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Infinitely many solutions for systems of n two-point Kirchhoff-type boundary value problems

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Abstract. Using Ricceri's variational principle, we establish the existence of infinitely many solutions for a class of two-point boundary value Kirchhoff-type systems.

1. Introduction. Let $K_i : [0, +\infty[\to \mathbb{R}$ for $1 \le i \le n$ be n continuous functions such that there exist n positive numbers m_i with $K_i(t) \ge m_i$ for all $t \ge 0$ for $1 \le i \le n$, and denote $\underline{m} := \min\{m_i;\ 1 \le i \le n\}$.

Consider the following double eigenvalue Kirchhoff-type system on a bounded interval $[a, b]$ in $\mathbb{R}$ $(a < b)$:

$$\begin{cases} -K_i\Big(\int_a^b |u_i'(x)|^2\,dx\Big)u_i'' = \lambda F_{u_i}(x, u_1, \dots, u_n) + \mu G_{u_i}(x, u_1, \dots, u_n), \\ u_i(a) = u_i(b) = 0, \end{cases} \tag{1.1}$$

for $1 \le i \le n$. In (1.1), λ is a positive parameter, μ is a non-negative parameter, $F : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ is a function such that $F(\cdot, t)$ is continuous in $[a, b]$ for all $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, $F(x, \cdot, \dots, \cdot)$ is C^1 in $\mathbb{R}^n$ for every $x \in [a, b]$, $F(x, 0, \dots, 0) = 0$ for all $x \in [a, b]$ and for every $\varrho > 0$,

$$\sup_{|t| \le \varrho} \sum_{i=1}^n |F_{t_i}(\cdot, t)| \in L^1([a, b]),$$

$G : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ is a function such that $G(\cdot, t)$ is measurable in $[a, b]$ for all $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, $G(x, \cdot, \dots, \cdot)$ is C^1 in $\mathbb{R}^n$ for every $x \in [a, b]$ and $G(x, 0, \dots, 0) = 0$ for all $x \in [a, b]$, and F_{u_i} and G_{u_i} denote the partial derivatives of F and G with respect to u_i for $1 \le i \le n$, respectively.

Basing on the variational principle of [25], we will prove the existence of infinitely many solutions for the system (1.1); see [5].

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Corresponding to K_i we introduce the functions $\tilde{K_i} : [0, +\infty[\to \mathbb{R}$ by

$$\tilde{K_i}(t) = \int_0^t K_i(s)\, ds \quad \text{for } t \geq 0 \text{ and } 1 \leq i \leq n.$$

For all $\gamma > 0$ we set

$$Q(\gamma) = \Big\{ t = (t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n |t_i| \leq \gamma \Big\}. \tag{1.2}$$

A special case of our main result is the following theorem.

THEOREM 1.1. *Let $f_i : \mathbb{R}^n \to \mathbb{R}$ be a continuous function for $1 \leq i \leq n$ such that the differential 1-form $w := \sum_{i=1}^n f_i(\xi_1, \dots, \xi_n) d\xi_i$ is integrable and let F be a primitive of w such that $F(\xi_1, \dots, \xi_n) \geq 0$ in $\mathbb{R}^n$. Assume that*

$$\liminf_{\xi \to +\infty} \frac{\max_{t \in Q(\xi)} F(t)}{\xi^2} = 0$$

and

$$\limsup_{(t_1, \dots, t_n) \to (+\infty, \dots, +\infty)} \frac{F(t_1, \dots, t_n)}{\sum_{i=1}^n \tilde{K_i}\big(\frac{8}{b-a} t_i^2\big)} = +\infty.$$

Then the system

$$\begin{cases} -K_i\Big(\int_a^b |u_i'(x)|^2\, dx\Big) u_i'' = f_i(u_1, \dots, u_n) \quad \text{in } (a, b), \\ u_i(a) = u_i(b) = 0, \end{cases} \tag{1.3}$$

for $1 \leq i \leq n$, has a sequence of pairwise distinct positive weak solutions.

Problems of Kirchhoff type have been widely investigated. We refer the reader to [1, 13, 16–20, 23, 24, 27, 29] and the references therein. For instance, B. Ricceri in an interesting paper [27] established the existence of at least three weak solutions to a class of Kirchhoff-type double eigenvalue boundary value problems using Theorem A of [26]. In [19], motivated by [27], based on a three critical points theorem proved in [2], the existence of two intervals of positive real parameters λ was established for which the boundary value problem of Kirchhoff type

$$\begin{cases} -K\Big(\int_a^b |u'(x)|^2\, dx\Big) u'' = \lambda f(x, u), \\ u(a) = u(b) = 0, \end{cases}$$

where $K : [0, +\infty[\to \mathbb{R}$ is a continuous function, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $\lambda > 0$, admits three weak solutions whose norms are uniformly bounded with respect to λ belonging to one of certain two intervals. In [16], the authors studied the existence of infinitely many non-negative solutions for a $p(x)$-Kirchhoff-type Dirichlet problem by applying Ricceri's variational principle [25] and the theory of variable exponent Sobolev spaces.

By a (*weak*) *solution* of the system (1.1), we mean any $u = (u_1, \dots, u_n) \in (W_0^{1,2}([a,b]))^n$ such that

$$\sum_{i=1}^{n} K_i\Big(\int_a^b |u_i'(x)|^2\,dx\Big)\int_a^b u_i'(x)v_i'(x)\,dx - \lambda\int_a^b\sum_{i=1}^{n} F_{u_i}(x,u_1(x),\dots,u_n(x))v_i(x)\,dx - \mu\int_a^b\sum_{i=1}^{n} G_{u_i}(x,u_1(x),\dots,u_n(x))v_i(x)\,dx = 0$$

for every $v = (v_1, \dots, v_n) \in (W_0^{1,2}([a,b]))^n$.

For a discussion of the existence of infinitely many solutions for some differential equations, applying a smooth version of Theorem 2.1 of [5], which is a more precise version of Ricceri's variational principle [25], we refer the reader to [5, 6, 7, 10]. A non-smooth version of Ricceri's variational principle due to Marano and Motreanu [22] is employed in [11]. Here, our motivation comes from the recent paper of Bonanno and Di Bella [4].

Below we recall Theorem 2.5 of [25] which is our main tool.

THEOREM 1.2. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r])}\Psi(v) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then:

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in\ ]0, 1/\varphi(r)[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point* (*local minimum*) *of I_λ in X.*

(b) *If $\gamma < +\infty$ then, for each $\lambda \in\ ]0, 1/\gamma[$, either*

  (b$_1$) *I_λ possesses a global minimum, or*

  (b$_2$) *there is a sequence $\{u_n\}$ of critical points* (*local minima*) *of I_λ such that*

$$\lim_{n\to\infty} \Phi(u_n) = +\infty.$$

(c) *If* $\delta < +\infty$ *then, for each* $\lambda \in\,]0, 1/\delta[$, *either*

(c$_1$) *there is a global minimum of* Φ *which is a local minimum of* I_λ, *or*

(c$_2$) *there is a sequence of pairwise distinct critical points (local minima) of* I_λ *which weakly converges to a global minimum of* Φ.

For other studies on the subject, we refer the reader to [8, 9, 14, 15].

