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On existence theorems for semilinear equations and applications

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Abstract. Existence results for semilinear operator equations without the assumption of normal cones are obtained by the properties of a fixed point index for A-proper semilinear operators established by Cremins. As an application, the existence of positive solutions for a second order m-point boundary value problem at resonance is considered.

1. Introduction and preliminaries. Coincidence degree theory appears to be a convenient framework for studying various types of equations of the form $Lx = Nx$ when L^{-1} does not exist (see [14]), in the same way as Leray–Schauder's degree is extremely useful for considering cases where L is invertible (see [11]). The concept of fixed point index for A-proper maps of the form $L - N$ in cones, with L a Fredholm operator of index zero and N some nonlinear operator, has been introduced in [3]. In [4], Cremins established the existence of positive solutions to semilinear operator equations defined on a quasinormal or normal cone in a Banach space. The purpose of this paper is to obtain existence results for semilinear operator equations without exploiting the notion of normal cones.

We first review some of the standard facts on A-proper mappings and Fredholm operators. Let X and Y be Banach spaces, D a linear subspace of X, $\{X_n\} \subset D$, and $\{Y_n\} \subset Y$ sequences of oriented finite-dimensional subspaces such that $Q_n y \to y$ in Y for every y and $\operatorname{dist}(x, X_n) \to 0$ for every $x \in D$ where $Q_n : Y \to Y_n$ and $P_n : X \to X_n$ are sequences of continuous linear projections. The projection scheme $\Gamma = \{X_n, Y_n, P_n, Q_n\}$ is then said to be *admissible* for maps from $D \subset X$ to Y.

A map $T : D \subset X \to Y$ is called *approximation-proper* (abbreviated *A-proper*) *at a point* $y \in Y$ with respect to the admissible scheme Γ if $T_n \equiv Q_n T|_{D\cap X_n}$ is continuous for each $n \in \mathbb{N}$ and whenever $\{x_{n_j} : x_{n_j} \in D \cap X_{n_j}\}$

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is bounded with $T_{n_j}x_{n_j} \to y$, then there exists a subsequence $\{x_{n_{j_k}}\}$ such that $x_{n_{j_k}} \to x \in D$ and $Tx = y$. T is simply called *A-proper* if it is A-proper at all points of Y.

$L : \operatorname{dom} L \subset X \to Y$ is a *Fredholm operator of index zero* if $\operatorname{Im} L$ is closed and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$. Then X and Y may be expressed as direct sums $X = X_0 \oplus X_1$, $Y = Y_0 \oplus Y_1$ with continuous linear projections $P : X \to \operatorname{Ker} L = X_0$ and $Q : Y \to Y_0$. The restriction of L to $\operatorname{dom} L \cap X_1$, denoted L_1, is a bijection onto $\operatorname{Im} L = Y_1$ with continuous inverse $L_1^{-1} : Y_1 \to \operatorname{dom} L \cap X_1$. Since X_0 and Y_0 have the same finite dimension, there exists a continuous bijection $J : Y_0 \to X_0$. If we let $H = L + J^{-1}P$, then $H : \operatorname{dom} L \subset X \to Y$ is a linear bijection with bounded inverse.

Cremins [3] defined a fixed point index $\operatorname{ind}_K([L, N], \Omega)$ for A-proper maps of the form $L - N$ acting on cones, which has the usual properties of the classical fixed point index, that is, existence, normalization, additivity and homotopy invariance. In this paper, we focus on some applications of this theory. Let K be a cone in the Banach space X, and $\Omega \subset X$ open and bounded such that $\Omega_K = \Omega \cap K \neq \emptyset$. We set $K_1 = H(K \cap \operatorname{dom} L)$. We make the following assumptions:

(A1) $L : \operatorname{dom} L \to Y$ is Fredholm of index zero.
(A2) $L - \lambda N$ is A-proper for $\lambda \in [0, 1]$.
(A3) N is bounded and $P + JQN + L_1^{-1}(I - Q)N$ maps K to K.

The following two lemmas will be used in this paper.

Lemma 1.1 ([3]). *Under assumptions* (A1)–(A3), *if moreover $\theta \in \Omega \subset X$ and $Lx \neq \mu Nx - (1 - \mu)J^{-1}Px$ on $\partial\Omega_K$ for $\mu \in [0, 1]$, then*

$$\operatorname{ind}_K([L, N], \Omega) = \{1\}.$$

Lemma 1.2 ([3], [17]). *Under assumptions* (A1)–(A3), *if moreover there exists $e \in K_1 \setminus \{\theta\}$ such that*

$$Lx - Nx \neq \mu e$$

for every $x \in \partial\Omega_K$ and all $\mu \geq 0$, then $\operatorname{ind}_K([L, N], \Omega) = \{0\}$.

