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Generalized problem of starlikeness for products of close-to-star functions

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Abstract. We consider functions of the type $F(z) = z\prod_{j=1}^{n}[f_j(z)/z]^{a_j}$, where a_j are real numbers and f_j are β_j-strongly close-to-starlike functions of order α_j. We look for conditions on the center and radius of the disk $\mathcal{D}(a,r) = \{z : |z-a| < r\}$, $|a| < r \le 1 - |a|$, ensuring that $F(\mathcal{D}(a,r))$ is a domain starlike with respect to the origin.

1. Introduction. Let $\widetilde{\mathcal{A}}$ denote the class of functions which are analytic in $\mathcal{D} = \mathcal{D}(0,1)$, where

$$\mathcal{D}(a,r) = \{z : |z-a| < r\},$$

and let $\mathcal{A}$ denote the class of functions $f \in \widetilde{\mathcal{A}}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{D}). \tag{1}$$

A function $f \in \mathcal{A}$ is said to be *starlike of order* α, $0 \le \alpha < 1$, in $\mathcal{D}(r) := \mathcal{D}(0,r)$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathcal{D}(r) \setminus f^{-1}(0)).$$

A function $f \in \mathcal{A}$ is said to be *convex of order* α, $0 \le \alpha < 1$, in $\mathcal{D}$ if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathcal{D} \setminus (f')^{-1}(0)).$$

We denote by $\mathcal{S}^c(\alpha)$ the class of all functions $f \in \mathcal{A}$ which are convex of order α in $\mathcal{D}$, and by $\mathcal{S}^*(\alpha)$ the class of all functions $f \in \mathcal{A}$ which are starlike of order α in $\mathcal{D}$.

Let $\mathcal{H}$ be a subclass of $\mathcal{A}$. We define the *radius of starlikeness* of $\mathcal{H}$ by

$$R^*(\mathcal{H}) = \inf_{f\in\mathcal{H}}(\sup\{r \in (0,1] : f \text{ is starlike of order 0 in } \mathcal{D}(r)\}).$$

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We denote by $\mathcal{P}(\beta),\ 0<\beta\le 1$, the class of functions $h\in\widetilde{\mathcal{A}}$ such that $h(0)=1$ and

$$h(\mathcal{D})\subset \Pi_\beta := \{w\in\mathbb{C}\setminus\{0\} : |\mathrm{Arg}\ w| < \beta\pi/2\},$$

where Arg w denotes the principal argument of the complex number w (i.e. from the interval $(-\pi,\pi]$). The class $\mathcal{P}:=\mathcal{P}(1)$ is the well known class of Carathéodory functions.

We say that a function $f\in\mathcal{A}$ belongs to the class $\mathcal{CS}^*(\alpha,\beta)$ if there exists a function $g\in\mathcal{S}^*(\alpha)$ such that $f/g\in\mathcal{P}(\beta)$. The class $\mathcal{CS}^*:=\mathcal{CS}^*(0,1)$ is the well-known class of close-to-star functions with argument 0.

Let a,c,m,M,N be positive real numbers and let $b\in[-m,m]$.

Silverman [8] introduced the class of functions F given by the formula

$$F(z)=z\prod_{j=1}^{n}(f_j(z)/z)^{a_j}\prod_{j=1}^{k}(g_j'(z))^{b_j},$$

where $f_j\in\mathcal{S}^*(\alpha)$, $g_j\in\mathcal{S}^c(\beta)$, and a_j,b_j are positive real numbers satisfying

$$\sum_{j=1}^{n}a_j=a,\qquad \sum_{j=1}^{k}b_j=b.$$

Dimkov [2] studied the class of functions F given by

$$F(z)=z\prod_{j=1}^{n}(f_j(z)/z)^{a_j}\qquad (f_j\in\mathcal{S}^*(\alpha_j),\ j=1,\dots,n),$$

where a_j $(j=1,\dots,n)$ are complex numbers satisfying

$$\sum_{j=1}^{n}(1-\alpha_j)|a_j|\le M.$$

Let n be a positive integer and let

$$\alpha=(\alpha_1,\dots,\alpha_n),\beta=(\beta_1,\dots,\beta_n),\mathbf{a}=(a_1,\dots,a_n)\in\mathbb{R}^n$$

be fixed vectors, with $0\le\alpha_j<1,\ 0<\beta_j\le 1$ $(j=1,\dots,n)$.

