ANNALES
POLONICI MATHEMATICI
99.2 (2010)

Weighted θ-incomplete pluripotential theory

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Abstract. Weighted pluripotential theory is a rapidly developing area; and Callaghan [Ann. Polon. Math. 90 (2007)] recently introduced θ-incomplete polynomials in $\mathbb{C}^n$ for $n > 1$. In this paper we combine these two theories by defining weighted θ-incomplete pluripotential theory. We define weighted θ-incomplete extremal functions and obtain a Siciak–Zahariuta type equality in terms of θ-incomplete polynomials. Finally we prove that the extremal functions can be recovered using orthonormal polynomials and we demonstrate a result on strong asymptotics of Bergman functions in the spirit of Berman [Indiana Univ. Math. J. 58 (2009)].

1. Introduction. The theory of θ-incomplete polynomials in $\mathbb{C}^n$ for $n > 1$ was recently developed by Callaghan [9]. It has many applications in approximation theory. He also defined interesting extremal functions in terms of θ-incomplete polynomials and related plurisubharmonic functions.

This paper has three goals. The first one is to further develop the θ-incomplete pluripotential theory of Callaghan. The second goal is to combine this theory with weighted pluripotential theory and get a unified theory by defining weighted θ-incomplete pluripotential theory in $\mathbb{C}^n$. If $\theta = 0$, we get weighted pluripotential theory, and for the weight $w = 1$, we get θ-incomplete pluripotential theory. Finally we show that extremal functions in these settings can be recovered asymptotically using orthonormal polynomials.

In this section we recall some definitions and major results of weighted pluripotential theory and we recall Berman's paper [2] which is a special case of weighted pluripotential theory. Our initial goal was to study Berman's recent work on globally defined weights within the framework of θ-incomplete pluripotential theory. We were able to prove many results for admissible weights defined on closed subsets of $\mathbb{C}^n$.

2010 *Mathematics Subject Classification*: Primary 32U35.
Key words and phrases: weighted pluripotential theory, θ-incomplete pluripotential theory, weighted Bergman kernels.

DOI: 10.4064/ap99-2-1

In the second section we recall some important results of θ-incomplete pluripotential theory. We improve a result of Callaghan and we extend a result of Bloom and Shiffman [7] to the θ-incomplete extremal function $V_{K,\theta}$ associated to a compact set K for $0 \leq \theta < 1$.

In the third section we work on closed subsets of $\mathbb{C}^n$. We define the weighted θ-incomplete extremal function $V_{K,Q,\theta}$ for a closed set K and an admissible weight function w and we give various properties of this extremal function. We also show that $V_{K,Q,\theta}$ can be obtained via taking the supremum of θ-incomplete polynomials whose weighted norm is less than or equal to 1 on K, generalizing the analogous result for $V_{K,\theta}$ (unweighted case) from the previous section. In particular we state analogous results in the case of global weights.

In the last section we recall the Bernstein–Markov property relating the sup norms and $L^2(\mu)$ norms of polynomials on a compact set K with measure μ. We define a version of the Bernstein–Markov property for θ-incomplete polynomials in the weighted setting. Then we prove results on asymptotics of orthonormal polynomials to extremal functions in the θ-incomplete and weighted setting. Finally in Theorem 4.8, we prove a result on strong asymptotics of Bergman functions analogous to the main theorem in [2].

1.1. Weighted pluripotential theory. We give some basic definitions from weighted pluripotential theory. A good reference is Saff and Totik's book [14] for $n = 1$ and Thomas Bloom's Appendix B of [14] for $n > 1$.

Let K be a nonpluripolar closed subset of $\mathbb{C}^n$. An upper semicontinuous function $w : K \to [0, \infty)$ is called an *admissible weight function* on K if

(i) the set $\{z \in K \mid w(z) > 0\}$ is not pluripolar,
(ii) if K is unbounded, then $|z|w(z) \to 0$ as $|z| \to \infty$, $z \in K$.

We define $Q = Q_w = -\log w$, and we will use Q and w interchangeably.

The weighted pluricomplex extremal function of K with respect to Q is defined as

$$V_{K,Q}(z) := \sup\{u(z) \mid u \in L,\ u \leq Q \text{ on } K\}, \tag{1.1}$$

where the *Lelong class* L is defined as

$$L := \{u \mid u \text{ is plurisubharmonic on } \mathbb{C}^n,\ u(z) \leq \log^+ |z| + C\}, \tag{1.2}$$

where C depends on u.

We recall that the *upper semicontinuous regularization* of a function v is defined by $v^*(z) := \limsup_{w\to z} v(w)$ and it is well known that the upper semicontinuous regularization of $V_{K,Q}$ is plurisubharmonic and in L^+ where

$$L^+ := \{u \in L \mid \log^+ |z| + C \leq u(z)\},$$

where C depends on u. By Lemma 2.3 of Bloom's Appendix B of [14], the support, S_w, of $(dd^c V^*_{K,Q})^n$ is a subset of $S^*_w := \{z \in K \mid V^*_{K,Q}(z) \geq Q(z)\}$.

Here $dd^cv = 2i\partial\bar{\partial}v$ and $(dd^cv)^n$ is the complex Monge–Ampère operator defined by $(dd^cv)^n = dd^cv \wedge \cdots \wedge dd^cv$ for plurisubharmonic functions which are C^2. For the cases considered in this paper see [12, 10] for the details of the definition.

A set E is called *pluripolar* if $E \subset \{z \in \mathbb{C}^n \mid u(z) = -\infty\}$ for some plurisubharmonic function u. If a property holds everywhere except on a pluripolar set we will say it holds *quasi everywhere*.

1.2. A special case of weighted pluripotential theory. We recall some definitions from Berman's paper [2], where the weight is defined globally in $\mathbb{C}^n$. Let ϕ be a lower semicontinuous function, and $\phi(z) \geq (1+\varepsilon)\log|z|$ for $z \gg 1$ for some fixed $\varepsilon > 0$. The weighted extremal function is defined as

$$V_\phi(z) := \sup\{u(z) \mid u \in L \text{ and } u \leq \phi \text{ on } \mathbb{C}^n\}. \tag{1.3}$$

We define

$$S^*_\phi := \{z \in \mathbb{C}^n \mid V^*_\phi(z) \geq \phi(z)\}, \qquad S_\phi := \operatorname{supp}((dd^cV^*_\phi)^n).$$

This is a special case of weighted pluripotential theory with $K = \mathbb{C}^n$ and $Q = \phi$. Hence $S_\phi \subset S^*_\phi$.

Berman [2] studied the case where $\phi \in C^{1,1}(\mathbb{C}^n)$. In this case we define

$$D_\phi = \{z \in \mathbb{C}^n \mid V_\phi(z) = \phi(z)\}, \tag{1.4}$$

$$P = \{z \in \mathbb{C}^n \mid dd^c\phi(z) \text{ exists and is positive}\}. \tag{1.5}$$

We remark that D_ϕ is a compact set and $S_\phi \subset D_\phi$. By Proposition 2.1 of [2], if $\phi \in C^{1,1}(\mathbb{C}^n)$, then $V_\phi \in C^{1,1}(\mathbb{C}^n)$ and $(dd^cV_\phi)^n = (dd^c\phi)^n$ on $D_\phi \cap P$ almost everywhere as (n,n) forms with L^∞ coefficients.

Example 1.1. Let $\phi(z) = |z|^2$. Then

$$V_\phi(z) = \begin{cases} |z|^2 & \text{if } |z| \leq 1/\sqrt{2}, \\ \log|z| + \frac{1}{2} - \frac{1}{2}\log\frac{1}{2} & \text{if } |z| \geq 1/\sqrt{2}. \end{cases} \tag{1.6}$$

Clearly the plurisubharmonic function, V, on the right hand side is less than or equal to ϕ, hence $V \leq V_\phi$. On the other hand the support of the Monge–Ampère measure of V is the closed ball of radius $1/\sqrt{2}$ centered at the origin. Since any competitor, u, for the extremal function is less than or equal to $|z|^2$ on this closed ball, by the domination principle (see Appendix B of [14] or Theorem 2.1 below) $u \leq V$ on $\mathbb{C}^n$. Therefore $V_\phi \leq V$ and hence equality holds.

2. θ-Incomplete pluripotential theory. We recall the basic notions of θ-incomplete pluripotential theory from [9]. We fix $0 \leq \theta \leq 1$. A *θ-incomplete polynomial* in $\mathbb{C}^n$ is a polynomial of the form

$$P(z) = \sum_{|\alpha|=\lceil N\theta \rceil}^{N} c_\alpha z^\alpha, \tag{2.1}$$

where $\lceil x \rceil$ is the least integer greater than or equal to x.

The set of all θ-incomplete polynomials of the form (2.1) will be denoted by $\pi_{N,\theta}$. We remark that when $\theta = 0$, $\pi_{N,\theta}$ is the set of all polynomials of degree at most N; and when $\theta = 1$, $\pi_{N,\theta}$ is the set of homogeneous polynomials of degree N.