2. Main results. We state our main result as follows:

THEOREM 2.1. *Assume that there exist positive constants* α *and* β *with* $\beta + \alpha < b - a$ *such that*

(A$_1$) $F(x,t) \geq 0$ *for each* $(x,t) \in ([a, a+\alpha] \cup [b-\beta, b]) \times \mathbb{R}^n$;

(A$_2$) $\displaystyle \liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in Q(\xi)} F(x,t)\,dx}{\xi^2} < \frac{4\underline{m}}{n^2(b-a)} \limsup_{t\to+\infty} \frac{\int_{a+\alpha}^{b-\beta} F(x,t)\,dx}{\sum_{i=1}^n \tilde{K}_i\big(\frac{\alpha+\beta}{\alpha\beta} t_i^2\big)}$

(*note* $t \to +\infty$ *means* $(t_1, \ldots, t_n) \to (+\infty, \ldots, +\infty)$). *Then, for each* $\lambda \in\,]\lambda_1, \lambda_2[$ *where*

$$\lambda_1 := \frac{1}{2 \limsup_{t\to+\infty} \frac{\int_{a+\alpha}^{b-\beta} F(x,t)\,dx}{\sum_{i=1}^n \tilde{K}_i\left(\frac{\alpha+\beta}{\alpha\beta} t_i^2\right)}},$$

$$\lambda_2 := \frac{\frac{2\underline{m}}{n^2(b-a)}}{\liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in Q(\xi)} F(x,t)\,dx}{\xi^2}},$$

for every non-negative function $G : [a,b] \times \mathbb{R}^n \to \mathbb{R}$, *measurable in* $[a,b]$, C^1 *in* $\mathbb{R}^n$ *and satisfying the condition*

$$G_\infty := \lim_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in Q(\xi)} G(x,t)\,dx}{\xi^2} < +\infty, \tag{2.1}$$

and for every $\mu \in [0, \mu_{G,\lambda}[$ *where*

$$\mu_{G,\lambda} := \frac{2\underline{m}}{n^2(b-a)G_\infty}\left(1 - \lambda \frac{n^2(b-a)}{2\underline{m}} \liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in Q(\xi)} F(x,t)\,dx}{\xi^2}\right),$$

system (1.1) *has an unbounded sequence of weak solutions in* $(W_0^{1,2}([a,b]))^n$.

Proof. To apply Theorem 1.2, let $X = (W_0^{1,2}([a,b]))^n$ be equipped with the norm

$$\|(u_1, \ldots, u_n)\| = \sum_{i=1}^n \|u_i\|_*$$

where $\|u_i\|_* = (\int_a^b (|u_i'(x)|^2)\,dx)^{1/2}$ for $1 \leq i \leq n$. Arguing as in [3], fix $\overline{\lambda} \in\,]\lambda_1, \lambda_2[$ and let G be a function satisfying (2.1). Since $\overline{\lambda} < \lambda_2$, one has

$$\mu_{G,\overline{\lambda}} := \frac{2\underline{m}}{n^2(b-a)G_\infty}\left(1 - \overline{\lambda}\frac{n^2(b-a)}{2\underline{m}} \liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in Q(\xi)} F(x,t)\,dx}{\xi^2}\right) > 0.$$

Fix $\overline{\mu} \in \,]0, \mu_{G,\overline{\lambda}}[$ and put

$$\nu_1 := \lambda_1 \quad \text{and} \quad \nu_2 := \frac{\lambda_2}{1 + \frac{n^2(b-a)}{2\underline{m}} \frac{\overline{\mu}}{\overline{\lambda}} \lambda_2 G_\infty}.$$

If $G_\infty = 0$, then clearly $\nu_1 = \lambda_1$, $\nu_2 = \lambda_2$ and $\lambda \in \,]\nu_1, \nu_2[$. If $G_\infty \neq 0$, since $\overline{\mu} < \mu_{G,\overline{\lambda}}$, we obtain

$$\frac{\overline{\lambda}}{\lambda_2} + \frac{n^2(b-a)}{2\underline{m}}\overline{\mu} G_\infty < 1,$$

and so

$$\frac{\lambda_2}{1 + \frac{n^2(b-a)}{2\underline{m}} \frac{\overline{\mu}}{\overline{\lambda}} \lambda_2 G_\infty} > \overline{\lambda},$$

that is, $\overline{\lambda} < \nu_2$. Hence, taking into account that $\overline{\lambda} > \lambda_1 = \nu_1$, one has $\overline{\lambda} \in \,]\nu_1, \nu_2[$.

Now, set

$$H(x,\xi) = F(x,\xi) + \frac{\overline{\mu}}{\overline{\lambda}} G(x,\xi)$$

for $x \in [a,b]$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. We define $\Phi, \Psi : X \to \mathbb{R}$ for $u = (u_1, \ldots, u_n) \in X$ as follows:

$$\Phi(u) = \frac{1}{2}\sum_{i=1}^{n} \tilde{K}_i(\|u_i\|_*^2), \qquad \Psi(u) = \int_a^b H(x, u_1(x), \ldots, u_n(x))\,dx.$$

Let us prove that Φ and Ψ satisfy the required conditions. It is well known that Ψ is a differentiable functional whose differential at $u \in X$ is

$$\Psi'(u)(v) = \int_a^b \sum_{i=1}^{n} H_{u_i}(x, u_1(x), \ldots, u_n(x)) v_i(x)\,dx$$

for every $v = (v_1, \ldots, v_n) \in X$; moreover, Ψ is sequentially weakly upper semicontinuous.

Furthermore, $\Psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X. For this, for fixed $(u_1, \ldots, u_n) \in X$ let $(u_{1k}, \ldots, u_{nk}) \to (u_1, \ldots, u_n)$ weakly in X as $k \to \infty$. Then $(u_{1k}, \ldots, u_{nk})$ converges uniformly to $(u_1, \ldots, u_n)$ on $[a,b]$ as $k \to \infty$ (see [30]). Since $H(x, \cdot, \ldots, \cdot)$ is C^1 in $\mathbb{R}^n$ for every $x \in [a,b]$, the derivatives of H are continuous in $\mathbb{R}^n$ for every $x \in [a,b]$, so for $1 \le i \le n$, $H_{u_i}(x, u_{1k}, \ldots, u_{nk}) \to H_{u_i}(x, u_1, \ldots, u_n)$ strongly as $k \to \infty$, which yields $\Psi'(u_{1k}, \ldots, u_{nk}) \to$

$\Psi'(u_1,\dots,u_n)$ strongly as $k\to\infty$. Thus we proved that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by Proposition 26.2 of [30].

Moreover, it is well known that Φ is sequentially weakly lower semicontinuous as well as continuously differentiable, and its differential at $u\in X$ is

$$\Phi'(u)(v)=\sum_{i=1}^{n}K_i\Big(\int_a^b|u_i'(x)|^2\,dx\Big)\int_a^b u_i'(x)v_i'(x)\,dx$$

for every $v\in X$.