2. Main results. In this section we will give the following existence theorems for semilinear equations, which, to the best of our knowledge, are new.

Theorem 2.1. *Under assumptions* (A1)–(A3), *if moreover $\theta \in \Omega \subset X$, and*

$$Nx \not\geq Lx \text{ for any } x \in \partial\Omega_K, \tag{2.1}$$

where the partial order is induced by the cone K_1 in Y, then

$$\operatorname{ind}_K([L, N], \Omega) = \{1\}.$$

Proof. We show that

$$(2.2)\qquad Lx \neq \mu Nx - (1-\mu)J^{-1}Px \quad \text{for any } x \in \partial\Omega_K,\ \mu \in [0,1].$$

Indeed, if there exist $x_1 \in \partial\Omega_K$ and $\mu_1 \in [0,1]$ such that $Lx_1 = \mu_1 Nx_1 - (1-\mu_1)J^{-1}Px_1$, then $(L+J^{-1}P)x_1 = \mu_1(N+J^{-1}P)x_1 \le (N+J^{-1}P)x_1$. So

$$Nx_1 \ge Lx_1,$$

which contradicts (2.1). Hence (2.2) is true, and so the proof is finished by Lemma 1.1. ∎

Theorem 2.2. *Under assumptions* (A1)–(A3), *if moreover*

$$(2.3)\qquad Nx \not\le Lx \quad \textit{for any } x \in \partial\Omega_K,$$

where the partial order is induced by the cone K_1 in Y, then

$$\mathrm{ind}_K([L,N],\Omega) = \{0\}.$$

Proof. We show that

$$(2.4)\qquad Lx - Nx \neq \mu e \quad \text{for any } x \in \partial\Omega_K,\ \mu \ge 0.$$

Indeed, if there exist $x_2 \in \partial\Omega_K$ and $\mu_2 \ge 0$ such that $Lx_2 - Nx_2 = \mu_2 e$, then we obtain $Lx_2 = Nx_2 + \mu_2 e \ge Nx_2$. So $Nx_2 \le Lx_2$, which contradicts (2.3). Hence (2.4) is true, and so the proof is finished by Lemma 1.2. ∎

Theorem 2.3. *Under assumptions* (A1)–(A3), *if moreover $\theta \in \Omega \subset X$, $Lx \neq Nx$ on $\partial\Omega_K$ and*

$$(2.5)\qquad \|Nx + J^{-1}Px\| \le \|Lx + J^{-1}Px\| \quad \textit{for any } x \in \partial\Omega_K,$$

then

$$\mathrm{ind}_K([L,N],\Omega) = \{1\}.$$

Proof. We show that

$$(2.6)\qquad Lx \neq \mu Nx - (1-\mu)J^{-1}Px \quad \text{for any } x \in \partial\Omega_K,\ \mu \in [0,1].$$

Indeed, if there exist $x_3 \in \partial\Omega_K$ and $\mu_3 \in [0,1]$ such that $Lx_3 = \mu_3 Nx_3 - (1-\mu_3)J^{-1}Px_3$, then $\mu_3 \in (0,1)$ and $(L+J^{-1}P)x_3 = \mu_3(N+J^{-1}P)x_3$. So

$$\|(N+J^{-1}P)x_3\| = \frac{1}{\mu_3}\|(L+J^{-1}P)x_3\| > \|(L+J^{-1}P)x_3\|,$$

which contradicts (2.5). Hence (2.6) is true, and so the proof is finished by Lemma 1.1. ∎

Theorem 2.4. *Under assumptions* (A1)–(A3), *suppose $Lx \neq Nx$ on $\partial\Omega_K$. Suppose that there exists $e \in K_1 \setminus \{\theta\}$ such that*

$$\|y + \mu e\| > \|y\| \quad \textit{for any } \mu > 0,\ y \in K_1.$$

If moreover

$$(2.7)\qquad \|Nx + J^{-1}Px\| \ge \|Lx + J^{-1}Px\| \quad \textit{for any } x \in \partial\Omega_K,$$

then

$$\mathrm{ind}_K([L,N],\Omega) = \{0\}.$$

Proof. We show that

$$Lx - Nx \neq \mu e \quad \text{for any } x \in \partial\Omega_K,\ \mu \geq 0. \tag{2.8}$$