Related classes of plurisubharmonic functions are defined as follows (see [9] for details):

$$L_\theta = \{u \in L \mid u(z) \le \theta \log|z| + C \text{ for } |z| < 1\},$$
$$L_\theta^+ = \{u \in L_\theta \mid \max(\theta \log|z|, \log|z|) + C \le u(z) \text{ for all } z \in \mathbb{C}^n\},$$

where C depends on u.

We remark that if $P \in \pi_{N,\theta}$ then $N^{-1} \log|P| \in L_\theta$. Another observation is that if $\theta_1 \ge \theta_2$, then $L_{\theta_2} \subset L_{\theta_1}$.

The next theorem gives a domination principle for L_θ classes.

Theorem 2.1 ([9, Theorem 3.15]). *Let $0 \le \theta < 1$. If $u \in L_\theta$ and $v \in L_\theta^+$ and if $u \le v$ almost everywhere with respect to $(dd^c v)^n$, then $u \le v$ on $\mathbb{C}^n$.*

We remark that, for $0 < \theta < 1$, we have $u(0) = v(0) = -\infty$ and the origin is a distinguished point as it is charged by $(dd^c v)^n$.

Callaghan [9] defined the following extremal function for a set $E \subset \mathbb{C}^n$:

$$V_{E,\theta}(z) := \sup\{u(z) \mid u \in L_\theta \text{ and } u \le 0 \text{ on } E\}. \tag{2.2}$$

We will call it the *θ-incomplete extremal function* of E. Its upper semicontinuous regularization, $V_{E,\theta}^*$, is in L_θ^+ if E is bounded and nonpluripolar by Lemma 3.7 of [9]. Also if K is a regular compact set in $\mathbb{C}^n$, then $V_{K,\theta}^* = V_{K,\theta}$. Hence it is continuous except at $z = 0$. Recall that regular means that the extremal function of K, $V_K := V_{K,0}$, is continuous. We remark that $(dd^c V_{E,\theta}^*)^n$ is supported in $\overline{E} \cup \{0\}$ if $\theta > 0$.

We define the following functions for a compact set K. For $N \ge 1$ we let

$$\Phi_{K,\theta,N}(z) = \sup\{|f(z)| \mid f \in \pi_{N,\theta},\ \|f\|_K \le 1\},$$
$$\Phi_{K,\theta} = \sup_N (\Phi_{K,\theta,N})^{1/N}. \tag{2.3}$$

The next proposition shows that the supremum in (2.3) is actually a limit.

Proposition 2.2. *With the above notation we have*

$$\sup_N \frac{1}{N} \log \Phi_{K,\theta,N} = \lim_{N\to\infty} \frac{1}{N} \log \Phi_{K,\theta,N}.$$

Hence, by [9] *we have* $\lim_{N\to\infty} N^{-1} \log \Phi_{K,\theta,N} = V_{K,\theta}$.

Proof. First of all, we have $\Phi_{K,\theta,J}\Phi_{K,\theta,I} \le \Phi_{K,\theta,J+I}$ for all integers $I,J \ge 0$. For if $P(z) = \sum_{|\alpha|=\lceil\theta J\rceil}^{J} a_\alpha z^\alpha$ and $Q(z) = \sum_{\alpha=\lceil\theta I\rceil}^{I} b_\alpha z^\alpha$, then $PQ(z) = \sum_{\alpha=\lceil\theta J\rceil+\lceil\theta I\rceil}^{J+I} c_\alpha z^\alpha$ is in $\pi_{J+I,\theta}$, since $\lceil\theta J\rceil + \lceil\theta I\rceil \ge \lceil\theta(J+I)\rceil$.

By taking logarithms, we get

$$\log \Phi_{K,\theta,J} + \log \Phi_{K,\theta,I} \le \log \Phi_{K,\theta,J+I}. \tag{2.4}$$

Therefore, by Theorem 4.9.19 of [1], the limit $\lim_{N\to\infty} N^{-1}\log\Phi_{K,\theta,N}$ exists and equals $\sup_N N^{-1}\log\Phi_{K,\theta,N}$. ∎

In the next section we will extend this result to the weighted case. This proposition also fixes a gap in the proof of Theorem 8.2 in [8] and we will use it in the proof of Theorem 4.3.

Let K be a compact set in $\mathbb{C}^n$. For $0 \le \theta \le 1$, we define the following θ-incomplete hulls:

$$\widehat{K}_{\theta,\Omega} = \{z \in \Omega \mid |P_N(z)| \le \|P_N\|_K \text{ where } P_N \in \pi_{N,\theta}\}. \tag{2.5}$$

Again if $\Omega = \mathbb{C}^n$, then we will drop Ω and write $\widehat{K}_\theta$. Using Theorem 4.4 of [9], we have the following theorem.

Theorem 2.3. *Let $K \subset \mathbb{C}^n$ and $0 \le \theta \le 1$. Then $V_{K,\theta} = V_{\widehat{K},\theta}$.*

It is clear that for $\theta > 0$, the origin always belongs to $\widehat{K}_\theta$ for any set K, so $\widehat{K}_\theta$ is often larger than the usual polynomially convex hull $\widehat{K} := \widehat{K}_0$. It is also easy to see that $\widehat{K}_\theta = \{z \in \mathbb{C}^n \mid V_{K,\theta} \le 0\}$.

The following theorem extends a result of Bloom and Shiffman [7] to the θ-incomplete case.

Theorem 2.4. *Let $0 < \theta \le 1$ and K be a regular compact set in $\mathbb{C}^n$. Then*

$$\frac{1}{N}\log\Phi_{K,\theta,N} \to V_{K,\theta}$$

uniformly on compact subsets of $\mathbb{C}^n \setminus \{0\}$.

Proof. Let $\varepsilon > 0$. We will show that for every $a \in \mathbb{C}^n \setminus \{0\}$ there exist $\delta = \delta(a) > 0$ and $N_0 := N_0(a) \ge 1$ such that

$$1 \le \frac{\Phi(z)}{\Phi_N^{1/N}(z)} \le e^\varepsilon \qquad \text{if } |z-a| < \delta$$

for all $N \ge N_0$, where $\Phi := \Phi_{K,\theta}$, $\Phi_N := \Phi_{K,N,\theta}$. Since any compact subset of $\mathbb{C}^n \setminus \{0\}$ can be covered by finitely many balls of the form $B(a,\delta(a))$, we are done.

Without loss of generality we may assume that $a_1 \ne 0$ where $a = (a_1,\dots,a_n)$. We define $p_1(z) := z_1/c_0$ such that $\|p_1\|_K \le 1$. Hence, $\Phi_r(z) \ge |p_1^{\lceil\theta r\rceil}(z)|$ in $\mathbb{C}^n$ for all $r \ge 1$.

By Proposition 2.2, we choose $M \geq 1$ such that $\Phi(a)/\Phi_M^{1/M}(a) \leq e^{\varepsilon/4}$. Next we take $\delta > 0$ small enough that $\Phi(z)/\Phi(a) \leq e^{\varepsilon/4}$ and $\Phi_M^{1/M}(a)/\Phi_M^{1/M}(z) \leq e^{\varepsilon/4}$ if $|z-a| < \delta$. These are possible since Φ and $\Phi_M^{1/M}$ are continuous and nonzero on $\mathbb{C}^n \setminus \{0\}$.

Let $N \geq M$. There exist k, r such that $N = kM + r$ where k and r depend on M, and $r < M$.

Now we have

$$1 \leq \frac{\Phi(z)}{\Phi_N^{1/N}(z)} \leq \frac{\Phi(z)}{(\Phi_M^k(z))^{1/N}} \frac{1}{\Phi_r^{1/N}(z)} = \frac{\Phi(z)}{(\Phi_M^{1/M}(z))^{kM/N}} \frac{1}{\Phi_r^{1/N}(z)},$$

which equals

$$\frac{\Phi(z)}{\Phi_M^{1/M}(z)} \frac{(\Phi_M^{1/M}(z))^{r/N}}{\Phi_r^{1/N}(z)},$$

which is less than or equal to

$$\frac{\Phi(z)}{\Phi(a)} \frac{\Phi(a)}{\Phi_M^{1/M}(a)} \frac{\Phi_M^{1/M}(a)}{\Phi_M^{1/M}(z)} \frac{(\Phi(z))^{r/N}}{\Phi_r^{1/N}(z)},$$

for all $z \in \mathbb{C}^n \setminus \{0\}$. Thus, for $z \in B(a,\delta)$ and $N \geq N_0$ we have

$$\frac{\Phi(z)}{\Phi(a)} \frac{\Phi(a)}{\Phi_M^{1/M}(a)} \frac{\Phi_M^{1/M}(a)}{\Phi_M^{1/M}(z)} \frac{(\Phi(z))^{r/N}}{\Phi_r^{1/N}(z)} < e^{\varepsilon},$$

where N_0 is chosen so large that $(m_1/m_0)^{M/N} < e^{\varepsilon/4}$ for $N > N_0$. Here $m_1 := \|\Phi\|_{B(a,\delta)}$ and $m_0 := \min(m_1, \inf\{|p_1(z)| \mid z \in B(a,\delta)\})$. ∎

Note that for $\theta = 0$, an analogous proof holds for all $a \in \mathbb{C}^n$. Hence, it gives uniform convergence on all compact subsets of $\mathbb{C}^n$, which is the original result of Bloom and Shiffman.