Put $I_{\overline{\lambda}}:=\Phi-\overline{\lambda}\Psi$. Clearly, the weak solutions of (1.1) are exactly the solutions of the equation $I'_{\overline{\lambda}}(u_1,\dots,u_n)=0$. Moreover, since $m_i\le K_i(s)$ for all $s\in[0,+\infty[$ and $1\le i\le n$, from the definition of Φ we have

$$\Phi(u)\ge\frac{1}{2}\sum_{i=1}^{n}m_i\|u_i\|_*^2\ge\frac{m}{2}\sum_{i=1}^{n}\|u_i\|_*^2\quad\text{for all } u\in X. \tag{2.2}$$

Now, let us verify that

$$\gamma<+\infty.$$

Let $\{\xi_k\}$ be a real sequence such that $\xi_k\to\infty$ as $k\to\infty$ and

$$\lim_{k\to\infty}\frac{\int_a^b\sup_{t\in Q(\xi_k)}H(x,t)\,dx}{\xi_k^2}=\liminf_{\xi\to\infty}\frac{\int_a^b\sup_{t\in Q(\xi)}H(x,t)\,dx}{\xi^2}.$$

Put $r_k=\frac{2m\xi_k^2}{n^2(b-a)}$ for all $k\in\mathbb{N}$. Since

$$\max_{x\in[a,b]}|u_i(x)|\le\frac{(b-a)^{1/2}}{2}\|u_i\|_*\quad\text{for all } u_i\in W_0^{1,2}([a,b]) \text{ and } 1\le i\le n,$$

we have

$$\sup_{x\in[a,b]}\sum_{i=1}^{n}|u_i(x)|^2\le\frac{b-a}{4}\sum_{i=1}^{n}\|u_i\|_*^2 \tag{2.3}$$

for each $u=(u_1,\dots,u_n)\in X$. So, from (2.2) and (2.3) we have

$$\begin{aligned}\Phi^{-1}(]-\infty,r_k])&=\Big\{u\in X;\ \frac{m}{2}\sum_{i=1}^{n}\|u_i\|_*^2\le r_k\Big\}\\&\subseteq\Big\{u\in X;\ \sum_{i=1}^{n}|u_i(x)|^2\le\frac{r_k(b-a)}{2m}\text{ for each } x\in[a,b]\Big\}\\&\subseteq\Big\{u\in X;\ \sum_{i=1}^{n}|u_i(x)|\le\xi_k\text{ for each } x\in[a,b]\Big\}.\end{aligned}$$

Hence, taking into account that $\Phi(0,\dots,0)=\Psi(0,\dots,0)=0$, for every k

large enough, one has

$$\begin{aligned}\varphi(r_k) &= \inf_{u\in\Phi^{-1}(]-\infty,r_k[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_k])}\Psi(v)-\Psi(u)}{r_k-\Phi(u)} \\ &\le \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_k])}\Psi(v)}{r_k} \le \frac{\int_a^b \sup_{t\in Q(\xi_k)} H(x,t)\,dx}{\frac{2\underline{m}\xi_k^2}{n^2(b-a)}} \\ &= \frac{\int_a^b \sup_{t\in Q(\xi_k)} \left[F(x,t)+\frac{\overline{\mu}}{\overline{\lambda}}G(x,t)\right]dx}{\frac{2\underline{m}\xi_k^2}{n^2(b-a)}} \\ &\le \frac{\int_a^b \sup_{t\in Q(\xi_k)} F(x,t)\,dx}{\frac{2\underline{m}\xi_k^2}{n^2(b-a)}} + \frac{\overline{\mu}}{\overline{\lambda}} \frac{\int_a^b \sup_{t\in Q(\xi_k)} G(x,t)\,dx}{\frac{2\underline{m}\xi_k^2}{n^2(b-a)}}\end{aligned}$$

Moreover, from Assumption (A2) and (2.1) one has

$$\lim_{k\to\infty} \frac{\int_a^b \sup_{t\in Q(\xi_k)} F(x,t)\,dx}{\frac{2\underline{m}\xi_k^2}{n^2(b-a)}} + \lim_{k\to\infty} \frac{\overline{\mu}}{\overline{\lambda}} \frac{\int_a^b \sup_{t\in Q(\xi_k)} G(x,t)\,dx}{\frac{2\underline{m}\xi_k^2}{n^2(b-a)}} < +\infty,$$

which implies

$$\lim_{k\to\infty} \frac{\int_a^b \sup_{t\in Q(\xi_k)} H(x,t)\,dx}{\xi_k^2} < +\infty.$$

Therefore,

$$\gamma \le \liminf_{k\to\infty} \varphi(r_k) \le \frac{n^2(b-a)}{2\underline{m}} \lim_{k\to\infty} \frac{\int_a^b \sup_{t\in Q(\xi_k)} H(x,t)\,dx}{\xi_k^2} < +\infty. \tag{2.4}$$

Since

$$\frac{\int_a^b \sup_{t\in Q(\xi_k)} H(x,t)\,dx}{\frac{2\underline{m}\xi_k^2}{n^2(b-a)}} \le \frac{\int_a^b \sup_{t\in Q(\xi_k)} F(x,t)\,dx}{\frac{2\underline{m}\xi_k^2}{n^2(b-a)}} + \frac{\overline{\mu}}{\overline{\lambda}} \frac{\int_a^b \sup_{t\in Q(\xi_k)} G(x,t)\,dx}{\frac{2\underline{m}\xi_k^2}{n^2(b-a)}},$$

taking (2.1) into account, one has

$$\liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in Q(\xi)} H(x,t)\,dx}{\xi^2} \le \liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in Q(\xi)} F(x,t)\,dx}{\xi^2} + \frac{\overline{\mu}}{\overline{\lambda}} G_\infty. \tag{2.5}$$

Moreover, since G is non-negative, from Assumption (A1) we obtain

$$\limsup_{\xi\to+\infty} \frac{\int_{a+\alpha}^{b-\beta} H(x,\xi,\dots,\xi)\,dx}{\xi^2} \ge \limsup_{\xi\to+\infty} \frac{\int_{a+\alpha}^{b-\beta} F(x,\xi,\dots,\xi)\,dx}{\xi^2}. \tag{2.6}$$

Therefore, from (2.5) and (2.6), we observe

$$\overline{\lambda} \in \,]\nu_1,\nu_2[\, \subseteq \,]\lambda_1,\lambda_2[.$$

Assumption (A2) in conjunction with (2.4), implies

$$]\lambda_1, \lambda_2[\subseteq]0, 1/\gamma[.$$

For fixed $\overline{\lambda}$, the inequality (2.4) ensures that condition (b) of Theorem 1.2 can be applied and either $I_{\overline{\lambda}}$ has a global minimum or there exists a sequence $\{u_k = (u_{1k}, \dots, u_{nk})\}$ of weak solutions of the system (1.1) such that $\lim_{k\to\infty} \|(u_{1k}, \dots, u_{nk})\| = +\infty$.