Motivated by Silverman's and Dimkov's definitions, we define the class $\mathcal{H}^n(\mathbf{a},\alpha,\beta)$ of functions F given by the formula

$$F(z)=z\prod_{j=1}^{n}(f_j(z)/z)^{a_j}\qquad (f_j\in\mathcal{CS}^*(\alpha_j,\beta_j),\ j=1,\dots,n). \tag{2}$$

We denote by $\mathcal{G}^n(m,b,c)$ the union of all classes $\mathcal{H}^n(\mathbf{a},\alpha,\beta)$ for which

$$\sum_{j=1}^{n}(1-\alpha_j)|a_j|=m,\quad \sum_{j=1}^{n}(1-\alpha_j)a_j=b,\quad \sum_{j=1}^{n}\beta_j|a_j|=c. \tag{3}$$

Finally, set

$$\mathcal{G}^n(M,N) := \bigcup_{\substack{c\in[0,N]\\ m\in[0,M]}} \bigcup_{b\in[-m,m]} \mathcal{G}^n(m,b,c). \tag{4}$$

It is clear that $\mathcal{G}^n(M,N)$ contains all functions F given by (2) for which

$$\sum_{j=1}^{n}(1-\alpha_j)|a_j| \le M, \qquad \sum_{j=1}^{n}\beta_j|a_j| \le N.$$

Aleksandrov [1] stated and solved the following problem.

PROBLEM 1. Let $\mathcal{H}$ be the class of all functions $f \in \mathcal{A}$ that are univalent in $\mathcal{D}$, and let $\Delta \subset \mathcal{D}$ be a domain starlike with respect to an inner point ω with smooth boundary given by the formula

$$z(t) = \omega + r(t)e^{it} \qquad (0 \le t \le 2\pi).$$

Find conditions on $r(t)$ ensuring that for each $f \in \mathcal{H}$ the image domain $f(\Delta)$ is starlike with respect to $f(\omega)$.

Świtoniak et al. [9, 10] and Dimkov and Dziok [3] (see also [4]) have investigated a similar problem for generalized starlikeness.

PROBLEM 2. Let $\mathcal{H} \subset \mathcal{A}$. Determine the set $B^*(\mathcal{H})$ of all pairs $(a,r) \in \mathcal{D}\times\mathbb{R}$ such that $|a| < r \le 1-|a|$ and every function $f \in \mathcal{H}$ maps the disk $\mathcal{D}(a,r)$ onto a domain starlike with respect to the origin. The set $B^*(\mathcal{H})$ is called the *set of generalized starlikeness* of the class $\mathcal{H}$.

We note that

$$R^*(\mathcal{H}) = \sup\{r : (0,r) \in B^*(\mathcal{H})\}. \tag{5}$$

In this paper we determine the sets of generalized starlikeness of the classes $\mathcal{H}^n(\mathbf{a},\alpha,\beta)$, $\mathcal{G}^n(m,b,c)$, $\mathcal{G}^n(M,N)$ and $\mathcal{CS}^*(\alpha,\beta)$. Moreover, we obtain the radii of starlikeness of these classes.

2. Main results. We begin by listing some lemmas which will be useful later on.

LEMMA 1 ([10]). *A function $f\in\mathcal{A}$ maps the disk $\mathcal{D}(a,r)$, $|a|<r\le 1-|a|$, onto a domain starlike with respect to the origin if and only if*

$$\operatorname{Re}\frac{e^{i\theta}f'(a+re^{i\theta})}{f(a+re^{i\theta})} \ge 0 \qquad (0\le\theta\le 2\pi). \tag{6}$$

For a function $f \in \mathcal{S}^*(\alpha)$ it is easy to verify that

$$\left|\frac{zf'(z)}{f(z)} - \alpha - (1-\alpha)\frac{1+|z|^2}{1-|z|^2}\right| \le \frac{2(1-\alpha)|z|}{1-|z|^2} \qquad (z\in\mathcal{D}).$$

Thus, after some calculations we get the following lemma.