3. Weighted θ-incomplete pluripotential theory. In this section, we define and develop two weighted versions of θ-incomplete pluripotential theory. The first one is the θ-incomplete version of the weighted pluripotential theory on closed subsets of $\mathbb{C}^n$ and the second one is the θ-incomplete version of the special case of weighted pluripotential theory studied in [2]. As in the $\theta = 0$ case, the second version is a special case of the first.

Let K be a closed set in $\mathbb{C}^n$ and w be an admissible weight on K. We define

$$V_{K,Q,\theta}(z) := \sup\{u(z) \mid u \in L_\theta,\ u \leq Q \text{ on } K\}. \tag{3.1}$$

We remark that $V_{K,Q,\theta_1} \leq V_{K,Q,\theta_2}$ if $\theta_1 > \theta_2$. The $\theta = 0$ case gives the classical weighted pluripotential theory. Following Siciak [15], it can be shown that $V_{K,Q,\theta} = V_{K,Q,\theta}^*$, so that $V_{K,Q,\theta}$ is continuous on $\mathbb{C}^n \setminus \{0\}$, for K locally regular and Q continuous. Recall that K *locally regular* means for all $a \in K$, $K \cap \overline{B(a,r)}$ is regular for all $r > 0$, where $B(a,r) := \{z \in \mathbb{C}^n \mid |z-a| < r\}$.

Comparing the defining families, we get the following obvious inequalities.

PROPOSITION 3.1. *Let $K_1 \subset K_2$ and let w be a function defined on K_2 which is an admissible weight on both K_1 and K_2. Then $V_{K_1,Q,\theta} \geq V_{K_2,Q,\theta}$.*

Using (ii) in the definition of admissibility from Section 1.1, we show that $V_{K,Q,\theta}$ coincides with the weighted θ-incomplete extremal function of a compact subset of K.

LEMMA 3.2. *Let K be a closed unbounded set in $\mathbb{C}^n$ and w be an admissible weight function on K. Then $V_{K_\rho,Q,\theta} = V_{K,Q,\theta}$ for some $\rho > 0$ where $K_\rho = \{z \in K \mid |z| \leq \rho\}$.*

Proof. Since $V^*_{K_\rho,Q,\theta} \in L$, there exist C and ρ such that

$$V_{K_\rho,Q,\theta}(z) \leq V^*_{K_\rho,Q,\theta}(z) \leq \log|z| + C \quad \text{for } |z| > \rho.$$

Now by the second condition of admissibility we may choose ρ large enough that

$$Q(z) - \log|z| \geq C + 1 \quad \text{for } z \in K \setminus K_\rho.$$

If $u \in L_\theta$ and $u \leq Q$ on K_ρ, so that $u \leq V_{K_\rho,Q,\theta}$, by the above inequalities we get $u \leq Q$ on K. Hence we get $V_{K_\rho,Q,\theta} \leq V_{K,Q,\theta}$. The other inequality is given by Proposition 3.1, which gives the equality. ∎

PROPOSITION 3.3. *Let K be a closed subset of $\mathbb{C}^n$ and let w be an admissible weight function on K. Then $V^*_{K,Q,\theta} \in L^+_\theta$.*

Proof. The case $\theta = 0$ is classical and well known. For $0 < \theta \leq 1$ we will follow the proof of Lemma 3.7 of [9].

Since $V^*_{K,Q,\theta} \leq V^*_{K,Q}$ and $V^*_{K,Q} \in L^+$, we have $V^*_{K,Q,\theta} \in L$.

Next we show that $V^*_{K,Q,\theta} \in L_\theta$. Let $M := \sup_{z\in B(0,1)} V^*_{K,Q,\theta}(z)$ and u be in the defining class for $V_{K,Q,\theta}$. Then $\theta^{-1}(u - M) \leq 0$ on $B(0,1)$. Hence it is a competitor for the pluricomplex Green function of the unit ball $B(0,1)$ with logarithmic pole at the origin. The pluricomplex Green function of a bounded domain Ω with logarithmic pole at $a \in \Omega$ is defined by

$$g_\Omega(z,a) := \sup\{u(z) \mid u \in \mathrm{PSH}(\Omega),\ u \leq 0 \text{ and } u(z) - \log|z-a| \leq C \text{ as } z \to a\}, \tag{3.2}$$

and $g_{B(0,1)}(z,0) = \log|z|$. Hence $\theta^{-1}(u-M) \leq \log|z|$ on the unit ball. Since u is arbitrary, we get $V^*_{K,Q,\theta}(z) \leq \theta\log|z| + M$ on $B(0,1)$. Thus $V^*_{K,Q,\theta} \in L_\theta$.

By Lemma 3.2, we may assume $K \subset B(0,R)$ for some R. Let $A := \sup_{z\in B(0,R)}(\theta\log|z| - Q(z))$. Then

$$u(z) = \max(\theta\log|z|, \log|z|) - A$$

is a competitor for the extremal function $V_{K,Q,\theta}$ and $u \in L_\theta^+$, which implies that $V_{K,Q,\theta}^* \in L_\theta^+$. ∎

We define the following sets:

$$S_{K,Q,\theta}^* := \{z \in K \mid V_{K,Q,\theta}^*(z) \geq Q(z)\},$$
$$S_{K,Q,\theta} := \operatorname{supp}((dd^c V_{K,Q,\theta}^*)^n).$$

LEMMA 3.4. *Let K be closed in $\mathbb{C}^n$ and let w be an admissible weight on K. Then $S_{K,Q,\theta} \subset S_{K,Q,\theta}^* \cup \{0\}$ if $0 < \theta \leq 1$ and $S_{K,Q,\theta} \subset S_{K,Q,\theta}^*$ if $\theta = 0$.*

Proof. The classical case, i.e., when $\theta = 0$, is Lemma 2.3 of Appendix B of [14]. Therefore, we assume $0 < \theta \leq 1$. Let z_0 be a point in $K \setminus \{0\}$ such that $V_{K,Q,\theta}^*(z_0) < Q(z_0) - \varepsilon$ for some positive ε. We will show that $V_{K,Q,\theta}^*$ is maximal in a neighborhood of z_0, i.e $(dd^c V_{K,Q,\theta}^*)^n = 0$ there.

Since Q is lower semicontinuous, the set $\{z \in K \mid Q(z) > Q(z_0) - \varepsilon/2\}$ is open relative to K. Similarly, $\{z \in \mathbb{C}^n \mid V_{K,Q,\theta}^*(z) < V_{K,Q,\theta}^*(z_0) + \varepsilon/2\}$ is open. Thus we may find a ball of radius r around z_0 such that

$$\sup_{z \in B(z_0,r)} V_{K,Q,\theta}^*(z) < \inf_{z \in B(z_0,r) \cap K} Q(z)$$

and $0 \notin B(z_0, r)$.

By Theorem 1.3 of Appendix B in [14], we can find a plurisubharmonic function u with $u \geq V_{K,Q,\theta}^*$ on $B(z_0, r)$, $u = V_{K,Q,\theta}^*$ on $\mathbb{C}^n \setminus B(z_0, r)$, and u maximal on $B(z_0, r)$. Then $u \leq V_{K,Q,\theta}^*$ because

$$u(z) \leq \sup_{z \in B(z_0,r)} V_{K,Q,\theta}^*(z) < \inf_{z \in B(z_0,r) \cap K} Q(z) \quad \text{for all } z \in B(z_0, r).$$

Since $B(z_0, r) \cap \{0\}$, we have $u \in L_\theta$. Hence $u \equiv V_{K,Q,\theta}^*$. Therefore, $V_{K,Q,\theta}^*$ is maximal in a neighborhood of z_0. Hence z_0 is not in $S_{K,Q,\theta}$. ∎

A special case is when the admissible weights are globally defined. Let $\phi : \mathbb{C}^n \to \mathbb{R}$ be an admissible weight function. Generalizing the case of [2], we define weighted θ-incomplete extremal functions by

$$V_{\phi,\theta}(z) = \sup\{u(z) \mid u \in L_\theta \text{ and } u \leq \phi\} \quad \text{for } 0 \leq \theta \leq 1. \tag{3.3}$$

Observe that $V_{\phi,\theta}^* = V_{\phi,\theta}$ if ϕ is continuous, for in this case $V_{\phi,\theta}^* \leq \phi$ on $\mathbb{C}^n$ so that $V_{\phi,\theta}^* \leq V_{\phi,\theta}$. We also remark that $\theta = 0$ gives $V_{\phi,0} = V_\phi$, and $V_{\phi,\theta_1} \leq V_{\phi,\theta_2}$ if $\theta_1 > \theta_2$ since $L_{\theta_1} \subset L_{\theta_2}$.