The next step is to show that for fixed $\overline{\lambda}$ the functional $I_{\overline{\lambda}}$ has no global minimum. Let us verify that $I_{\overline{\lambda}}$ is unbounded from below. Since

$$\frac{1}{\overline{\lambda}} < 2 \limsup_{t\to+\infty} \frac{\int_{a+\alpha}^{b-\beta} F(x,t)\,dx}{\sum_{i=1}^n \tilde{K}_i\big(\frac{\alpha+\beta}{\alpha\beta} t_i^2\big)} \le 2 \limsup_{t\to+\infty} \frac{\int_{a+\alpha}^{b-\beta} H(x,t)\,dx}{\sum_{i=1}^n \tilde{K}_i\big(\frac{\alpha+\beta}{\alpha\beta} t_i^2\big)},$$

we can consider a real sequence $\{d_k\}$ and a positive constant τ such that $d_k \to \infty$ as $k \to \infty$ and

$$\frac{1}{\overline{\lambda}} < \tau < 2\frac{\int_{a+\alpha}^{b-\beta} H(x, d_k, \dots, d_k)\,dx}{\sum_{i=1}^n \tilde{K}_i\big(\frac{\alpha+\beta}{\alpha\beta} d_k^2\big)} \tag{2.7}$$

for each $k \in \mathbb{N}$ large enough. Let $\{w_k = (w_{1k}, \dots, w_{nk})\}$ be the sequence in X defined by

$$w_{ik}(x) = \begin{cases} \dfrac{d_k}{\alpha}(x-a) & \text{if } a \le x < a+\alpha, \\ d_k & \text{if } a+\alpha \le x \le b-\beta, \\ \dfrac{d_k}{\beta}(b-x) & \text{if } b-\beta < x \le b, \end{cases} \tag{2.8}$$

for $1 \le i \le n$. For any fixed $k \in \mathbb{N}$, it is easy to see that $w_k \in X$, in particular,

$$\|w_{ik}\|_*^2 = \frac{\alpha+\beta}{\alpha\beta} d_k^2 \quad \text{for } 1 \le i \le n,$$

and so

$$\Phi(w_k) = \frac{1}{2}\sum_{i=1}^n \tilde{K}_i\bigg(\frac{\alpha+\beta}{\alpha\beta} d_k^2\bigg). \tag{2.9}$$

On the other hand, bearing in mind Assumption (A1), since G is non-negative, from the definition of Ψ we infer

$$\Psi(w_k) \ge \int_{a+\alpha}^{b-\beta} H(x, d_k, \dots, d_k)\,dx. \tag{2.10}$$

So, according to (2.7), (2.9) and (2.10),

$$\begin{aligned} I_{\overline{\lambda}}(w_k) &\le \frac{1}{2}\sum_{i=1}^n \tilde{K}_i\bigg(\frac{\alpha+\beta}{\alpha\beta} d_k^2\bigg) - \overline{\lambda} \int_{a+\alpha}^{b-\beta} H(x, d_k, \dots j, d_k)\,dx \\ &< \frac{1}{2}(1-\overline{\lambda}\tau)\sum_{i=1}^n \tilde{K}_i\bigg(\frac{\alpha+\beta}{\alpha\beta} d_k^2\bigg) \end{aligned}$$

for every $k \in \mathbb{N}$ large enough. Hence, $I_{\overline{\lambda}}$ is unbounded from below, and so has no global minimum. Therefore, applying Theorem 1.2 we deduce that there is a sequence $\{u_k = (u_{1k}, \dots, u_{nk})\} \subset X$ of critical points of $I_{\overline{\lambda}}$ such that $\lim_{k\to\infty} \|(u_{1k}, \dots, u_{nk})\| = +\infty$. Hence, the conclusion is achieved. ∎

Remark 2.2. Under the conditions

$$\liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in Q(\xi)} F(x,t)\,dx}{\xi^2} = 0, \qquad \limsup_{t\to+\infty} \frac{\int_{a+\alpha}^{b-\beta} F(x,t)\,dx}{\sum_{i=1}^n \tilde{K}_i\big(\frac{\alpha+\beta}{\alpha\beta} t_i^2\big)} = +\infty,$$

from Theorem 2.1 we see that for every $\lambda > 0$ and $\mu \in \big[0, \frac{2\underline{m}}{n^2(b-a)G_\infty}\big[$ the system (1.1) admits infinitely many weak solutions in $(W_0^{1,2}([a,b]))^n$. Moreover, if $G_\infty = 0$, the result holds for every $\lambda > 0$ and $\mu > 0$.

The following result is a special case of Theorem 2.1 with $\mu = 0$.

Theorem 2.3. *Assume that all the assumptions of Theorem* 2.1 *hold. Then, for each* λ *in*

$$\Lambda := \left] \frac{1}{2\limsup_{t\to+\infty} \frac{\int_{a+\alpha}^{b-\beta} F(x,t)\,dx}{\sum_{i=1}^n \tilde{K}_i\left(\frac{\alpha+\beta}{\alpha\beta} t_i^2\right)}}, \frac{\frac{2\underline{m}}{n^2(b-a)}}{\liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{t\in Q(\xi)} F(x,t)\,dx}{\xi^2}} \right[$$

system (1.1) *has an unbounded sequence of weak solutions in* $(W_0^{1,2}([a,b]))^n$.

Now we present the following existence result in which instead of Assumption (A2) a more general condition is assumed.

Theorem 2.4. *Assume that all the hypotheses of Theorem* 2.1 *hold except for Assumption* (A2). *Suppose that*

(A3) *there exist sequences* $\{a_k\}$ *and* $\{b_k\}$ *with*

$$\sum_{i=1}^n \tilde{K}_i\left(\frac{\alpha+\beta}{\alpha\beta} a_k^2\right) < \frac{4\underline{m} b_k^2}{n^2(b-a)} \quad \textit{for every } k \in \mathbb{N}$$

and $\lim_{k\to\infty} b_k = +\infty$ *such that*

$$\lim_{k\to\infty} \frac{\int_a^b \sup_{t\in Q(b_k)} F(x,t)\,dx - \int_{a+\alpha}^{b-\beta} F(x,a_k,\dots,a_k)\,dx}{\frac{2\underline{m} b_k^2}{n^2(b-a)} - \frac{1}{2}\sum_{i=1}^n \tilde{K}_i\left(\frac{\alpha+\beta}{\alpha\beta} a_k^2\right)} < 2\limsup_{t\to\infty} \frac{\int_{a+\alpha}^{b-\beta} F(x,t)\,dx}{\sum_{i=1}^n \tilde{K}_i\left(\frac{\alpha+\beta}{\alpha\beta} t_i^2\right)}.$$

Then, for each λ *in*

$$\Lambda' := \left]\frac{1}{2\limsup_{t\to+\infty}\frac{\int_{a+\alpha}^{b-\beta}F(x,t)\,dx}{\sum_{i=1}^{n}\tilde{K}_i(\frac{\alpha+\beta}{\alpha\beta}t_i^2)}}, \frac{\frac{2\underline{m}}{n^2(b-a)}}{\lim_{k\to\infty}\frac{\int_a^b\sup_{t\in Q(b_k)}F(x,t)\,dx-\int_{a+\alpha}^{b-\beta}F(x,a_k,\ldots,a_k)\,dx}{\frac{2\underline{m}b_k^2}{n^2(b-a)}-\frac{1}{2}\sum_{i=1}^{n}\tilde{K}_i(\frac{\alpha+\beta}{\alpha\beta}a_k^2)}}\right[$$

system (1.1) *has an unbounded sequence of weak solutions in* $(W_0^{1,2}([a,b]))^n$*.*