LEMMA 2. *Let* $f \in \mathcal{S}^*(\alpha)$, $a, \theta \in \mathbb{R}$, $z \in \mathcal{D}_0 := \mathcal{D} \setminus \{0\}$. *Then*

$$\operatorname{Re}\left[ae^{i\theta}\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right)\right] \ge \operatorname{Re}\frac{2(1-\alpha)|z|^2 ae^{i\theta}}{(1-|z|^2)z} - \frac{2(1-\alpha)|a|}{1-|z|^2}.$$

LEMMA 3 ([6]). *If* $h \in \mathcal{P}(\beta)$, *then*

$$\left|\frac{h'(z)}{h(z)}\right| \le \frac{2\beta}{1-|z|^2} \quad (z \in \mathcal{D}).$$

THEOREM 1. *Let* m, b, c *be defined by* (3) *and set*

$$\mathcal{B}' = \left\{ (a,r) \in \mathbb{C} \times \mathbb{R} : \begin{array}{l} (0 \le r \le r_1 \wedge |a| < r) \vee \\ (r_1 < r < r_2 \wedge |a| \le \varphi(r)) \vee \\ (r_2 \le r < q \wedge |a| \le q - r) \end{array} \right\}, \tag{7}$$

$$\mathcal{B}'' = \{(a,r) \in \mathbb{C} \times \mathbb{R} : |a| < r \le q - |a|\}, \tag{8}$$

where

$$r_1 = \frac{1}{4(m+c)}, \tag{9}$$

$$r_2 = \frac{m+c}{(m+c+\sqrt{(m+c)^2-2b+1})^2}, \tag{10}$$

$$q = \frac{1}{m+c+\sqrt{(m+c)^2-2b+1}}, \tag{11}$$

$$\varphi(r) = \sqrt{r^2 - \frac{(1-2\sqrt{r(m+c)})^2}{2b-1}}. \tag{12}$$

Moreover, set

$$\mathcal{B} = \begin{cases} \mathcal{B}' & \text{for } b > 1/2, \\ \mathcal{B}'' & \text{for } b \le 1/2. \end{cases} \tag{13}$$

If $(a,r) \in \mathcal{B}$, *then every function* $F \in \mathcal{H}^n(\mathbf{a}, \alpha, \beta)$ *maps the disk* $\mathcal{D}(a,r)$ *onto a domain starlike with respect to the origin. The result is sharp for* $b \le 1/2$, *and for* $b > 1/2$ *the set* $\mathcal{B}$ *cannot be larger than* $\mathcal{B}''$. *This means that*

$$\mathcal{B}' \subset B^*(\mathcal{H}^n(\mathbf{a}, \alpha, \beta)) \subset \mathcal{B}'' \quad (b > 1/2), \tag{14}$$

$$B^*(\mathcal{H}^n(\mathbf{a}, \alpha, \beta)) = \mathcal{B}'' \quad (b \le 1/2). \tag{15}$$

Proof. Let $F \in \mathcal{H}^n(\mathbf{a}, \alpha, \beta)$ and $z = a + re^{i\theta} \in \mathcal{D}$. The functions

$$g_{j,s}(z) = e^{-is} f_j(e^{is} z) \quad (z \in \mathcal{D};\ j = 1, \dots, n,\ s \in \mathbb{R})$$

belong to $\mathcal{CS}^*(\alpha_j, \beta_j)$ together with the functions f_j. Thus, by (2), the functions

$$G_s(z) = e^{-is} F(e^{is} z) \quad (z \in \mathcal{D};\ s \in \mathbb{R}),$$

belong to $\mathcal{H}^n(\mathbf{a},\alpha,\beta)$ together with F. Consequently,

(16)

$$(a,r)\in B^*(\mathcal{H}^n(\mathbf{a},\alpha,\beta)) \;\Leftrightarrow\; (|a|,r)\in B^*(\mathcal{H}^n(\mathbf{a},\alpha,\beta)) \quad (a\in\mathcal{D},\ r\geq 0).$$

Therefore, without loss of generality we may assume that a is a nonnegative real number. Since $f_j\in\mathcal{CS}^*(\alpha_j,\beta_j)$, there exist $g_j\in\mathcal{S}^*(\alpha_j)$ and $h_j\in\mathcal{P}(\beta_j)$ such that

$$\frac{f_j(z)}{g_j(z)}=h_j(z) \quad (z\in\mathcal{D}),$$

or equivalently

$$f_j(z)=g_j(z)h_j(z) \quad (z\in\mathcal{D}). \tag{17}$$

From (2) we obtain

$$\frac{F'(z)}{F(z)}=\frac{1}{z}+\sum_{j=1}^{n}a_j\left(\frac{f_j'(z)}{f_j(z)}-\frac{1}{z}\right) \quad (z\in\mathcal{D}_0).$$

Thus, using (17) we have

$$\begin{aligned}\operatorname{Re}\frac{e^{i\theta}F'(z)}{F(z)} &= \operatorname{Re}\frac{e^{i\theta}}{z}+\sum_{j=1}^{n}\operatorname{Re}\left(a_je^{i\theta}\left(\frac{g_j'(z)}{g_j(z)}-\frac{1}{z}\right)\right)\\ &\quad+\sum_{j=1}^{n}\operatorname{Re}\left(a_je^{i\theta}\frac{h'_j(z)}{h_j(z)}\right) \quad (z\in\mathcal{D}_0).\end{aligned}$$