We define the following sets:

$$D_{\phi,\theta} := \{z \in \mathbb{C}^n \mid V_{\phi,\theta}^*(z) \geq \phi(z)\} \quad \text{and} \quad S_{\phi,\theta} := \operatorname{supp}((dd^c V_{\phi,\theta}^*)^n).$$

If $\theta = 0$, we will write $D_{\phi,0} = D_\phi$ and $S_{\phi,0} = S_\phi$. If ϕ is continuous, then $V_{\phi,\theta}$ is continuous and we have

$$D_{\phi,\theta} = \{z \in \mathbb{C}^n \mid V_{\phi,\theta}(z) = \phi(z)\}.$$

If ϕ is a globally defined admissible weight function, then we define $K := D_{\phi,\theta}$ and $Q := \phi|_K$. Clearly, $V^*_{\phi,\theta} \le Q$ quasi everywhere in K so $V^*_{\phi,\theta} \le V^*_{K,Q,\theta}$.

Conversely, on K, $V_{K,Q,\theta} \le Q = \phi = V_{\phi,\theta}$ quasi everywhere. Since $(dd^c V^*_{\phi,\theta})^n$ is supported on $K \cup \{0\}$, by Theorem 2.1 we have $V^*_{K,Q,\theta} \le V^*_{\phi,\theta}$. Hence $V^*_{K,Q,\theta} = V^*_{\phi,\theta}$. This shows that we may reduce the global weighted situation to the compact case by considering the sets $D_{\phi,\theta}$.

As a consequence of the above definitions, Lemma 3.4 and earlier results of this section we have the following corollary.

Corollary 3.5. *Let ϕ be a globally defined admissible weight. Then:*

(i) $S_{\phi,\theta} = \operatorname{supp}((dd^c V^*_{\phi,\theta})^n) \subset D_{\phi,\theta} \cup \{0\}$ *if* $\theta > 0$, *and* $\operatorname{supp}((dd^c V^*_{\phi})^n) \subset D_\phi$ *for* $\theta = 0$,

(ii) $D_{\phi,1} \subset D_{\phi,\theta_1} \subset D_{\phi,\theta_2} \subset D_{\phi,0} = D_\phi$ *when* $\theta_1 > \theta_2$,

(iii) $V_{\phi,\theta}$ *is in* L^+_θ *for* $0 \le \theta \le 1$,

(iv) *if* $u \in L_\theta$ *and* $u \le \phi$ *on* $D_{\phi,\theta}$ *then* $u \le V_{\phi,\theta}$.

The next lemma shows the monotonicity of the extremal functions under increasing and decreasing θ.

Lemma 3.6. *Let $K \subset \mathbb{C}^n$ be a closed set and let w be an admissible weight on K. For $0 \le \theta_0 < 1$, $V^*_{K,Q,\theta}$ increases to V^*_{K,Q,θ_0} quasi everywhere as $\theta \searrow \theta_0$, and $V^*_{K,Q,\theta}$ decreases to V^*_{K,Q,θ_0} as $\theta \nearrow \theta_0$.*

Proof. The last statement is clear, thus we consider $\theta \searrow \theta_0$. Clearly, we have monotonicity of the $V^*_{K,Q,\theta}$. Since $V^*_{K,Q,\theta}$ is bounded above by V^*_{K,Q,θ_0}, $V^*_{K,Q,\theta}$ increases to a function, v, whose upper semicontinuous regularization v^* is plurisubharmonic and again bounded above by V^*_{K,Q,θ_0}.

Since $V^*_{K,Q,\theta} \in L^+_\theta$, we have $V^*_{K,Q,\theta}(z) \ge \max(\theta \log|z|, \log|z|) + M_\theta$ where M_θ is a constant depending on θ. As $\theta \searrow \theta_0$, we get $v^* \in L^+_{\theta_0}$ since $v^* \le V^*_{K,Q,\theta_0}$. Also by monotonicity, $(dd^c V^*_{K,Q,\theta})^n \to (dd^c v^*)^n$ weak*.

We write $S := \operatorname{supp}((dd^c v^*)^n) \setminus \{0\}$ and $S' := \{z \in K \mid v^*(z) \ge Q(z)\}$. By lower semicontinuity of Q, and upper semicontinuity of v^*, the set S' is closed. Next we will show that $v^* \ge Q$ on S by showing that $S \subset S'$.

Since $(dd^c V^*_{K,Q,\theta})^n \to (dd^c v^*)^n$, we have $S \subset \overline{\bigcup_{\theta>\theta_0} S_{K,Q,\theta}} \setminus \{0\}$. By Lemma 3.4,

$$\bigcup_{\theta>\theta_0} S_{K,Q,\theta} \setminus \{0\} \subset \bigcup_{\theta>\theta_0} S^*_{K,Q,\theta} \setminus \{0\} \subset \{z \in K \mid v(z) \ge Q(z)\} \subset S'.$$

Since S' is closed, it follows that $\overline{\bigcup_{\theta>\theta_0} S_{K,Q,\theta_0}} \setminus \{0\} \subset S'$. Therefore, $S \subset S'$. Since $V^*_{K,Q,\theta_0} \le Q$ quasi everywhere on K and $(dd^c v^*)^n$ does not charge pluripolar sets except the origin, we have $V^*_{K,Q,\theta_0} \le v^*$ almost everywhere with respect to $(dd^c v^*)^n$ on the support of $(dd^c v^*)^n$. Here we recall that if $\theta > 0$, then $V^*_{K,Q,\theta_0}(0) = v^*(0) = -\infty$. Therefore, by the domination principle (Theorem 2.1), $V^*_{K,Q,\theta_0} \le v^*$ on $\mathbb{C}^n$, so that $V^*_{K,Q,\theta_0} = v^*$. ∎

COROLLARY 3.7. *Let ϕ be a globally defined admissible weight. Let $0 \le \theta_0 < 1$. Then $V^*_{\phi,\theta}$ increases to V^*_{ϕ,θ_0} quasi everywhere as $\theta \searrow \theta_0$, and $V^*_{\phi,\theta}$ decreases to V^*_{ϕ,θ_0} as $\theta \nearrow \theta_0$.*

The following example illustrates the above corollary.

EXAMPLE 3.8. Let $\phi(z) = |z|^2$. Then for $0 < \theta < 1$ we have

$$V_{\phi,\theta}(z) = \begin{cases} \theta \log|z| + \frac{\theta}{2} - \frac{\theta}{2}\log\frac{\theta}{2} & \text{if } |z| < \sqrt{\theta/2}, \\ |z|^2 & \text{if } \sqrt{\theta/2} \le |z| \le \sqrt{1/2}, \\ \log|z| + \frac{1}{2} - \frac{1}{2}\log\frac{1}{2} & \text{if } |z| \ge \sqrt{1/2}. \end{cases}$$

If $\theta = 1$, we get

$$V_{\phi,\theta}(z) = V_{\phi,1}(z) = \log|z| + \frac{1}{2} - \frac{1}{2}\log\frac{1}{2}.$$

We had given $V_{\phi,0}$ earlier in (1.6).

Note that we have $D_{\phi,\theta} = \overline{B(0,1/\sqrt{2})} \setminus B(0,\sqrt{\theta/2})$, which increases to $\overline{B(0,1/\sqrt{2})} \setminus \{0\}$ as θ decreases to 0.

We define the following notions. Let $K \subset \mathbb{C}^n$ be compact and w be an admissible weight on K. We define

$$\Phi^N_{K,Q,\theta}(z) := \sup\{|P(z)|^{1/N} \mid \|w^N P_N\|_K \le 1 \text{ where } P_N \in \pi_{N,\theta}\}, \tag{3.4}$$

$$\Phi_{K,Q,\theta} := \sup_N \Phi^N_{K,Q,\theta} = \lim_{N\to\infty} \Phi^N_{K,Q,\theta}. \tag{3.5}$$

We can see that the supremum is actually a limit by following the proof of Proposition 2.2.

THEOREM 3.9. *Let $0 \le \theta \le 1$. Let $K \subset \mathbb{C}^n$ be a compact set and w be a continuous admissible weight on K. Then $V_{K,Q,\theta} = \log \Phi_{K,Q,\theta}$.*

Proof. Let $P_N \in \pi_{N,\theta}$ satisfy $\|w^N P_N\|_K \le 1$. Then

$$\frac{1}{N}\log|P_N(z)| \le Q(z) \quad \text{on } K.$$

Hence we get

$$\log \Phi_{K,Q,\theta} \le V_{K,Q,\theta}. \tag{3.6}$$

The rest of the proof essentially follows the proof of Callaghan [9]. We will modify the last step using a result of Brelot–Cartan instead of the Hartogs lemma.

We fix $\varepsilon > 0$ such that $\theta + \varepsilon < 1$. Let $u \in L_{\theta+\varepsilon}$ and $u \le Q$ on K. By Theorem 2.9 of Appendix B of [14], we have

$$u(z) = \lim_{j\to\infty} \frac{1}{N_j} \max_{1\le k\le t_j} \log |P_{k,j}(z)|,$$

where the sequence N_j is decreasing and each $P_{k,j}$ is a polynomial of degree at most N_j. Here t_j is a finite number depending on j.

As in [9], we write

$$P_{k,j}(z) := \sum_{|\alpha|=0}^{N_j} c_{\alpha,k,j} z^\alpha, \qquad P'_{k,j}(z) := \sum_{|\alpha|=0}^{\lfloor N_j\theta\rfloor} c_{\alpha,k,j} z^\alpha,$$

where $\lfloor x\rfloor$ is the largest integer less than or equal to x.