Proof. Clearly, from (A3) we obtain (A2), by choosing $a_k = 0$ for all $k\in\mathbb{N}$. Moreover, if we assume (A3) instead of (A2) and set $r_k = \frac{2\underline{m}b_k^2}{n^2(b-a)}$ for all $k\in\mathbb{N}$, by the same reasoning as in the proof of Theorem 2.1 with $\mu = 0$, we obtain

$$\begin{aligned}\varphi(r_k) &= \inf_{u\in\Phi^{-1}(]-\infty,r_k[)}\frac{\sup_{v\in\Phi^{-1}(]-\infty,r_k])}\Psi(v)-\Psi(u)}{r_k-\Phi(u)}\\ &\le \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_k])}\Psi(v)-\int_a^b F(x,w_{1k}(x),\ldots,w_{nk}(x))\,dx}{r_k-\frac{1}{2}\sum_{i=1}^{n}\tilde{K}_i(\|w_{ik}\|_*^2)}\\ &\le \frac{\int_a^b\sup_{t\in Q(b_k)}F(x,t)\,dx-\int_{a+\alpha}^{b-\beta}F(x,a_k,\ldots,a_k)\,dx}{\frac{2\underline{m}b_k^2}{n^2(b-a)}-\frac{1}{2}\sum_{i=1}^{n}\tilde{K}_i\big(\frac{\alpha+\beta}{\alpha\beta}a_k^2\big)}\end{aligned}$$

where $w_k = (w_{1k},\ldots,w_{nk})$ with w_{ik} for $1\le i\le n$ as given in (2.8) with a_k instead of d_k. So, we have the desired conclusion. ∎

Here we point out the following consequence of Theorem 2.3.

COROLLARY 2.5. *Assume that there exist positive constants* α *and* β *with* $\beta+\alpha<b-a$ *such that Assumption* (A1) *holds. Suppose that*

(B1) $\liminf_{\xi\to+\infty}\frac{\int_a^b\sup_{t\in Q(\xi)}F(x,t)\,dx}{\xi^2} < \frac{2\underline{m}}{n^2(b-a)}$;

(B2) $\limsup_{t\to+\infty}\frac{\int_{a+\alpha}^{b-\beta}F(x,t)\,dx}{\sum_{i=1}^{n}\tilde{K}_i\big(\frac{\alpha+\beta}{\alpha\beta}t_i^2\big)} > \frac{1}{2}$.

Then the system

$$\begin{cases}-K_i\Big(\int_a^b|u_i'(x)|^2\,dx\Big)u_i'' = F_{u_i}(x,u_1,\ldots,u_n),\\ u_i(a)=u_i(b)=0,\end{cases}\tag{2.11}$$

for $1\le i\le n$*, has an unbounded sequence of weak solutions in* $(W_0^{1,2}([a,b]))^n$*.*

REMARK 2.6. Theorem 1.1 in the Introduction is a consequence of Corollary 2.5, obtained by setting $F(x,t) = F(t)$ for all $x \in [a,b]$ and $t \in \mathbb{R}^n$ for $1 \le i \le n$, and choosing $\alpha = \beta = (b-a)/4$.

In the same way as in the proof of Theorem 2.1 but using conclusion (c) of Theorem 1.2 instead of (b), we will obtain the following result.

THEOREM 2.7. *Assume that all the hypotheses of Theorem* 2.1 *hold except for Assumption* (A2). *Suppose that*

(A4) $$\liminf_{\xi\to 0^+} \frac{\int_a^b \sup_{t\in Q(\xi)} F(x,t)\,dx}{\xi^2} < \frac{4\underline{m}}{n^2(b-a)} \limsup_{t\to 0^+} \frac{\int_{a+\alpha}^{b-\beta} F(x,t)\,dx}{\sum_{i=1}^n \tilde{K}_i\big(\frac{\alpha+\beta}{\alpha\beta}t_i^2\big)}$$

(*note* $t \to 0^+$ *means* $(t_1,\dots,t_n) \to (0^+,\dots,0^+)$).

Then, for each $\lambda \in]\lambda_3, \lambda_4[$ *where*

$$\lambda_3 := \frac{1}{2\limsup\limits_{t\to 0^+} \frac{\int_{a+\alpha}^{b-\beta} F(x,t)\,dx}{\sum_{i=1}^n \tilde{K}_i(\frac{\alpha+\beta}{\alpha\beta}t_i^2)}},$$

$$\lambda_4 := \frac{\frac{2\underline{m}}{n^2(b-a)}}{\liminf\limits_{\xi\to 0^+} \frac{\int_a^b \sup_{t\in Q(\xi)} F(x,t)\,dx}{\xi^2}},$$

for every non-negative function $G : [a,b]\times\mathbb{R}^n \to \mathbb{R}$, *measurable in* $[a,b]$, C^1 *in* $\mathbb{R}^n$ *and satisfying the condition*

$$G_0 := \lim_{\xi\to 0^+} \frac{\int_a^b \sup_{t\in Q(\xi)} G(x,t)\,dx}{\xi^2} < +\infty, \tag{2.12}$$

and for every $\mu \in [0, \mu_{G,\lambda}[$ *where*

$$\mu_{G,\lambda} := \frac{2\underline{m}}{n^2(b-a)G_0}\bigg(1 - \lambda\frac{n^2(b-a)}{2\underline{m}} \liminf_{\xi\to 0^+} \frac{\int_a^b \sup_{t\in Q(\xi)} F(x,t)\,dx}{\xi^2}\bigg),$$

the system (1.1) *has a sequence of weak solutions which strongly converges to* 0 *in* $(W_0^{1,2}([a,b]))^n$.

Proof. Fix $\overline{\lambda} \in]\lambda_3, \lambda_4[$ and let G be a function satisfying (2.12). Since $\overline{\lambda} < \lambda_2$, one has

$$\mu_{G,\overline{\lambda}} := \frac{2\underline{m}}{n^2(b-a)G_0}\bigg(1 - \overline{\lambda}\frac{n^2(b-a)}{2\underline{m}} \liminf_{\xi\to 0^+} \frac{\int_a^b \sup_{t\in Q(\xi)} F(x,t)\,dx}{\xi^2}\bigg) > 0.$$

Fix $\overline{\mu} \in]0, \mu_{G,\overline{\lambda}}[$ and put

$$\nu_1 := \lambda_3 \quad \text{and} \quad \nu_2 := \frac{\lambda_4}{1 + \frac{n^2(b-a)}{2\underline{m}}\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2 G_0}.$$

If $G_0 = 0$, then clearly $\nu_1 = \lambda_3$, $\nu_2 = \lambda_4$ and $\lambda \in]\nu_1, \nu_2[$. If $G_0 \neq 0$, since $\overline{\mu} < \mu_{G,\overline{\lambda}}$, we obtain

$$\frac{\overline{\lambda}}{\lambda_2} + \frac{n^2(b-a)}{2\underline{m}}\overline{\mu} G_0 < 1,$$

and so

$$\frac{\lambda_2}{1 + \frac{n^2(b-a)}{2\underline{m}}\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2 G_0} > \overline{\lambda},$$

that is, $\overline{\lambda} < \nu_2$. Hence, bearing in mind that $\overline{\lambda} > \lambda_3 = \nu_1$, one has $\overline{\lambda} \in]\nu_1, \nu_2[$. Now, set

$$H(x,\xi) = F(x,\xi) + \frac{\overline{\mu}}{\overline{\lambda}} G(x,\xi)$$

for $x \in [a,b]$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. Since

$$\frac{\int_a^b \sup_{t\in Q(\xi_k)} H(x,t)\,dx}{\frac{2\underline{m}\xi_k^2}{n^2(b-a)}} \leq \frac{\int_a^b \sup_{t\in Q(\xi_k)} F(x,t)\,dx}{\frac{2\underline{m}\xi_k^2}{n^2(b-a)}} + \frac{\overline{\mu}}{\overline{\lambda}} \frac{\int_a^b \sup_{t\in Q(\xi_k)} G(x,t)\,dx}{\frac{2\underline{m}\xi_k^2}{n^2(b-a)}},$$

taking into account (2.12) one has

(2.13)

$$\liminf_{\xi\to 0^+} \frac{\int_a^b \sup_{t\in Q(\xi)} H(x,t)\,dx}{\xi^2} \leq \liminf_{\xi\to 0^+} \frac{\int_a^b \sup_{t\in Q(\xi)} F(x,t)\,dx}{\xi^2} + \frac{\overline{\mu}}{\overline{\lambda}} G_0.$$