By Lemmas 2 and 3 we obtain

$$\begin{aligned}\operatorname{Re}\frac{e^{i\theta}F'(z)}{F(z)} \geq \operatorname{Re}\frac{e^{i\theta}}{z}+\frac{2|z|^2}{1-|z|^2}\sum_{j=1}^{n}(1-\alpha_j)a_j\operatorname{Re}\frac{e^{i\theta}}{z}\\ -\frac{2}{1-|z|^2}\sum_{j=1}^{n}(1-\alpha_j)|a_j|-\frac{2}{1-|z|^2}\sum_{j=1}^{n}\beta_j|a_j| \quad (z\in\mathcal{D}_0).\end{aligned}$$

Using (3) and setting $z=a+re^{i\theta}$ in the above inequality yields

$$\begin{aligned}\operatorname{Re}\frac{e^{i\theta}F'(a+re^{i\theta})}{F(a+re^{i\theta})} \geq \operatorname{Re}\frac{e^{i\theta}}{a+re^{i\theta}}\\ +\frac{2}{1-|a+re^{i\theta}|^2}\left(\operatorname{Re}\frac{be^{i\theta}|a+re^{i\theta}|^2}{a+re^{i\theta}}-m-c\right).\end{aligned}$$

We have to require that the right-hand side above be nonnegative, that is,

$$\operatorname{Re}\frac{1}{r+ae^{-i\theta}}+\frac{2}{1-|r+ae^{-i\theta}|^2}\left(\operatorname{Re}\frac{b|r+ae^{-i\theta}|^2}{r+ae^{-i\theta}}-m-c\right)\geq 0. \tag{18}$$

If we put

$$r+ae^{-i\theta}=x+yi,$$

then we obtain

$$\frac{x}{x^2+y^2}+2\frac{bx-m-c}{1-x^2-y^2}\geq 0.$$

Thus, using the equality

$$(x-r)^2+y^2=a^2 \tag{19}$$

we obtain

$$\begin{aligned} w(x) = 2r(2b-1)x^2 - \big((2b-1)(r^2-a^2)+4r(m+c)-1\big)x \\ + 2(m+c)(r^2-a^2) \geq 0. \end{aligned} \tag{20}$$

The discriminant Δ of $w(x)$ is given by

$$\begin{aligned} \Delta = \big((2b-1)(r^2-a^2)+4r(m+c)-1\big)^2 \\ - 16r(2b-1)(m+c)(r^2-a^2) =: A_1A_2, \end{aligned} \tag{21}$$

where

$$A_1 = \left[1+2\sqrt{r(m+c)}\right]^2 + (1-2b)(r^2-a^2), \tag{22}$$

$$A_2 = \left[1-2\sqrt{r(m+c)}\right]^2 + (1-2b)(r^2-a^2). \tag{23}$$

Let

$$D = \{(a,r)\in\mathbb{R}^2 : 0\leq a<r\leq 1-a\}. \tag{24}$$

First, we discuss the case $b>1/2$. If we put

$$x_0 = \frac{(2b-1)(r^2-a^2)+4r(m+c)-1}{4r(2b-1)}, \tag{25}$$

then the inequality (20) is satisfied for every $x\in[r-a,r+a]$ if one of the following conditions is fulfilled:

1° $\Delta\leq 0$,
2° $\Delta>0, w(r-a)\geq 0$ and $x_0\leq r-a$,
3° $\Delta>0, w(r+a)\geq 0$ and $x_0\geq r+a$.

Case 1°. Since $A_1>0$, by (21) the condition $\Delta\leq 0$ is equivalent to $A_2\leq 0$. Then

$$\mathcal{B}_1 := \{(a,r)\in D : \Delta\leq 0\} = \{(a,r)\in D : A_2\leq 0\} = \{(a,r)\in D : a\leq\varphi(r)\},$$

where φ is defined by (12). Let

$$\gamma = \{(a,r)\in\overline{D} : a=\varphi(r)\}.$$

Then γ is a curve which is tangent to the straight lines $a=r$ and $a=q-r$ at the points