We remark that $P_{k,j} - P'_{k,j}$ is a θ-incomplete polynomial. By Callaghan's asymptotic estimates, we get

$$u(z) = \lim_{j\to\infty} \frac{1}{N_j} \max_{1\le k\le t_j} \log |P_{k,j}(z) - P'_{k,j}(z)|$$

pointwise on $\mathbb{C}^n$.

By Theorem 3.4.3(c) of [13], for $\varepsilon_1 > 0$, there exists j_1 such that for $j \ge j_1$ we have

$$\frac{1}{N_j} \max_{1\le k\le t_j} \log |P_{k,j}(z) - P'_{k,j}(z)| \le Q + \varepsilon_1 \quad \text{on } K,$$

since Q is continuous. Now we have

$$u(z) = \lim_{j\to\infty} \frac{1}{N_j} \max_{1\le k\le t_j} \log |P_{k,j}(z) - P'_{k,j}(z)| \le \log \Phi_{K,Q,\theta}(z) + \varepsilon_1$$

for any ε_1 and therefore $u(z) \le \log \Phi_{K,Q,\theta}(z)$. Hence

$$V_{K,Q,\theta+\varepsilon}(z) \le \log \Phi_{K,Q,\theta}(z).$$

By Lemma 3.6, as $\varepsilon \to 0$ we get

$$V_{K,Q,\theta}(z) \le \log \Phi_{K,Q,\theta}(z). \tag{3.7}$$

Combining (3.7) with (3.6), we get the desired result. ∎

Note that if $\theta = 0$, we recover

$$V_{K,Q} = \log \Phi_{K,Q} \quad \text{where} \quad \Phi_{K,Q} := \Phi_{K,Q,0}. \tag{3.8}$$

Corollary 3.10. *Let $0 \le \theta < 1$. Let ϕ be a globally defined continuous admissible weight. Then $V_{\phi,\theta} = \log \Phi_{\phi,\theta}$, where*

$$\Phi^N_{\phi,\theta}(z) := \sup\{|P(z)|^{1/N} \mid \|e^{-N\phi} P_N\|_{D_{\phi,\theta}} \le 1 \text{ where } P_N \in \pi_{N,\theta}\}, \tag{3.9}$$

$$\Phi_{\phi,\theta} := \sup_N \Phi^N_{\phi,\theta}. \tag{3.10}$$

Corollary 3.11. *Let $0 \le \theta < 1$. Let ϕ be a globally defined continuous admissible weight. Then $V_{\phi,\theta} = \log \widetilde{\Phi}_{\phi,\theta}$, where*

$$\widetilde{\Phi}^N_{\phi,\theta}(z) := \sup\{|P(z)|^{1/N} \mid \|e^{-N\phi}P_N\|_{\mathbb{C}^n} \le 1 \text{ where } P_N \in \pi_{N,\theta}\}, \tag{3.11}$$

$$\widetilde{\Phi}_{\phi,\theta} := \sup_N \widetilde{\Phi}^N_{\phi,\theta}. \tag{3.12}$$

Proof. It is sufficient to show that for any $P_N \in \pi_{N,\theta}$, $\|e^{-N\phi}P_N\|_{\mathbb{C}^n} \le 1$ if and only if $\|e^{-N\phi}P_N\|_{D_{\phi,\theta}} \le 1$. The "only if" direction is trivial. For the other direction, let $P_N \in \pi_{N,\theta}$ and $\|e^{-N\phi}P_N\|_{D_{\phi,\theta}} \le 1$. We will show that $\|e^{-N\phi}P_N\|_{\mathbb{C}^n} \le 1$. We have $e^{-N\phi(z)}|P_N(z)| \le 1$ for $z \in D_{\phi,\theta}$ so we get $N^{-1}\log|P_N(z)| \le \phi(z)$ on $D_{\phi,\theta}$. Hence, $N^{-1}\log|P_N(z)|$ is a competitor for the extremal function $V_{\phi,\theta}$, and so $N^{-1}\log|P_N(z)| \le V_{\phi,\theta}(z) \le \phi(z)$ for all $z \in \mathbb{C}^n$. Therefore, $e^{-N\phi(z)}P_N(z) \le 1$ for all $z \in \mathbb{C}^n$. ∎

4. Asymptotics. Let K be a compact set in $\mathbb{C}^n$ and μ be a Borel probability measure with support in K. We say that the pair (K,μ) has the *Bernstein–Markov property* if for any $\varepsilon > 0$, there exists $C > 0$ such that

$$\|P\|_K \le Ce^{\varepsilon N}\|P\|_{L^2(\mu)} \tag{4.1}$$

for all polynomials P of degree at most N. Equivalently, there exists $M_N > 0$ with $M_N^{1/N} \to 1$ as $N \to \infty$ such that for all polynomials P of degree at most N,

$$\|P\|_K \le M_N\|P\|_{L^2(\mu)}. \tag{4.2}$$

We remark that if K is a regular compact set, then $(K,(dd^cV_K)^n)$ has the Bernstein–Markov property. See [16] for details.

We fix $0 \le \theta \le 1$. If these inequalities are satisfied for all $P \in \pi_{N,\theta}$ for all $N \ge 0$, then we say the pair (K,μ) has the *Bernstein–Markov property for θ-incomplete polynomials.*

Let μ be a measure such that (K,μ) has the Bernstein–Markov property for θ-incomplete polynomials. Let $\{P_j\}$ be an orthonormal basis of $\pi_{N,\theta}$ with respect to the inner product $\langle f,g\rangle := \int f\bar{g}\,d\mu$. We define the Nth *Bergman function* $K_{N,\theta}(z,w) := \sum_{j=1}^{d(N,\theta)} P_j(z)\,\overline{P_j(w)}$, where $d(N,\theta)$ is the dimension of $\pi_{N,\theta}$.

The following two lemmas are generalizations of results of Bloom and Shiffman [7].

Lemma 4.1. *If (K,μ) has the Bernstein–Markov property for θ-incomplete polynomials, then for all $\varepsilon > 0$, there exists $C > 0$ such that*

$$\frac{(\Phi_{K,\theta,N}(z))^2}{d(N,\theta)} \le K_{N,\theta}(z,z) \le Ce^{\varepsilon N}(\Phi_{K,\theta,N}(z))^2 d(N,\theta) \tag{4.3}$$

for all $z \in \mathbb{C}^n$.

Proof. To show the first inequality, we take $P \in \pi_{N,\theta}$ and $\|P\|_K \le 1$. Then

$$\begin{aligned}
|P(z)| &= \Big|\int_K K_{N,\theta}(z,w)P(w)\,d\mu(w)\Big| \le \int_K |K_{N,\theta}(z,w)|\,d\mu(w)\\
&\le \int_K (K_{N,\theta}(z,z))^{1/2}(K_{N,\theta}(w,w))^{1/2}\,d\mu(w)\\
&\le (K_{N,\theta}(z,z))^{1/2}\|(K_{N,\theta}(w,w))^{1/2}\|_{L^1(\mu)}\\
&\le (K_{N,\theta}(z,z))^{1/2}\|1\|_{L^2(\mu)}\|K_{N,\theta}(w,w)\|_{L^2(\mu)}\\
&= (K_{N,\theta}(z,z))^{1/2}d(N,\theta)^{1/2}.
\end{aligned}$$

Taking the supremum over all P as above, we have

$$\Phi_{K,\theta,N}(z) \le (K_{N,\theta}(z,z))^{1/2}d(N,\theta)^{1/2},$$

which gives the first inequality.

For the second inequality, let $\{P_j\}$ be an orthonormal basis of $\pi_{N,\theta}$. Then by the Bernstein–Markov property we have $\|P_j\|_K \le Ce^{\varepsilon N}$, hence

$$|P_j(z)| \le \|P_j\|_K\Phi_{K,\theta,N}(z) \le Ce^{\varepsilon N}\Phi_{K,\theta,N}(z) \quad \text{for all } j.$$

Thus, we have

$$K_{N,\theta}(z,z) = \sum_{j=1}^{d(N,\theta)} |P_j(z)|^2 \le d(N,\theta)C^2e^{2\varepsilon N}(\Phi_{K,\theta,N}(z))^2.$$

Hence, we get the second inequality. ∎

LEMMA 4.2. *Let $0 < \theta < 1$. Let K be a regular compact set in $\mathbb{C}^n$. If (K,μ) has the Bernstein–Markov property for θ-incomplete polynomials, then*

$$\frac{1}{2N}\log K_{N,\theta}(z,z) \to V_{K,\theta}(z)$$

uniformly on compact subsets of $\mathbb{C}^n \setminus \{0\}$.

Proof. We remark that $d(N,\theta) \le d(N) := d(N,0)$ and $d(N) = \binom{N+n}{n} \le (N+n)^n$.

