, then \( R_k \circ A_h \) is closable and (i), (ii), (iv) hold.

A sufficient condition for the assumptions of Theorem 7 to hold is presented in the following

**Proposition 8.** If \( \mathcal{H} = [0, T] \), all the operators of the family \( (A_t^*)_{t \in [0, T]} \) have the same domain \( \mathcal{D}^* \) and for every \( y^* \in \mathcal{D}^* \) the mapping

\[
(17) \quad [0, T] \ni t \mapsto A_t^* y^* \in X^*
\]

is of class \( C^1 \) then the family \( (A_t)_{t \in [0, T]} \) satisfies the assumptions of Theorem 7.

**Proof.** By [4, Ch. II, Lemma 1.5], the family \( (A_t^* \circ R_s^*)_{s, t \in [0, T]} \) of bounded operators is continuous with respect to \((s, t)\). Since also \( A_t^* \circ R_s^* = (R_s \circ A_t)^* \)
and $D((R_s \circ A_t)^*) = X^*$, the mapping $R_s \circ A_t$ is closable, and by [3, Ch. IV, Theorem 2.23], the continuity of the family $(R_s \circ A_t)^*$ with respect to $(s, t)$ implies the continuity of $(R_s \circ A_t)$. ■

3.1. Some remarks on the case of differential operators. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$, and $\mathcal{H}$ a connected metric space. Let

$$A_h = \sum_{|\alpha| \leq m} a_\alpha(x, h)D^\alpha \quad \text{for } h \in \mathcal{H}$$

be a family of differential operators of order $m$ with coefficients $a_\alpha$ continuous in $\overline{\Omega} \times \mathcal{H}$. Closedness of $A_h$ and continuity of the mapping $h \mapsto A_h$ depend on the domain $D(A_h)$, the space $X$ in which $D(A_h)$ is contained, and the space $Y$ of values of $A_h$.

- If $D(A_h) = X = H^m(\Omega)$ and $Y = \mathcal{L}^2(\Omega)$ then $A_h$ is bounded and the mapping $\mathcal{H} \ni h \mapsto A_h \in \mathcal{B}(X, Y)$ is continuous.

- Let $X = Y = \mathcal{L}^2(\Omega)$ and let $D$ be a closed subspace of $H^m(\Omega)$ such that $D$ is dense in $\mathcal{L}^2(\Omega)$, and the mapping $A_h : D \to \mathcal{L}^2(\Omega)$ is one-to-one and onto for $h \in \mathcal{H}$. Then $R_h = A_h^{-1} \in \mathcal{B}(Y, X)$ and the mapping $\mathcal{H} \ni h \mapsto A_h \in C(X, Y)$ is continuous. This situation often occurs when considering strongly elliptic operators $A_h$ with boundary operators independent of $h$,

$$B_j = \sum_{|\alpha| \leq m_j} b_{j\alpha}(x)D^\alpha, \quad 1 \leq j \leq m/2,$$

which cover $A_h$ for each $h \in \mathcal{H}$. If additionally we know that $D(A_h^*) = \mathcal{D}^*$ is independent of $h$ then $R_k \circ A_h$ is closable for each $h, k$.

Now we show an example of a family $(\tilde{A}_t)_{t \in T}$ of elliptic operators with pairwise different domains for which the corresponding family $(\mathcal{D}_t)_{t \in T}$ of domains is of class $C_{a_k}$ and the family of domains of the conjugate operators is independent of $t$.

Keep the notation of Section 2.1 and assume that $a(x, t) = t$. The sets $\mathcal{D}_t$ given by (3) are dense linear subspaces of $\mathcal{L}^2(\Omega)$ such that $\mathcal{D}_t \neq \mathcal{D}_\tau$ for $t \neq \tau \in [0, T]$ and $\mathcal{D}_0 = \mathcal{D}$, where $\mathcal{D}$ is given by (4). The operator

$$A = -\Delta + \lambda I$$

is well defined on $H^2(\Omega)$; when considered as defined only on $\mathcal{D}_t$, it is closed, and for $\lambda$ large enough, it is onto and one-to-one. By the closed graph theorem its inverse is bounded. Let $A_t$ denote the operator given by (20) with domain $\mathcal{D}_t$.

**Example 2.** The family

$$\tilde{A}_t = A_0 \circ \Phi_t : \mathcal{D}_t \to \mathcal{L}^2(\Omega)$$
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parametrized by $t \in [0,T]$ is a continuous (with respect to $t$) family of closed densely defined linear differential operators with pairwise different domains. Indeed, since $(B_t = A_t \circ \Psi_t)_{t \in [0,T]}$ is a family of closed differential operators of order two with coefficients continuous with respect to both $x$ and $t$, and with domains independent of $t$, the mapping $[0,T] \ni t \mapsto B_t \in C(L^2(\Omega))$ is continuous and, by Lemma 4, the mapping $[0,T] \ni t \mapsto A_t = B_t \circ \Phi_t \in C(L^2(\Omega))$ is also continuous.

By Lemma 6, the domain $D(A^*_t) = D(A^*)$ is the same for all $t \in [0,T]$.

The next example shows that in Theorem 7 the assumption of continuity of the mapping (16) cannot be replaced by the continuity of the family $(A_h)_{h \in H}$.

EXAMPLE 3. Let $(A_t)_{t \in [0,T]}$ be a family of self-adjoint operators with pairwise different domains, and with the same property for the family $A_t^*$. Since $C^\infty_0(\Omega) \subset \bigcap_{t \in [0,T]} D_t$, $C^\infty_0$ is dense in $L^2(\Omega)$, $(R_\tau \circ A_t)u = (R_\tau \circ A_t)u = u$ for $u \in C^\infty_0(\Omega)$ and $(R_\tau \circ A_t)u \neq u$ for $u \in D_t \setminus D_\tau$, it follows that the operator $R_\tau \circ A_t$ is not closable for $t \neq \tau$. Thus, the mapping (16) is even not well defined.

References


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