Indeed, if there exist $x_4 \in \partial\Omega_K$ and $\mu_4 \geq 0$ such that $Lx_4 - Nx_4 = \mu_4 e$, then $\mu_4 > 0$ and

$$\|Lx_4 + J^{-1}Px_4\| = \|Nx_4 + J^{-1}Px_4 + \mu_4 e\| > \|Nx_4 + J^{-1}Px_4\|,$$

which contradicts (2.7). Hence (2.8) is true, and so the proof is finished by Lemma 1.2. ∎

Theorem 2.5. *Assume* (A1)–(A3) *hold. Suppose Ω_1 and Ω_2 are bounded open sets in X such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, $\Omega_2 \cap K \cap \mathrm{dom}\,L \neq \emptyset$. If one of the following two conditions is satisfied:*

(C_1) *$Nx \ngeq Lx$ for all $x \in \partial\Omega_1 \cap K$ and $Nx \nleq Lx$ for all $x \in \partial\Omega_2 \cap K$,*
(C_2) *$Nx \nleq Lx$ for all $x \in \partial\Omega_1 \cap K$ and $Nx \ngeq Lx$ for all $x \in \partial\Omega_2 \cap K$,*

then there exists $x \in (\overline{\Omega}_2 \setminus \Omega_1) \cap K$ such that $Lx = Nx$.

Proof. This follows from Theorem 2.1 and 2.2. ∎

Theorem 2.6. *Assume* (A1)–(A3) *hold. Suppose Ω_1 and Ω_2 are bounded open sets in X such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, $\Omega_2 \cap K \cap \mathrm{dom}\,L \neq \emptyset$. Moreover, suppose that there exists $e \in K_1 \setminus \{\theta\}$ such that*

$$\|y + \mu e\| > \|y\| \quad \text{for any } \mu > 0,\ y \in K_1.$$

If one of the following two conditions is satisfied:

(C_3) *$\|Nx + J^{-1}Px\| \leq \|Lx + J^{-1}Px\|$ for all $x \in \partial\Omega_1 \cap K$ and $\|Nx + J^{-1}Px\| \geq \|Lx + J^{-1}Px\|$ for all $x \in \partial\Omega_2 \cap K$,*
(C_4) *$\|Nx + J^{-1}Px\| \geq \|Lx + J^{-1}Px\|$ for all $x \in \partial\Omega_1 \cap K$ and $\|Nx + J^{-1}Px\| \leq \|Lx + J^{-1}Px\|$ for all $x \in \partial\Omega_2 \cap K$,*

then there exists $x \in (\overline{\Omega}_2 \setminus \Omega_1) \cap K$ such that $Lx = Nx$.

Proof. This follows from Theorems 2.3 and 2.4. ∎

Remark 2.1. When using Theorems 2.4 and 2.6, we find that the condition that $\|y + \mu e\| > \|y\|$ for any $\mu > 0$ and $y \in K_1$ is easily satisfied. For example, let Y be the space of continuous functions, and K_1 be the cone of positive functions. It is worth mentioning that the condition $\|y + \mu e\| > \|y\|$ for any $\mu > 0$, $y \in K_1$ is slightly stronger than the quasinormality condition by Remark 2 of [4].

Remark 2.2. In Theorems 2.1−2.3, 2.5, we do not use the assumption that the cones are normal.

3. Positive solutions to an m-point boundary value problem at resonance. The goal of this section is to apply Theorem 2.5 to discuss the existence of positive solutions for the following m-point boundary value problem at resonance:

$$-x''(t) = f(t, x(t), x'(t), x''(t)), \qquad t \in (0,1), \tag{3.1}$$

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\eta_i), \tag{3.2}$$

where $m \geq 3$ is an integer, $a_i \geq 0$, $\eta_i \in (0,1)$ $(i = 1, \ldots, m-2)$ are constants satisfying $\sum_{i=1}^{m-2} a_i = 1$, $0 < \eta_1 < \cdots < \eta_{m-2} < 1$, in which the highest order derivative may appear nonlinearly.

The study of multi-point boundary value problems for linear second order differential equations was initiated by Bitsadze and Samarskiĭ [2] and continued by Il'in and Moĭseev [8]. Since then, nonlinear multi-point boundary value problems have been studied by many authors: for example, see [5], [6], [12], [13], [16], [20] and the references therein. However, as far positive solutions are concerned, most of the results pertain to non-resonance problems; to the best of our knowledge, only few papers deal with the existence of positive solutions of multi-point boundary value problems at resonance: see [1], [3], [7], [10], [15], [16]–[19].