Taking logarithms in (4.3), we obtain

$$-\frac{\log d(N,\theta)}{N} \le \frac{\log\big(\frac{K_{N,\theta}(z,z)}{(\Phi_{K,\theta,N}(z))^2}\big)}{N} \le \frac{\log(Ce^{\varepsilon N}d(N,\theta))}{N}.$$

By the above observation, we get

$$-\frac{n}{N}\log(N+n) \le \frac{1}{N}\log\bigg(\frac{K_{N,\theta}(z,z)}{(\Phi_{K,\theta,N}(z))^2}\bigg) \le \frac{\log C}{N} + \varepsilon + \frac{n}{N}\log(N+n).$$

Since ε is arbitrary, we have $N^{-1}\log(K_{N,\theta}(z,z)/(\Phi_{K,\theta,N}(z))^2) \to 0$, which gives the desired result by Theorem 2.4. ∎

Let K be a compact set with an admissible weight w on K. Let μ be a Borel probability measure on K. We say the triple (K, μ, w) has the *weighted Bernstein–Markov property* if there exist $M_N > 0$ with $M_N^{1/N} \to 1$ such that for any polynomial P_N of degree N,

$$\|w^N P_N\|_K \le M_N \|w^N P_N\|_{L^2(\mu)}. \tag{4.4}$$

We remark that if K is locally regular and Q is continuous then by Corollary 3.1 of [5], $(K, (dd^c V_{K,Q})^n, w)$ has the weighted Bernstein–Markov property. Also, $(D_\phi, (dd^c V_\phi)^n, e^{-\phi})$ has the weighted Bernstein–Markov property if ϕ is continuous by Theorem 4.5 of [4].

THEOREM 4.3. *Let K be a compact set with a continuous admissible weight w on K. Let μ be a probability measure on K such that (K, μ, w) has the weighted Bernstein–Markov property. Then*

$$\lim_{N\to\infty} \sup_{k=1,\dots,d(N)} |B_{k,N}(z)|^{1/N} = e^{V_{K,Q}(z)}, \tag{4.5}$$

where $\{B_{k,N}\}_{k=1}^{d(N)}$ is an orthonormal basis for the polynomials with degree at most N with respect to the measure $w^{2N}\mu$.

We remark that unlike the unweighted case, where $w = 1$, each time N changes, the basis and the L^2 norms change.

Proof. By the weighted Bernstein–Markov property,

$$\|w^N B_{k,N}\|_K \le M_N \|w^N B_{k,N}\|_{L^2(\mu)} = M_N,$$

so

$$\frac{1}{N} \log \frac{|B_{k,N}(z)|}{M_N} \le Q(z) \quad \text{on } K.$$

Hence

$$\frac{1}{N} \log \frac{|B_{k,N}(z)|}{M_N} \le V_{K,Q}(z) \quad \text{on } \mathbb{C}^n.$$

Since $M_N^{1/N} \to 1$, we have

$$\limsup_{N\to\infty} \sup_{k=1,\dots,d(N)} |B_{k,n}(z)|^{1/N} \le \limsup_{N\to\infty} e^{V_{K,Q}(z)} M_N^{1/N} \le e^{V_{K,Q}(z)}.$$

Now we want to show that

$$\liminf_{N\to\infty} \sup_{k=1,\dots,d(N)} |B_{k,N}(z)|^{1/N} \ge e^{V_{K,Q}(z)}.$$

Let P be a polynomial of degree at most N such that $\|w^N P\|_K \le 1$. We write $w = e^{-Q}$. Since $\{B_{k,N}\}_{k=1}^{d(N)}$ is an orthonormal basis, we have

$$P(z) = \sum_{j=1}^{d(N)} \Big(\int_K P \bar{B}_{j,N} e^{-2NQ}\, d\mu \Big) B_{j,N}(z).$$

By the triangle inequality,

$$|P(z)| \le \sum_{j=1}^{d(N)} \Big| \int_K P\bar{B}_{j,N} e^{-2NQ}\, d\mu \Big|\, |B_{j,N}(z)|.$$

By the Cauchy–Schwarz inequality,

$$|P(z)| \le \sum_{j=1}^{d(N)} \Big| \Big(\int_K |P|^2 e^{-2NQ}\, d\mu\Big)^{1/2} \Big(\int_K |B_{j,N}|^2 e^{-2NQ}\, d\mu\Big)^{1/2} \Big|\, |B_{j,N}(z)|.$$

Now since $\|w^N P\|_K \le 1$ and $\{B_{k,N}\}_{k=1}^{d(N)}$ is an orthonormal basis,

$$|P(z)| \le \sum_{j=1}^{d(N)} |B_{j,N}(z)|.$$

This implies that

$$|P(z)| \le d(N) \sup_{k=1,\dots,d(N)} |B_{j,N}(z)| \quad \text{for any } z \in \mathbb{C}^n. \tag{4.6}$$

We fix $z \in \mathbb{C}^n$. Then

$$\begin{aligned} e^{V_{K,Q}(z)} &\le \liminf_{N\to\infty} \sup_{P\in\pi_{N,0},\, \|w^N P\|_K\le 1} |P(z)|^{1/N} \\ &\le \liminf_{N\to\infty} d(N)^{1/N} \sup_{k=1,\dots,d(N)} |B_{j,N}(z)|^{1/N}. \end{aligned}$$

Here, the first inequality follows from (3.8). Now since $d(N)^{1/N} \to 1$, we get the result. ∎

Corollary 4.4. *Let ϕ be a globally defined continuous admissible weight and μ be a Borel probability measure on D_ϕ such that $(D_\phi, \mu, e^{-\phi})$ has the weighted Bernstein–Markov property. Then*

$$\lim_{N\to\infty} \sup_{k=1,\dots,d(N)} |B_{k,N}(z)|^{1/N} = e^{V_\phi(z)}. \tag{4.7}$$

Here $\{B_{k,N}\}_{k=1}^{d(N)}$ is an orthonormal basis for the polynomials with degree at most N with respect to the measure $e^{-2N\phi}\mu$.

Remark 4.5. The proof of Theorem 4.3 shows that if additionally $V_{K,Q}$ is continuous, then the convergence in (4.5) is locally uniform. Thus, the convergence in (4.7) is locally uniform.

If (4.4) holds for any $P_N \in \pi_{N,\theta}$ then we say that (K,μ,w) has the *weighted Bernstein–Markov property for θ-incomplete polynomials.*

We remark that if a triple (K,μ,w) has the weighted Bernstein–Markov property, then it has the weighted Bernstein–Markov property for θ-incomplete polynomials.

Using only the orthonormal basis for $\pi_{N,\theta}$ and using Theorem 3.9 instead of (3.8) we get the following theorem by the same proof as for Theorem 4.3.

THEOREM 4.6. *Let $0 \le \theta < 1$. Let K be a compact set with a continuous admissible weight w on K. Let μ be a measure on K such that (K, μ, w) has the weighted Bernstein–Markov property for θ-incomplete polynomials. Then*

$$\lim_{N\to\infty} \sup_{k=1,\dots,d(N)} |B^{\theta}_{k,N}(z)|^{1/N} = e^{V_{K,Q,\theta}(z)},$$

where $\{B^{\theta}_{k,N}\}_{k=1}^{d(N,\theta)}$ is an orthonormal basis for $\pi_{N,\theta}$ with respect to the measure $w^{2N}\mu$.

COROLLARY 4.7. *Let $0 \le \theta < 1$. Let ϕ be a globally defined continuous admissible weight. If $(D_\phi, \mu, e^{-\phi})$ has the weighted Bernstein–Markov property then*

$$\lim_{N\to\infty} \sup_{k=1,\dots,d(N)} |B^{\theta}_{k,N}(z)|^{1/N} = e^{V_{\phi,\theta}(z)}, \tag{4.8}$$

where $\{B^{\theta}_{k,N}\}_{k=1}^{d(N,\theta)}$ is an orthonormal basis for $\pi_{N,\theta}$ with respect to the measure $e^{-2N\phi}\mu$.

Finally, we prove the strong Bergman asymptotics in the weighted θ-incomplete setting, following [2] closely. We fix $0 \le \theta < 1$. Let ϕ be a globally defined admissible weight and $\phi(z) \ge (1+\varepsilon)\log|z|$ if $|z| \gg 1$. Let $\{p_1, \dots, p_{d(N,\theta)}\}$ be an orthonormal basis for $\pi_{N,\theta}$ with respect to the inner product $\langle f, g\rangle := \int_{\mathbb{C}^n} f\bar{g}e^{-2N\phi}\omega_n$ where $\omega_n(z) = (dd^c|z|^2)^n/4^n n!$ on $\mathbb{C}^n$. We denote the L^2 norm by $\|p_N\|^2_{N\phi} := \|p_N\|^2_{\omega_n,N\phi} = \int_{\mathbb{C}^n} |p_N(z)|^2 e^{-2N\phi(z)}\omega_n(z)$. We define the Nth θ-incomplete Bergman function by

$$K_N(z) := K^{\phi}_{N,\theta}(z,z) = \sum_{j=1}^{d(N,\theta)} |p_j(z)|^2 e^{-2N\phi(z)}. \tag{4.9}$$

By the reproducing property of the Bergman functions we have

$$K_N(z) = \sup_{p_N\in\pi_{N,\theta}\setminus\{0\}} |p_N(z)|^2 e^{-2N\phi(z)}/\|p_N\|^2_{N\phi}. \tag{4.10}$$