$$S_1=(r_1,r_1) \quad\text{and}\quad S_2=(q-r_2,r_2), \tag{26}$$

where r_1, r_2, q are defined by (9), (10), (11), respectively. Moreover γ cuts

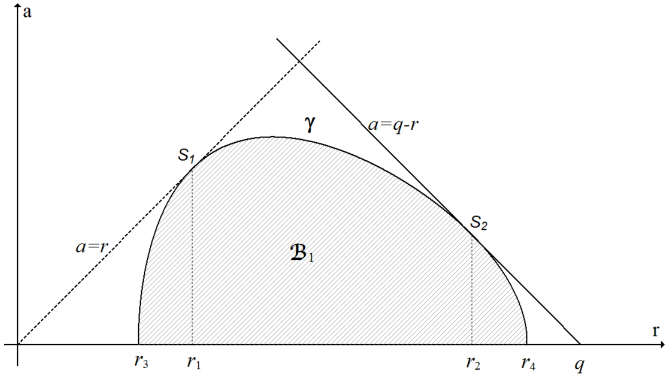

Fig. 1

the straight line $a = 0$ at the points

$$r_3 = \left(\sqrt{m+c+\sqrt{2b-1}} + \sqrt{m+c}\right)^{-2},$$
$$r_4 = \left(\sqrt{m+c-\sqrt{2b-1}} + \sqrt{m+c}\right)^{-2}.$$

Since

$$0 < r_3 < r_1 < r_2 < r_4 < q,$$

we have

$$\gamma = \{(a,r) \in \mathbb{R}^2 : r_3 \le r \le r_4,\ a = \varphi(r)\},$$

and consequently

(27) $$\mathcal{B}_1 = \{(a,r) \in \mathbb{R}^2 : r_3 \le r \le r_4,\ 0 \le a \le \varphi(r)\},$$

where φ is defined by (12) (see Fig. 1).

Case 2°. Let

$$\mathcal{B}_2 := \{(a,r) \in D : \Delta > 0 \wedge w(r-a) \ge 0 \wedge x_0 \le r-a\}.$$

It is easy to verify that

$$\begin{aligned} w(r-a) &= (r-a)\big((2b-1)(r-a)^2 - 2(m+c)(r-a) + 1\big) \\ &= (2b-1)(r-a)(r-a-q')(r-a-q), \end{aligned}$$

where q is defined by (11) and

(28) $$q' = \left(m+c-\sqrt{(m+c)^2-2b+1}\right)^{-1}.$$

Since

(29) $$0 < q < 1 < q' \quad (1/2 < b \le m,\ (a,r) \in D),$$

we see that

$$(r-a)(r-a-q') < 0 \quad ((a,r) \in D).$$

Thus, $w(r-a) \geq 0$ if $a \geq r-q$. The inequality $x_0 \leq r-a$ may be written in the form

$$(2b-1)a^2 + 3(2b-1)r^2 - 4(m+c)r - 4(2b-1)ar + 1 \geq 0. \tag{30}$$

The hyperbola h_1 which is the boundary of the set of all pairs $(a,r) \in \mathbb{R}^2$ satisfying (30) cuts the boundary of D at the point S_1 defined by (26) and at $(r_5, 0)$, where

$$r_5 = \left(2(m+c) + \sqrt{4(m+c)^2 - 3(2b-1)}\right)^{-1}. \tag{31}$$

It is easy to verify that

$$r_3 < r_5 < r_4 < q.$$

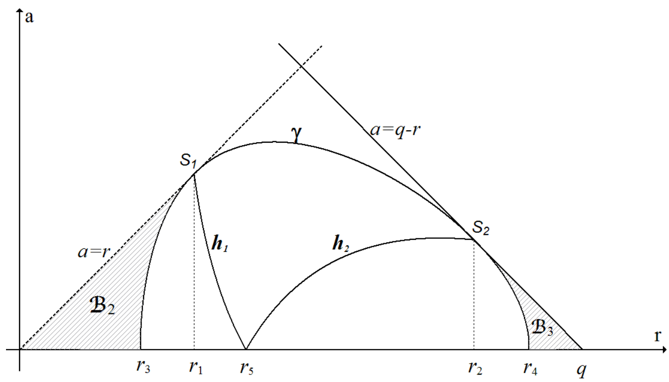

Fig. 2

Thus

$$\mathcal{B}_2 = \left\{ (a,r) \in \mathbb{R}^2 : \begin{array}{l} (0 \leq r \leq r_3 \wedge 0 \leq a < r) \vee \\ (r_3 < r < r_1 \wedge \varphi(r) < a < r) \end{array} \right\}, \tag{32}$$

where φ is defined by (12) (see Fig. 2).