Recently, for (3.1), when the nonlinear term f does not depend on the derivative, Infante and Zima [10] proved the existence of positive solutions of multi-point boundary value problems at resonance via the Leggett–Williams norm-type theorem. In [9], Infante also studied the existence of positive solutions of (3.1) under the boundary value conditions

$$x(0) = 0, \qquad \alpha x(\eta) = x(1), \qquad 0 < \eta < 1, \qquad \alpha\eta < 1,$$

by means of the theory of fixed point index for weakly inward A-proper maps.

Let

$$X = C^2[0,1] \cap \Big\{x : x'(0) = 0,\, x(1) = \sum_{i=1}^{m-2} a_i x(\eta_i)\Big\}, \qquad Y = C[0,1].$$

For every $x \in X$, denote its norm by

$$\|x\|_X = \max\{\sup_{t\in[0,1]} |x(t)|, \sup_{t\in[0,1]} |x'(t)|, \sup_{t\in[0,1]} |x''(t)|\},$$

and for every $y \in Y$, denote its norm by $\|y\|_Y = \sup_{t\in[0,1]} |y(t)|$. We can prove that X and Y are Banach spaces. Let $K = \{x \in X : x(t) \geq 0,\ t \in [0,1]\}$; then K is a cone of X.

For notational convenience, we set

$$l_i(s) := \begin{cases} 1-\eta_i, & 0 \le s \le \eta_i, \\ 1-s, & \eta_i < s \le 1, \end{cases}$$

for $i = 1, \dots, m-2$, and

$$G(t,s) := \begin{cases} \dfrac{(1-s)^2}{2} + \dfrac{5+3t^2}{3\sum_{i=1}^{m-2} a_i(1-\eta_i^2)} \displaystyle\sum_{i=1}^{m-2} a_i l_i(s), & 0 \le t \le s \le 1, \\ \dfrac{(1-s)^2}{2} + s - t + \dfrac{5+3t^2}{3\sum_{i=1}^{m-2} a_i(1-\eta_i^2)} \displaystyle\sum_{i=1}^{m-2} a_i l_i(s), & 0 \le s \le t \le 1. \end{cases}$$

Note that $G(t,s) \ge 0$ for all $t, s \in [0,1]$, and

$$1 - \frac{2\mathcal{K}}{\sum_{i=1}^{m-2} a_i(1-\eta_i^2)} \sum_{i=1}^{m-2} a_i l_i(s) \ge 0, \qquad s \in [0,1],$$

for every $\mathcal{K} \in (0, (1+\eta_1)/2]$. We also set

$$\mathcal{K} := \min\left\{ \frac{1+\eta_1}{2}, \frac{1}{\max_{t,s\in[0,1]} G(t,s)} \right\};$$

obviously $\mathcal{K} < 1$.

We define

$$\operatorname{dom} L = X, \qquad L : \operatorname{dom} L \to Y, \qquad Lx(t) = -x''(t),$$
$$N : X \to Y, \qquad Nx(t) = f(t, x(t), x'(t), x''(t)).$$

Then BVP (3.1), (3.2) can be written

$$Lx = Nx, \qquad x \in K.$$

It is easy to check that

$$\operatorname{Ker} L = \{x \in \operatorname{dom} L : x(t) \equiv c \text{ on } [0,1],\ c \in \mathbb{R}\},$$
$$\operatorname{Im} L = \Big\{ y \in Y : \sum_{i=1}^{m-2} a_i \int_0^1 l_i(s) y(s)\, ds = 0 \Big\},$$
$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 1,$$

so that L is a Fredholm operator of index zero, with kernel the subspace of constant functions in X, and range the subspace of functions with zero mean value, so that the corresponding operators P and Q can be chosen as

$$P : X \to \operatorname{Ker} L, \qquad Px = \int_0^1 x(s)\,ds,$$

$$Q : Y \to Y, \qquad Qy = \frac{2}{\sum_{i=1}^{m-2} a_i(1-\eta_i^2)} \cdot \sum_{i=1}^{m-2} a_i \int_0^1 l_i(s)y(s)\,ds.$$

Furthermore, we define the isomorphism $J : \operatorname{Im} Q \to \operatorname{Im} P$ as $Jy = \beta y$, where $\beta > 0$ is a constant. It is easy to verify that the inverse operator $L_1^{-1} : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ of $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is $(L_1^{-1}y)(t) = \int_0^1 k(t,s)y(s)\,ds$, where

$$k(t,s) = \begin{cases} (1-s)^2/2, & 0 \le t \le s \le 1, \\ (1-s)^2/2 + s - t, & 0 \le s \le t \le 1. \end{cases}$$