THEOREM 4.8. *Let $\phi \in C^2(\mathbb{C}^n)$ with $\phi(z) \ge (1+\varepsilon)\log|z|$ for $|z| \gg 1$. If $V_{\phi,\theta} \in C^{1,1}(\mathbb{C}^n \setminus \{0\})$, then $(dd^cV_{\phi,\theta})^n$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{C}^n \setminus \{0\}$ and $\det(dd^c\phi)\omega_n = (dd^cV_{\phi,\theta})^n$ on $\mathbb{C}^n \setminus \{0\}$ as (n,n) forms with $L^\infty_{\mathrm{loc}}(\mathbb{C}^n)$ coefficients. For a compact set K, we have a local bound*

$$\frac{1}{d(N,\theta)} K_N(z) \le C = C(K) \quad \text{for } z \in K. \tag{4.11}$$

Moreover,

$$\frac{1}{d(N,\theta)}\,K_N \to \frac{1}{1-\theta^n}\,\chi_{D_{\phi,\theta}\cap P}\,\frac{\det(dd^c\phi)}{(2\pi)^n} \qquad \text{in } L^1(\mathbb{C}^n) \tag{4.12}$$

and

$$\frac{1}{d(N,\theta)}\,K_N\omega_n \to \frac{1}{1-\theta^n}\,\frac{(dd^cV_{\phi,\theta})^n}{(2\pi)^n} \qquad \text{weak}^* \text{ on } D_{\phi,\theta}\cap P. \tag{4.13}$$

Here $\det(dd^cu) := (dd^cu)^n/\omega_n$ and for a twice continuously differentiable function u, we have $\det(dd^cu) = 2i\det[\partial^2u/\partial z_j\partial\bar z_k]_{j,k=1,\dots,n}$. We remark that we assume $V_{\phi,\theta} \in C^{1,1}(\mathbb{C}^n\setminus\{0\})$.

We will use the following lemma from measure theory in the proof of the theorem.

LEMMA 4.9 ([3, Lemma 2.2]). *Let (X,μ) be a measure space and let $\{f_N\}$ be a sequence of uniformly bounded, integrable functions on X. If f is a bounded, integrable function on X with*

(1) $\lim_{N\to\infty}\int_X f_N\,d\mu = \int_X f\,d\mu$,
(2) $\limsup_{N\to\infty} f_N \le f$ *a.e. with respect to μ,*

then f_N converges to f in $L^1(X,\mu)$.

Proof of Theorem 4.8. The $\theta = 0$ case is proven by Berman in [2], so we assume $0<\theta<1$.

By assumption, $V_{\phi,\theta} = \phi$ on $D_{\phi,\theta}\cap P$ and both are $C^{1,1}$ on $D_{\phi,\theta}\cap P$. Therefore, $\det(dd^c\phi)\omega_n = (dd^cV_{\phi,\theta})^n$ on $D_{\phi,\theta}\cap P$ almost everywhere as (n,n) forms with L^∞ coefficients by the argument in Section 12 of [11].

First of all, to prove an asymptotic upper bound on $(1/d(N,\theta))K_N(z)$ at a point $z_0 = (z_1^0,\dots,z_n^0)$, we can assume that near z_0, ϕ is of the form

$$\phi(z) = \sum_{j=1}^n \lambda_j|z_j - z_j^0|^2 + O(|z-z_0|^3) \tag{4.14}$$

as in [2]. Namely, we assume that $\phi(z_0) = 0$ and the first order partial derivatives of ϕ vanish at z_0.

Following [2], for each $z_0\in\mathbb{C}^n$ there exist $R>0$ and a constant C such that

$$|\phi(z)| \le C|z-z_0|^2 \qquad \text{on } B(z_0,R), \tag{4.15}$$

and for any $R>0$, we have

$$\lim_{N\to\infty}\sup_{z\in B(0,R)}\Big|N\phi(z/\sqrt{N}+z_0) - \sum_{j=1}^n\lambda_j|z_j|^2\Big| = 0. \tag{4.16}$$

We fix a point z_0 in $\mathbb{C}^n$. We take a polynomial $p_N\in\pi_{N,\theta}$ satisfying the extremal property (4.10) at z_0. Then

$$\frac{1}{d(N,\theta)}\,K_N(z_0)=\frac{|p_N(z_0)|^2e^{-2N\phi(z_0)}}{d(N,\theta)\|p_N\|^2_{N\phi}}=\frac{|p_N(z_0)|^2}{d(N,\theta)\int_{\mathbb{C}^n}|p_N(z)|^2e^{-2N\phi(z)}\omega_n(z)}.$$

By positivity of the integrand,

$$\frac{1}{d(N,\theta)}\,K_N(z_0)\le\frac{|p_N(z_0)|^2}{d(N,\theta)\int_{|z-z_0|\le R/\sqrt{N}}|p_N(z)|^2e^{-2N\phi(z)}\omega_n(z)}.$$

We choose R as in (4.15) so that we can replace $\phi(z)$ by $C|z-z_0|^2$ in the integrand, and thus

$$\frac{1}{d(N,\theta)}\,K_N(z_0)\le\frac{|p_N(z_0)|^2}{d(N,\theta)\int_{|z-z_0|\le R/\sqrt{N}}|p_N(z)|^2e^{-2NC|z-z_0|^2}\omega_n(z)}.$$

We apply the subaveraging property to the subharmonic function $|p_N|^2$ on the ball $\{|z-z_0|\le R/\sqrt{N}\}$ with respect to the radial probability measure (with center z_0) $e^{-2NC|z-z_0|^2}\omega_n(z)/\int_{|z-z_0|\le R/\sqrt{N}}e^{-2NC|z-z_0|^2}\omega_n(z)$ to obtain

$$\begin{aligned}\frac{1}{d(N,\theta)}\,K_N(z_0)&\le\frac{1}{d(N,\theta)\int_{|z-z_0|\le R/\sqrt{N}}e^{-2NC|z-z_0|^2}\omega_n(z)}\\&\le\frac{N^n}{d(N,\theta)\int_{|z'|\le R}e^{-2C|z'|^2}\omega_n(z')}.\end{aligned}$$

For the last inequality, we used a change of variable $z\mapsto z':=(z-z_0)\sqrt{N}$, where $\omega_n(z')=N^n\omega_n(z)$. Since $d(N,\theta)\asymp(1-\theta^n)d(N,0)$, we have $d(N,\theta)\ge(1-\tilde{\theta}^n)d(N,0)$ for all $N\ge N_0$ for some $\widetilde{\theta}\ge\theta$. Now using the estimate $d(N,\theta)\ge(1-\tilde{\theta}^n)d(N,0)=(1-\tilde{\theta}^n)\binom{n+N}{n}\ge(1-\tilde{\theta}^n)N^n/n!$ for all $N\ge N_0$, we get

$$\frac{1}{d(N,\theta)}\,K_N(z_0)\le\frac{n!}{(1-\tilde{\theta}^n)\int_{|z'|\le R}e^{-2C|z'|^2}\omega_n(z')}\qquad\text{for all } N\ge N_0.$$

The right hand side of the inequality is uniformly bounded. As z_0 varies on the compact set K, we get a constant $C(K)$ giving a local bound for all $N\ge N_0$. By the continuity of $(1/d(N,\theta))K_N(z)$, and considering

$$\max_{N=1,\dots,N_0}\sup_{z\in K}\frac{1}{d(N,\theta)}\,K_N(z),$$

we get the local bound (4.11) at each point of K.

For the rest of the proof, we fix z_0 and start with the inequality

$$\frac{1}{d(N,\theta)}\,K_N(z_0)\le\frac{|p_N(z_0)|^2}{d(N,\theta)\int_{|z-z_0|\le R/\sqrt{N}}|p_N(z)|^2e^{-2N\phi(z)}\omega_n(z)},$$

which holds for any $R>0$. By using the same change of variable and estimates as above, we get

$$\frac{1}{d(N,\theta)}\,K_N(z_0) \le \frac{n!|p_N(z_0)|^2}{(1-\tilde{\theta}^n)\int_{|z'|\le R}|p_N(z'/\sqrt{N}+z_0)|^2 e^{-2N\phi(z'/\sqrt{N}+z_0)}\omega_n(z')}$$

for all $N \ge N_0$ where $\tilde{\theta} \ge \theta$. Multiplying the integrand by

$$e^{-2\sum_{j=1}^n \lambda_j |z_j'|^2} e^{2\sum_{j=1}^n \lambda_j |z_j'|^2}$$

and taking the infimum of $\exp[-2|N\phi(z'/\sqrt{N}) - \sum_{j=1}^n \lambda_j |z_j'|^2|]$ on $B(0,R)$ out of the integral, we get

$$\frac{1}{d(N,\theta)}\,K_N(z_0) \le \frac{n!|p_N(z_0)|^2 \exp[2\sup_{|z'|\le R}|N\phi(z'/\sqrt{N}) - \sum_{j=1}^n \lambda_j |z_j'|^2|]}{(1-\tilde{\theta}^n)\int_{|z'|\le R}|p_N(z'/\sqrt{N}+z_0)|^2 e^{-2\sum_{j=1}^n \lambda_j |z_j'|^2}\omega_n(z')}$$

for all $N \ge N_0$. We apply the subaveraging property to the subharmonic function $|p_N(z'/\sqrt{N}+z_0)|^2$ with respect to the radial probability measure

$$\frac{e^{-2\sum_{j=1}^n \lambda_j |z_j'|^2}\omega_n(z')}{\int_{|z'|\le R} e^{-2\sum_{j=1}^n \lambda_j |z_j'|^2}\omega_n(z')}$$

and we get

$$\frac{1}{d(N,\theta)}\,K_N(z_0) \le \frac{n!\exp[2\sup_{|z'|\le R}|N\phi(z'/\sqrt{N}) - \sum_{j=1}^n \lambda_j |z_j'|^2|]}{(1-\tilde{\theta}^n)\int_{|z'|\le R} e^{-2\sum_{j=1}^n \lambda_j |z_j'|^2}\omega_n(z')}$$

for all $N \ge N_0$. By (4.16),

$$\exp\Big[2\sup_{|z'|\le R}\Big|N\phi(z'/\sqrt{N}) - \sum_{j=1}^n \lambda_j |z_j'|^2\Big|\Big] \to 1 \qquad \text{as } N \to \infty.$$