Case 3°. Let

$$\mathcal{B}_3 := \{(a,r) \in D : \varDelta > 0 \wedge w(r+a) \geq 0 \wedge x_0 \geq r+a\}$$

and let q and q' be defined by (11) and (28), respectively. Then

$$\begin{aligned} w(r+a) &= (r+a)[(2b-1)(r+a)^2 - 2(m+c)(r+a) + 1] \\ &= (2b-1)(r+a)(r+a-q')(r+a-q). \end{aligned}$$

Moreover, by (29) we have

$$(r+a)(r+a-q') < 0 \qquad ((a,r) \in D).$$

Thus, we conclude that $w(r+a) \geq 0$ if $a \leq q-r$. The inequality $x_0 \geq r+a$ may be written in the form

$$(2b-1)a^2 + 3(2b-1)r^2 - 4(m+c)r + 4(2b-1)ar + 1 \leq 0. \tag{33}$$

The hyperbola h_2 which is the boundary of the set of all pairs $(a,r) \in \mathbb{R}^2$ satisfying (33) cuts the boundary of D at the point S_2 defined by (26) and at $(r_5, 0)$, where r_5 is defined by (31). Thus,

$$\mathcal{B}_3 = \left\{ (a,r) \in \mathbb{R}^2 : \begin{array}{l} (r_2 < r < r_4 \wedge \varphi(r) < a \leq q-r) \vee \\ (r_4 < r < q \wedge 0 \leq a \leq q-r) \end{array} \right\}, \tag{34}$$

where φ is defined by (12) (see Fig. 2). The union of the sets $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ defined by (27), (32), (34) gives the set

$$\widetilde{\mathcal{B}}' = \left\{ (a,r) \in \mathbb{R}^2 : \begin{array}{l} (0 \leq r \leq r_1 \wedge 0 \leq a < r) \vee \\ (r_1 < r < r_2 \wedge 0 \leq a \leq \varphi(r)) \vee \\ (r_2 \leq r < q \wedge 0 \leq a \leq q-r) \end{array} \right\}.$$

Thus, by (16) we have

$$\mathcal{B}' \subset B^*(\mathcal{H}^n(\mathbf{a}, \alpha, \beta)) \quad (1/2 < b), \tag{35}$$

where $\mathcal{B}'$ is defined by (7).

Now, let $b < 1/2$. Then (20) is satisfied for every $x \in [r-a, r+a]$ if

$$w(r-a) \geq 0 \quad \text{and} \quad w(r+a) \geq 0. \tag{36}$$

We see that

$$\begin{aligned} w(r+a) &= (2b-1)(r+a)(r+a-q')(r+a-q), \\ w(r-a) &= (2b-1)(r-a)(r-a-q')(r-a-q), \end{aligned}$$

where q and q' are defined by (11) and (28), respectively. Since

$$q' < 0 < q < 1 \quad (b < 1/2),$$

the condition (36) is satisfied if $(a,r) \in D$ and

$$a \leq q - r. \tag{37}$$

Let $b = 1/2$. Then, by (20) we obtain

$$(1 - 4r(m+c))x + 2(m+c)(r^2 - a^2) \geq 0.$$

The above inequality holds for every $x \in [r-a, r+a]$ if $(a,r) \in D$ and

$$r + a \leq \frac{1}{2(m+c)},$$

or equivalently (37) holds. Thus, by (16) we have

$$\mathcal{B}'' \subset B^*(\mathcal{H}^n(\mathbf{a}, \alpha, \beta)) \quad (b \leq 1/2), \tag{38}$$

where $\mathcal{B}''$ is defined by (8). Because the function

$$(39)\quad F(z) = z\prod_{j=1}^{n}\left(\frac{1}{(1+\operatorname{sgn}(a_j)z)^{2(1-\alpha_j)}}\left(\frac{1-z}{1+z}\right)^{\beta_j \operatorname{sgn}(a_j)}\right)^{a_j} \qquad (z\in\mathcal{D})$$

belongs to $\mathcal{H}^n(\mathbf{a},\alpha,\beta)$, and for $z = a + r$, $\theta = 0$, $q < a + r < 1$ we have

$$\operatorname{Re}\frac{e^{i\theta}F'(z)}{F(z)} = \frac{1-2(m+c)(a+r)+(2b-1)(a+r)^2}{(a+r)(1-(a+r)^2)} < 0,$$

Lemma 1 yields

$$(40)\qquad B^*(\mathcal{H}^n(\mathbf{a},\alpha,\beta)) \subset \mathcal{B}''.$$

From (35) and (40) we have (14), while (38) and (40) give (15). ∎

Since the set $\mathcal{B}$ depends only on m, b, c, the following result is an immediate consequence of Theorem 1.