Moreover, since G is non-negative, from Assumption (A1) we obtain

$$\limsup_{\xi\to 0^+} \frac{\int_{a+\alpha}^{b-\beta} H(x,\xi,\ldots,\xi)\,dx}{\xi^2} \geq \limsup_{\xi\to 0^+} \frac{\int_{a+\alpha}^{b-\beta} F(x,\xi,\ldots,\xi)\,dx}{\xi^2}. \tag{2.14}$$

Therefore, from (2.13) and (2.14),

$$\overline{\lambda} \in]\nu_1, \nu_2[\subseteq]\lambda_3, \lambda_4[.$$

We take Φ, Ψ and $I_{\overline{\lambda}}$ as in the proof of Theorem 2.1. We verify that $\delta < +\infty$. For this, let $\{\xi_k\}$ be a sequence of positive numbers such that $\xi_k \to 0^+$ as $k \to \infty$ and

$$\lim_{k\to\infty} \frac{\int_a^b \sup_{t\in Q(\xi_k)} H(x,t)\,dx}{\xi_k^2} < +\infty.$$

Put $r_k = \frac{2\underline{m}\xi_k^2}{n^2(b-a)}$ for $k \in \mathbb{N}$. Let us show that the functional $I_{\overline{\lambda}}$ does not have a local minimum at zero. For this, let $\{d_k\}$ be a sequence of positive numbers such that $d_k \to 0^+$ as $k \to \infty$ and pick $\tau > 0$ such that

$$\frac{1}{\overline{\lambda}} < \tau < 2\frac{\int_{a+\alpha}^{b-\beta} H(x, d_k, \ldots, d_k)\, dx}{\sum_{i=1}^{n} \tilde{K}_i\big(\frac{\alpha+\beta}{\alpha\beta} d_k^2\big)} \tag{2.15}$$

for each $k \in \mathbb{N}$ large enough. Let $\{w_k\} = \{(w_{1k}, \ldots, w_{nk})\}$ be a sequence in X with w_{ik} defined in (2.8). According to (2.9), (2.10) and (2.15), we have

$$\begin{aligned} I_{\overline{\lambda}}(w_k) &= \Phi(w_k) - \overline{\lambda}\Psi(w_k) \\ &\le \frac{1}{2}\sum_{i=1}^{n} \tilde{K}_i\bigg(\frac{\alpha+\beta}{\alpha\beta} d_k^2\bigg) - \overline{\lambda} \int_{a+\alpha}^{b-\beta} H(x, d_k, \ldots, d_k)\, dx \\ &< \frac{1}{2}\sum_{i=1}^{n} \tilde{K}_i\bigg(\frac{\alpha+\beta}{\alpha\beta} d_k^2\bigg)(1 - \overline{\lambda}\tau) < 0 \end{aligned}$$

for every $k \in \mathbb{N}$ large enough. Since $I_{\overline{\lambda}}(0) = 0$, this implies that the functional $I_{\overline{\lambda}}$ does not have a local minimum at zero.

Hence, part (c) of Theorem 1.2 ensures that there exists a sequence $\{u_k = (u_{1k}, \ldots, u_{nk})\}$ in X of critical points of $I_{\overline{\lambda}}$ such that $\|u_k\| \to 0$ as $k \to \infty$, and the proof is complete. ■

Remark 2.8. Under the conditions

$$\liminf_{\xi\to 0^+} \frac{\int_a^b \sup_{t\in Q(\xi)} F(x,t)\, dx}{\xi^2} = 0, \qquad \limsup_{t\to 0^+} \frac{\int_{a+\alpha}^{b-\beta} F(x,t)\, dx}{\sum_{i=1}^{n} \tilde{K}_i\big(\frac{\alpha+\beta}{\alpha\beta} t_i^2\big)} = +\infty,$$

Theorem 2.7 ensures that for every $\lambda > 0$ and $\mu \in \big[0, \frac{2\underline{m}}{n^2(b-a)G_0}\big[$ the system (1.1) admits infinitely many weak solutions in $(W_0^{1,2}([a,b]))^n$. Moreover, if $G_0 = 0$, the result holds for every $\lambda > 0$ and $\mu > 0$.

Now we present the following example to illustrate the above result:

Example 2.9. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$F(x, t_1, t_2) = \begin{cases} 0 \quad \text{for all } (x, t_1, t_2) \in [0,1] \times \{0\}^2, \\ g(x)t_1^2\big(1 - \sin(\ln(|t_1|))\big) + h(x)t_2^2\big(1 - \cos(\ln(|t_2|))\big) \\ \qquad \text{for all } (x, t_1, t_2) \in [0,1] \times (\mathbb{R} - \{0\})^2, \end{cases}$$

where $g, h : [0,1) \to \mathbb{R}$ are non-negative continuous functions. We observe that

$$\liminf_{\xi\to 0^+} \frac{\int_a^b \sup_{|t_1|+|t_2|\le\xi} F(x, t_1, t_2)\, dx}{\xi^2} = 0$$

and

$$\limsup_{(t_1,t_2)\to(0^+,0^+)} \frac{\int_{1/4}^{3/4} F(x, t_1, t_2)\, dx}{8(t_1^2 + t_2^2) + 32(t_1^4 + t_2^4)} = +\infty.$$

Hence, by Remark 2.8, for every $(\lambda, \mu) \in]0, +\infty[\times [0, +\infty[$ the system

$$\begin{cases} -\Big(1 + \int_0^1 |u'(x)|^2\,dx\Big)u'' \\ \qquad = \lambda g(x)u\big(2 - 2\sin(\ln(|u|)) - \cos(\ln(|u|))\big) + \mu G_u(x,u,v), \\ -\Big(1 + \int_0^1 |v'(x)|^2\,dx\Big)v'' \\ \qquad = \lambda h(x)v\big(2 - 2\cos(\ln(|v|)) + \sin(\ln(|v|))\big) + \mu G_v(x,u,v), \\ u(0) = u(1) = v(0) = v(1) = 0, \end{cases}$$

where

$$G(x, t_1, t_2) = \frac{e^{-t_1^+}(t_1^+)^\gamma + e^{-t_2^+}(t_2^+)^\eta}{1 + x}$$

with $t_i^+ = \max\{t_i, 0\}$ for $i = 1, 2$, and γ and η positive real numbers, for all $(x, t_1, t_2) \in [0, 1] \times \mathbb{R}^2$, has a sequence of weak solutions which strongly converges to 0 in $W_0^{1,2}([0,1]) \times W_0^{1,2}([0,1])$.