Therefore,

$$\limsup_{N\to\infty}\frac{1}{d(N,\theta)}\,K_N(z_0) \le \frac{n!}{(1-\tilde{\theta}^n)\int_{|z'|\le R} e^{-2\sum_{j=1}^n \lambda_j |z_j'|^2}\omega_n(z')}.$$

As $R \to \infty$, the Gaussian integral on the right hand side goes to $\frac{\pi^n}{2^n\lambda_1\cdots\lambda_n}$ if all $\lambda_j > 0$ and to $+\infty$ otherwise. Since $\det(dd^c\phi(z_0)) = 4^n n!\lambda_1\cdots\lambda_n$, we have

$$\limsup_{N\to\infty}\frac{1}{d(N,\theta)}\,K_N(z) \le \frac{1}{1-\tilde{\theta}^n}\,\chi_P\,\frac{\det(dd^c\phi)}{(2\pi)^n} \qquad \text{a.e. on } \mathbb{C}^n.$$

Letting $\tilde{\theta} \to \theta$, we obtain

$$\limsup_{N\to\infty}\frac{1}{d(N,\theta)}\,K_N(z) \le \frac{1}{1-\theta^n}\,\chi_P\,\frac{\det(dd^c\phi)}{(2\pi)^n} \qquad \text{a.e. on } \mathbb{C}^n.$$

By the extremal property (4.10) and the local bound (4.11), we get

$$\frac{1}{N^n}\,|p_N(z)|^2 e^{-2N\phi(z)}/\|p_N\|_{N\phi}^2 \le C := C(D_{\phi,\theta}) \tag{4.17}$$

on $D_{\phi,\theta}$ for any $p_N \in \pi_{N,\theta}$. Next, we will show that

$$\frac{1}{N^n} K_N(z) \le C_N e^{-2N(\phi(z)-V_{\phi,\theta}(z))} \quad \text{on } \mathbb{C}^n. \tag{4.18}$$

Let $p_N \in \pi_{N,\theta}$ be such that $\|p_N\|^2_{N\phi} = N^{-n}$. Then by (4.17) we have

$$|p_N(z)|^2 e^{-2N\phi(z)} \le C \quad \text{on } D_{\phi,\theta}.$$

By taking logarithms, we get

$$\frac{1}{2N} \log |p_N(z)|^2 \le \phi(z) + \frac{1}{2N} \log C \quad \text{on } D_{\phi,\theta}$$

and thus

$$\frac{1}{2N} \log |p_N(z)|^2 \le V_{\phi,\theta}(z) + \frac{1}{2N} \log C \quad \text{on } \mathbb{C}^n.$$

So from the extremal property of Bergman functions (4.10),

$$\frac{1}{N^n} K_N(z) = \sup_{\|p_N\|^2_{N\phi}=N^{-n}} |p_N(z)|^2 e^{-2N\phi(z)} \le C e^{-2N(\phi(z)-V_{\phi,\theta}(z))} \quad \text{on } \mathbb{C}^n.$$

Since $\phi(z) > V_{\phi,\theta}(z)$ on $\mathbb{C}^n \setminus D_{\phi,\theta}$, we find that

$$\lim_{N\to\infty} \frac{1}{N^n} K_N(z) = 0 \quad \text{on } \mathbb{C}^n \setminus D_{\phi,\theta}.$$

Using $d(N,\theta) \asymp (1-\theta^n) d(N,0)$, we obtain

$$\lim_{N\to\infty} \frac{1}{d(N,\theta)} K_N(z) = 0 \quad \text{on } \mathbb{C}^n \setminus D_{\phi,\theta},$$

giving

$$\limsup_{N\to\infty} \frac{1}{d(N,\theta)} K_N(z) \le \frac{1}{(1-\theta^n)} \chi_{D_{\phi,\theta}\cap P} \frac{\det(dd^c\phi)}{(2\pi)^n} \quad \text{a.e. on } \mathbb{C}^n. \tag{4.19}$$

From (4.18) and the growth assumption on ϕ, for a sufficiently large R, there is a C with

$$\frac{1}{N^n} K_N(z) \le C|z|^{-2N\epsilon} \quad \text{for } |z| > R. \tag{4.20}$$

By combining the local bound (4.11) and above estimate (4.20), we get a global bound for $(1/d(N,\theta))K_N$. Therefore, Lebesgue's dominated convergence theorem gives

$$\lim_{N\to\infty} \int_{\mathbb{C}^n\setminus D_{\phi,\theta}} \frac{1}{d(N,\theta)} K_N \omega_n = 0. \tag{4.21}$$

Next, we show that

$$\lim_{N\to\infty} \int_{D_{\phi,\theta}\cap P} \frac{1}{d(N,\theta)} K_N \omega_n = \frac{1}{1-\theta^n} \int_{D_{\phi,\theta}\cap P} \frac{\det(dd^c\phi)}{(2\pi)^n} \omega_n. \tag{4.22}$$

Indeed, we know that $\int_{\mathbb{C}^n} K_N\omega_n = d(N,\theta)$, and so using (4.21), we have

$$1 = \lim_{N\to\infty} \int_{\mathbb{C}^n} \frac{1}{d(N,\theta)} K_N\omega_n = \lim_{N\to\infty} \int_{D_{\phi,\theta}\cap P} \frac{1}{d(N,\theta)} K_N\omega_n.$$

On the other hand, using the positivity of the integrand and applying (4.19) on $D_{\phi,\theta}$, we have

$$1 = \lim_{N\to\infty} \int_{D_{\phi,\theta}} \frac{1}{d(N,\theta)} K_N\omega_n \le \frac{1}{1-\theta^n} \int_{D_{\phi,\theta}\cap P} \frac{\det(dd^c\phi)}{(2\pi)^n}\,\omega_n.$$

By the first part of this theorem, we can replace $\det(dd^c\phi)\omega_n$ by $(dd^cV_{\phi,\theta})^n$, which has total mass $(2\pi)^n(1-\theta^n)$ on $D_{\phi,\theta}\cap P$; hence,

$$\begin{aligned}1 &= \lim_{N\to\infty} \int_{D_{\phi,\theta}\cap P} \frac{1}{d(N,\theta)} K_N\omega_n \\ &\le \frac{1}{1-\theta^n} \int_{D_{\phi,\theta}\cap P} \frac{(dd^cV_{\phi,\theta})^n}{(2\pi)^n} = \frac{(2\pi)^n(1-\theta^n)}{(2\pi)^n(1-\theta^n)} = 1.\end{aligned}$$

This gives (4.22).

We will use this relation, together with (4.12), to show that

$$\frac{1}{d(N,\theta)} K_N \to \frac{1}{1-\theta^n}\,\chi_{D_{\phi,\theta}\cap P}\,\frac{\det(dd^c\phi)}{(2\pi)^n} \quad \text{in } L^1(\mathbb{C}^n). \tag{4.23}$$

We set

$$f_N := \frac{1}{d(N,\theta)} K_N \quad \text{and} \quad f := \frac{1}{1-\theta^n}\,\chi_{D_{\phi,\theta}\cap P}\,\frac{\det(dd^c\phi)}{(2\pi)^n}.$$

By the upper bound (4.19), we have $\limsup_{N\to\infty} f_N \le f$ almost everywhere and by (4.21) and (4.22),

$$\lim_{N\to\infty} \int_{\mathbb{C}^n} f_N\omega_n = \int_{\mathbb{C}^n} f\omega_n.$$

Thus, by Lemma 4.9, we get (4.23). This implies the weak* convergence in (4.13) and completes the proof of the theorem. ∎

Acknowledgements. I thank Professor Norman Levenberg for his excellent support and help. The main part of this research was done during my visit to the University of Laval. I thank Professor Thomas Ransford and the University of Laval for their hospitality.

I thank the referee for his careful scrutiny. His comments improved the overall quality of the paper and helped me to prove a stronger version of Theorem 2.4 and to obtain Remark 4.5.

This research was partially supported by Glenn Schober Memorial Travel Award, Department of Mathematics, Indiana University Bloomington.

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Received 27.5.2009
and in final form 11.12.2009 (2023)