Theorem 2. *Let* $\mathcal{B}$ *be defined by* (13). *If* $(a, r) \in \mathcal{B}$, *then every* $F \in \mathcal{G}^n(m, b, c)$ *maps the disk* $\mathcal{D}(a, r)$ *onto a domain starlike with respect to the origin. The result is sharp for* $b \le 1/2$, *and for* $b > 1/2$ *the set* $\mathcal{B}$ *cannot be larger than* $\mathcal{B}''$, *where* $\mathcal{B}''$ *is defined by* (7). *This means that*

$$B^*(\mathcal{G}^n(m,b,c)) \subset \mathcal{B}'' \qquad (b > 1/2),$$
$$B^*(\mathcal{G}^n(m,b,c)) = \mathcal{B} \qquad (b \le 1/2).$$

The functions described by (39) *with* (3) *are extremal functions.*

Theorem 3.

$$(41)\qquad B^*(\mathcal{G}^n(M,N)) = \{(a,r) \in \mathbb{C}\times\mathbb{R} : |a| < r \le q - |a|\},$$

where

$$q = \frac{1}{M+N+\sqrt{(M+N)^2+2M+1}}.$$

Equality is realized by the function F *of the form*

$$(42)\qquad F(z) = z\frac{(1-z)^{2M+N}}{(1+z)^N} \qquad (z\in\mathcal{D}).$$

Proof. Let M, N be positive real numbers and let $\mathcal{B}' = \mathcal{B}'(m, b, c)$, $\mathcal{B}'' = \mathcal{B}''(m, b, c)$, $q = q(m, b, c)$ and $\varphi(r) = \varphi(r; m, b, c)$ be defined by (7), (8), (11) and (12), respectively. It is easy to verify that

$$\varphi(r; m, b, c) \ge q(m, 1/2, c) - r$$

whenever

$$1/(2q(m,1/2,c)) \le r \le q(m,1/2,c), \qquad 1/2 < b \le m.$$

Moreover, the function $q = q(m, b, c)$ is decreasing with respect to m and c, and increasing with respect to b. Thus, from Theorems 1 and 2 we have (see Fig. 3)

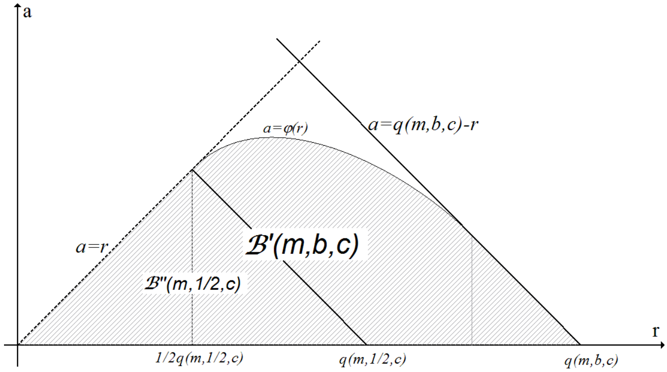

Fig. 3

$$B^*(\mathcal{G}^n(m,1/2,c)) = \mathcal{B}''(m,1/2,c) \subset \mathcal{B}'(m,b,c) \subset B^*(\mathcal{G}^n(m,b,c)) \qquad (m \in [0,M],\ c \in [0,N],\ b \in (1/2,m])$$

and

$$B^*(\mathcal{G}^n(M,-M,N)) \subset B^*(\mathcal{G}^n(m,b,c)) \subset B^*(\mathcal{G}^n(m,1/2,c)) \qquad (m \in [0,M],\ c \in [0,N],\ b \in [-m,1/2]).$$

Therefore, by (4) we obtain

$$B^*(\mathcal{G}^n(M,N)) = B^*(\mathcal{G}^n(M,-M,N)) \tag{43}$$

and by Theorem 2 we get (41). Putting $m = M, b = -M$ in (3) we see that $a_1, \ldots, a_n$ are negative real numbers. Thus, the extremal function (39) has the form

$$F(z) = z\prod_{j=1}^{n}\left(\frac{1}{(1-z)^{2(1-\alpha_j)}}\left(\frac{1+z}{1-z}\right)^{\beta_j}\right)^{a_j} \qquad (z \in \mathcal{D})$$

or equivalently

$$F(z) = \frac{z}{(1-z)^{-2\sum_{j=1}^{n}(1-\alpha_j)|a_j|}}\left(\frac{1+z}{1-z}\right)^{-\sum_{j=1}^{n}\beta_j|a_j|} \qquad (z \in \mathcal{D}).$$

Consequently, using (3) we obtain

$$F(z) = z\frac{1}{(1-z)^{-2M}}\left(\frac{1+z}{1-z}\right)^{-N} \qquad (z \in \mathcal{D}),$$

which is the function (42), and the proof is complete. ■

Since $\mathcal{H}^1((1),(\alpha),(\beta)) = \mathcal{CS}^*(\alpha,\beta)$, by Theorem 1 we obtain the following theorem.