The following existence result is a special case of Theorem 2.7 with $\mu = 0$.

Theorem 2.10. *Assume that all the assumptions of Theorem 2.7 hold. Then, for each λ in*

$$\Lambda'' := \left] \frac{1}{2 \limsup\limits_{t\to 0^+} \frac{\int_{a+\alpha}^{b-\beta} F(x,t)\,dx}{\sum_{i=1}^n \tilde{K}_i(\frac{\alpha+\beta}{\alpha\beta} t_i^2)}}, \frac{\frac{2m}{n^2(b-a)}}{\liminf\limits_{\xi\to 0^+} \frac{\int_a^b \sup_{t\in Q(\xi)} F(x,t)\,dx}{\xi^2}} \right[$$

the system (1.1) *has a sequence of weak solutions which strongly converges to* 0 *in* $(W_0^{1,2}([a,b]))^n$.

Remark 2.11. We easily observe that by assuming, in Theorem 2.4,

$$\lim_{k\to\infty} b_k = 0$$

instead of $\lim_{k\to\infty} b_k = +\infty$ and replacing $t \to +\infty$ with $t \to 0^+$, by the same argument, applying Theorem 2.7, for every $\lambda \in \Lambda'$ the system (1.1) has a sequence of weak solutions which strongly converges to 0 in $(W_0^{1,2}([a,b]))^n$.

We point out a remarkable particular case of Theorem 2.1.

Corollary 2.12. *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function, and denote $F(x,t) = \int_0^t f(x,\xi)\,d\xi$ for all $(x,t) \in [a,b] \times \mathbb{R}$. Assume that there exist four positive constants α, β, p and q with $\beta + \alpha < b - a$ such that*

(C1) $F(x,t) \geq 0$ *for each* $(x,t) \in ([a, a+\alpha] \cup [b-\beta, b]) \times \mathbb{R}$;

(C2) $\liminf_{\xi \to +\infty} \frac{\int_a^b \sup_{|t| \leq \xi} F(x,t)\,dx}{\xi^2} < \frac{4p}{b-a} \limsup_{t \to +\infty} \frac{\int_{a+\alpha}^{b-\beta} F(x,t)\,dx}{pt^2\left(\frac{\alpha+\beta}{\alpha\beta}\right) + \frac{q}{2} t^4 \frac{\alpha+\beta}{\alpha\beta}^2}$.

Then, for each $\lambda \in]\lambda_5, \lambda_6[$ *where*

$$\lambda_5 := \frac{1}{2 \limsup\limits_{t \to +\infty} \frac{\int_{a+\alpha}^{b-\beta} F(x,t)\,dx}{pt^2 \frac{\alpha+\beta}{\alpha\beta} + \frac{q}{2} t^4 (\frac{\alpha+\beta}{\alpha\beta})^2}},$$

$$\lambda_6 := \frac{\frac{2p}{b-a}}{\liminf\limits_{\xi \to +\infty} \frac{\int_a^b \sup_{|t| \leq \xi} F(x,t)\,dx}{\xi^2}},$$

for every L^1*-Carathéodory function* $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ *whose potential* $G(x,t) = \int_0^t g(x,\xi)\,d\xi$ *for* $(x,t) \in [a,b] \times \mathbb{R}$ *is a non-negative function satisfying the condition*

$$G_\infty := \lim_{\xi \to +\infty} \frac{\int_a^b \sup_{|t| \leq \xi} G(x,t)\,dx}{\xi^2} < +\infty, \tag{2.16}$$

and for every $\mu \in [0, \mu_{G,\lambda}[$ *where*

$$\mu_{G,\lambda} := \frac{2p}{(b-a)G_\infty} \left(1 - \lambda \frac{b-a}{2p} \liminf_{\xi \to +\infty} \frac{\int_a^b \sup_{|t| \leq \xi} F(x,t)\,dx}{\xi^2}\right),$$

the problem

$$\begin{cases} -\left(p + q \int_a^b |u'(x)|^2\,dx\right) u'' = \lambda f(x,u) + \mu g(x,u), \\ u(a) = u(b) = 0, \end{cases} \tag{2.17}$$

has an unbounded sequence of weak solutions in $W_0^{1,2}([a,b])$.

Proof. Let $n = 1$. For fixed $p, q > 0$, set $K_1(t) = p + qt$ for all $t \geq 0$. Bearing in mind that $m_1 = p$, all assumptions in Theorem 2.1 are satisfied, which yields the conclusion. ∎

REMARK 2.13. We note that in Corollary 2.12, replacing $\xi \to +\infty$ and $t \to +\infty$ with $\xi \to 0^+$ and $t \to 0^+$, respectively, by the same argument, applying Theorem 2.7, for every $\lambda \in]\lambda_5, \lambda_6[$, the problem (2.17) has a sequence of weak solutions which strongly converges to 0 in $W_0^{1,2}([a,b])$.

We present two examples to illustrate these results:

EXAMPLE 2.14. Let α and β be positive constants such that $\beta + \alpha < b - a$. Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x,t)=\begin{cases} g(x)t^4\big(5+10\sin^2(\ln t)+2\sin(2\ln t)\big) \\ \qquad \text{if } (x,t)\in[a,b]\times\,]0,+\infty[, \\ 0 \quad \text{if } (x,t)\in[a,b]\times\,]-\infty,0[, \end{cases}$$

where $g:[a,b]\to\mathbb{R}$ is a non-negative continuous function. Consider the problem

$$\begin{cases} -\Big(1+\int_a^b |u'(x)|^2\,dx\Big)u''=\lambda f(x,u), \\ u(a)=u(b)=0. \end{cases} \tag{2.18}$$

A direct calculation yields

$$F(x,t)=\begin{cases} g(x)t^5(1+2\sin^2(\ln t)) & \text{if } (x,t)\in[a,b]\times\,]0,+\infty[, \\ 0 & \text{if } (x,t)\in[a,b]\times\,]-\infty,0[. \end{cases}$$

Put

$$a_k=\begin{cases} k & \text{if } k \text{ is even}, \\ e^{-k\pi} & \text{if } k \text{ is odd}, \end{cases} \quad \text{and} \quad b_k=e^{k\pi} \quad \text{for every } k\in\mathbb{N}.$$

Then

$$\lim_{k\to\infty}\frac{\int_a^b \sup_{|t|\le a_k}F(x,t)\,dx}{a_k^2}=\begin{cases} 0 & \text{if } k \text{ is odd}, \\ +\infty & \text{if } k \text{ is even}, \end{cases}$$

and

$$\limsup_{k\to\infty}\frac{\int_{a+\alpha}^{b-\beta}F(x,t)\,dx}{b_k^2\frac{\alpha+\beta}{\alpha\beta}+\frac{1}{2}b_k^4\big(\frac{\alpha+\beta}{\alpha\beta}\big)^2}=+\infty.$$

So,

$$\liminf_{\xi\to+\infty}\frac{\int_a^b \sup_{|t|\le\xi}F(x,t)\,dx}{\xi^2}=0, \qquad \limsup_{t\to+\infty}\frac{\int_{a+\alpha}^{b-\beta}F(x,t)\,dx}{t^2\frac{\alpha+\beta}{\alpha\beta}+\frac{1}{2}t^4\big(\frac{\alpha+\beta}{\alpha\beta}\big)^2}=+\infty.$$

Hence, all assumptions of Corollary 2.12 with $\mu=0$ are satisfied. So, for every $\lambda\in\,]0,+\infty[$ the problem (2.18) has an unbounded sequence of weak solutions in $W_0^{1,2}([a,b])$.