THEOREM 3.1. *Suppose*

- (H_1) *there exist* $R \in (0,\infty)$ *and* $k \in (0,1)$ *such that* $f : [0,1] \times [0,R] \times [-R,R] \times \mathbb{R}^- \to \mathbb{R}$ *is continuous and* $|f(t,p,q,-s_1) - f(t,p,q,-s_2)| \le k|s_1 - s_2|$ *for* $t \in [0,1]$, $p \in [0,R]$, $q \in [-R,R]$, *and* $s_1, s_2 \in [0,R]$,
- (H_2) $f(t,p,q,s) > -\mathcal{K}p$ *for* $(t,p,q,s) \in [0,1] \times [0,R] \times [-R,R] \times \mathbb{R}^-$,
- (H_3) $f(t,p,q,-R) < R$ *for* $t \in [0,1]$, $p \in [0,R]$, *and* $q \in [-R,R]$,
- (H_4) *there exists* $r \in (0,R)$ *such that* $f(t,p,q,-r) > r$ *for* $t \in [0,1]$, $p \in [0,r]$, *and* $q \in [-r,r]$.

Then there exists at least one positive solution $x \in K$ *to problem* (3.1), (3.2) *with* $r \le \|x\|_X \le R$.

Proof. First, we note that L is Fredholm of index zero and condition (H_1) above implies that N is k-ball contractive so that $L - \lambda N$ is A-proper for $\lambda \in [0,1]$. Now we verify the hypotheses of Theorem 2.5.

First we show $P + JQN + L_1^{-1}(I-Q)N : K \to K$. For the isomorphism $Jy = \beta y$, take $\beta = 1$. For each $x \in K$, from condition (H_2) and $\beta = 1$ it follows that

$$\begin{aligned}
&(P + JQN + L_1^{-1}(I-Q)N)(x) \\
&\quad = \int_0^1 x(s)\,ds + \frac{2}{\sum_{i=1}^{m-2} a_i(1-\eta_i^2)} \cdot \sum_{i=1}^{m-2} a_i \int_0^1 l_i(s) f(s, x(s), x'(s), x''(s))\,ds \\
&\qquad + \int_0^1 k(t,s)\bigg[f(s, x(s), x'(s), x''(s)) \\
&\qquad - \frac{2}{\sum_{i=1}^{m-2} a_i(1-\eta_i^2)} \cdot \sum_{i=1}^{m-2} a_i \int_0^1 l_i(\tau) f(\tau, x(\tau), x'(\tau), x''(\tau))\,d\tau \bigg]\,ds
\end{aligned}$$

$$= \int_0^1 x(s)ds + \int_0^1 G(t,s)f(s,x(s),x'(s),x''(s))\,ds$$
$$\geq \int_0^1 x(s)ds - \mathcal{K}\int_0^1 G(t,s)x(s)\,ds = \int_0^1 (1-\mathcal{K}G(t,s))x(s)\,ds \geq 0.$$

Next, we show

$$Nx \not\geq Lx \quad \text{for any } x \in K \cap \partial B_R, \tag{3.3}$$

where $B_R = \{x \in X : \|x\|_X \leq R\}$.

In fact, if not, there exists $x_5 \in K \cap \partial B_R$ such that $Nx_5 \geq Lx_5$ and $\|x_5\|_X = R$. Then $\|Lx_5\|_Y = \|-x_5''\|_Y = R$ and there exists $t_1 \in [0,1]$, such that $-x_5''(t_1) = R$. Thus we have $t_1 \in [0,1]$, $x_5(t_1) \in [0,R]$, $x_5'(t_1) \in [-R,R]$, $-x_5''(t_1) = R$. From condition (H_3) we obtain

$$f(t_1, x_5(t_1), x_5'(t_1), -R) < R.$$

For every $t \in [0,1]$ (including t_1), we have $Nx_5 \geq Lx_5$. This would give

$$R = -x_5''(t_1) \leq f(t_1, x_5(t_1), x_5'(t_1), -R) < R,$$

which is a contradiction. Thus (3.3) holds.

Similarly, from condition (H_4), we get

$$Nx \not\leq Lx \quad \text{for any } x \in K \cap \partial B_r,$$

where $B_r = \{x \in X : \|x\|_X \leq r\}$. Thus all conditions of Theorem 2.5 are satisfied and there exists $x \in K$ such that $Lx = Nx$ and $r \leq \|x\|_X \leq R$. ∎

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