THEOREM 4. *Let* $0 \le \alpha < 1$, $0 < \beta \le 1$ *and let*

$$\mathcal{B}' = \left\{ (a,r) \in \mathbb{C} \times \mathbb{R} : \begin{array}{c} (0 \le r \le r_1 \wedge |a| < r) \vee \\ (r_1 < r < r_2 \wedge |a| \le \varphi(r)) \vee \\ (r_2 \le r < q \wedge |a| \le q - r) \end{array} \right\},$$

$$\mathcal{B}'' = \{(a,r) \in \mathbb{C} \times \mathbb{R} : |a| < r \le q - |a|\},$$

where

$$r_1 = \frac{1}{4(\beta - \alpha + 1)},$$

$$r_2 = \frac{\beta - \alpha + 1}{(\beta - \alpha + 1 + \sqrt{(\beta - \alpha)^2 + 2\beta})^2},$$

$$q = \frac{1}{\beta - \alpha + 1 + \sqrt{(\beta - \alpha)^2 + 2\beta}},$$

$$\varphi(r) = \sqrt{r^2 - \frac{(1 - 2\sqrt{r(\beta - \alpha + 1)})^2}{1 - 2\alpha}}.$$

Moreover, set

$$\mathcal{B} = \begin{cases} \mathcal{B}' & \text{for } \alpha < 1/2, \\ \mathcal{B}'' & \text{for } \alpha \ge 1/2. \end{cases}$$

If $(a,r) \in \mathcal{B}$, *then every function* $f \in \mathcal{CS}^*(\alpha,\beta)$ *maps the disk* $\mathcal{D}(a,r)$ *onto a domain starlike with respect to the origin. The result is sharp for* $\alpha \ge 1/2$, *and for* $\alpha < 1/2$ *the set* $\mathcal{B}$ *cannot be larger than* $\mathcal{B}''$. *This means that*

$$\mathcal{B}' \subset B^*(\mathcal{CS}^*(\alpha,\beta)) \subset \mathcal{B}'' \quad (\alpha < 1/2),$$
$$B^*(\mathcal{CS}^*(\alpha,\beta)) = \mathcal{B} \quad (\alpha \ge 1/2).$$

The function

$$f(z) = z\frac{(1+z)^\beta}{(1-z)^{2-2\alpha+\beta}} \quad (z \in \mathcal{D})$$

is an extremal function.

Using (5) and Theorems 1–4, we obtain the radii of starlikeness of the classes $\mathcal{H}^n(\mathbf{a},\alpha,\beta)$, $\mathcal{G}^n(m,b,c)$, $\mathcal{G}^n(M,N)$ and $\mathcal{CS}^*(\alpha,\beta)$.

COROLLARY 1. *We have*

$$R^*(\mathcal{H}^n(\mathbf{a},\alpha,\beta)) = \frac{1}{m + c + \sqrt{(m+c)^2 - 2b + 1}},$$

where m, c *are defined by* (3), *and*

$$R^*(\mathcal{G}^n(m,b,c)) = \frac{1}{m+c+\sqrt{(m+c)^2-2b+1}},$$
$$R^*(\mathcal{G}^n(M,N)) = \frac{1}{M+N+\sqrt{(M+N)^2+2M+1}},$$
$$R^*(\mathcal{CS}^*(\alpha,\beta)) = \frac{1}{\beta-\alpha+1+\sqrt{(\beta-\alpha)^2+2\beta}}.$$

REMARK. Putting $\beta = 1$ in Corollary 1 we get the radius of starlikeness of the class $\mathcal{CS}^*(\alpha) = \mathcal{CS}^*(\alpha, 1)$ obtained by Ratti [7]. Moreover, putting $\alpha = 0$ we get the radius of starlikeness of the class $\mathcal{CS}^* = \mathcal{CS}^*(0,1)$ obtained by MacGregor [5].

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