Example 2.15. Let α and β be positive constants such that $\beta+\alpha<b-a$. Let $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x,t)=\begin{cases} h(x)t\big(2-2\sin(\ln(|t|))-\cos(\ln(|t|))\big) \\ \qquad \text{if } (x,t)\in[a,b]\times(\mathbb{R}-\{0\}), \\ 0 \quad \text{if } (x,t)\in[a,b]\times\{0\}, \end{cases}$$

where $h:[a,b]\to\mathbb{R}$ is a non-negative continuous function. A direct calculation shows

$$F(x,t)=\begin{cases} h(x)t^2\big(1-\sin(\ln(|t|))\big) & \text{if } (x,t)\in[a,b]\times(\mathbb{R}-\{0\}), \\ 0 & \text{if } (x,t)\in[a,b]\times\{0\}, \end{cases}$$

and so

$$\liminf_{\xi\to 0^+}\frac{\int_a^b \sup_{|t|\le\xi} F(x,t)\,dx}{\xi^2}=0,\qquad \limsup_{t\to 0^+}\frac{\int_{a+\alpha}^{b-\beta} F(x,t)\,dx}{t^2\frac{\alpha+\beta}{\alpha\beta}+\frac{1}{2}t^4\left(\frac{\alpha+\beta}{\alpha\beta}\right)^2}=+\infty.$$

Hence, taking into account Remark 2.13 with $\mu=0$, we see that for all $\lambda\in\,]0,+\infty[$ the problem (2.18) has a sequence of weak solutions which strongly converges to 0 in $W_0^{1,2}([a,b])$.

REMARK 2.16. We point out that the result of Example 2.15 holds with f as given in [4, Example 3.1] for every $\lambda\in\,]0,+\infty[$. Indeed, by the same reasoning as in [4, Example 3.1], one has

$$\liminf_{\xi\to 0^+}\frac{\int_0^1 \sup_{|t|\le\xi} F(x,t)\,dx}{\xi^2}=0,\qquad \limsup_{t\to 0^+}\frac{\int_{\alpha}^{1-\beta} F(x,t)\,dx}{t^2\frac{\alpha+\beta}{\alpha\beta}+\frac{1}{2}t^4\left(\frac{\alpha+\beta}{\alpha\beta}\right)^2}=+\infty.$$

Finally, we give the following consequence of the main result:

COROLLARY 2.17. *Let $g_1:[a,b]\to\mathbb{R}$ be a non-negative continuous function, and denote $G_1(t)=\int_0^t g_1(\xi)\,d\xi$ for all $t\in\mathbb{R}$. Assume that there exist positive constants α, β, p and q with $\beta+\alpha<b-a$ such that*

(D1) $\displaystyle\liminf_{\xi\to+\infty}\frac{G_1(\xi)}{\xi^2}<+\infty$;

(D2) $\displaystyle\limsup_{t\to+\infty}\frac{G_1(t)}{pt^2\frac{\alpha+\beta}{\alpha\beta}+\frac{q}{2}t^4\left(\frac{\alpha+\beta}{\alpha\beta}\right)^2}=+\infty.$

Then, for every $\alpha_i\in L^1([a,b])$ for $1\le i\le n$, with $\min_{x\in[a,b]}\{\alpha_i(x);\,1\le i\le n\}\ge 0$ and with $\alpha_1\ne 0$, and for any non-negative continuous functions $g_i:\mathbb{R}\to\mathbb{R}$ for $2\le i\le n$ satisfying

$$\max\Big\{\sup_{\xi\in\mathbb{R}} G_i(\xi);\ 2\le i\le n\Big\}\le 0$$

and

$$\min\Big\{\liminf_{\xi\to+\infty}\frac{G_i(\xi)}{\xi^2};\ 2\le i\le n\Big\}>-\infty$$

where $G_i(t)=\int_0^t g_i(\xi)\,d\xi$ for all $t\in\mathbb{R}$ and $2\le i\le n$, for each λ in

$$\left]0,\ \frac{\frac{2p}{b-a}}{\left(\int_a^b\alpha_1(x)\,dx\right)\liminf_{\xi\to+\infty}\frac{G_1(\xi)}{\xi^2}}\right[,$$

the problem

$$\begin{cases}-\Big(p+q\int_a^b|u'(x)|^2\,dx\Big)u''=\lambda\sum_{i=1}^n\alpha_i(x)g_i(u),\\ u(a)=u(b)=0,\end{cases}\tag{2.19}$$

has an unbounded sequence of weak solutions in $W_0^{1,2}([a,b])$.

Proof. Set $f(x,t)=\sum_{i=1}^{n}\alpha_i(x)g_i(t)$ for all $(x,t)\in[a,b]\times\mathbb{R}$. Assumption (D2) along with the condition

$$\min\left\{\liminf_{\xi\to+\infty}\frac{G_i(\xi)}{\xi^2};\ 2\le i\le n\right\}>-\infty$$

ensures

$$\limsup_{t\to+\infty}\frac{\int_{a+\alpha}^{b-\beta}F(x,t)\,dx}{pt^2\frac{\alpha+\beta}{\alpha\beta}+\frac{q}{2}t^4\left(\frac{\alpha+\beta}{\alpha\beta}\right)^2}=\limsup_{t\to+\infty}\frac{\sum_{i=1}^{n}G_i(\xi)\int_{a+\alpha}^{b-\beta}\alpha_i(x)\,dx}{pt^2\frac{\alpha+\beta}{\alpha\beta}+\frac{q}{2}t^4\left(\frac{\alpha+\beta}{\alpha\beta}\right)^2}=+\infty.$$

Moreover, Assumption (D1) together with the condition

$$\max\left\{\sup_{\xi\in\mathbb{R}}G_i(\xi);\ 2\le i\le n\right\}\le 0$$

implies

$$\liminf_{\xi\to+\infty}\frac{\int_a^b\sup_{|t|\le\xi}F(x,t)\,dx}{\xi^2}\le\Big(\int_a^b\alpha_1(x)\,dx\Big)\liminf_{\xi\to+\infty}\frac{G_1(\xi)}{\xi^2}<+\infty.$$

Hence, applying Corollary 2.12 with $\mu=0$ we obtain the result. ∎

REMARK 2.18. In Corollary 2.17, replacing $t\to+\infty$ and $\xi\to+\infty$ with $t\to 0^+$ and $\xi\to 0^+$, respectively, by the same reasoning we find that for every λ in

$$\left]0,\ \frac{\frac{2p}{b-a}}{\left(\int_a^b\alpha_1(x)\,dx\right)\liminf_{\xi\to 0^+}\frac{G_1(\xi)}{\xi^2}}\right[$$

the problem (2.19) has a sequence of weak solutions which strongly converges to 0 in $W_0^{1,2}([a,b])$.

REMARK 2.19. Our statements mainly depend upon the choice of the test function w_k. With our choice of $w_k=(w_{1k},\dots,w_{nk})$ with w_{ik} given in (2.8) we have the present structure of the results. Other candidates for w_k can be considered to have other versions of the statements.

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