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On para-Nordenian structures

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Abstract. The aim of this paper is to investigate para-Nordenian properties of the Sasakian metrics in the cotangent bundle.

1. Introduction. Let (M^n, g) be an n-dimensional Riemannian manifold, T^*M^n its cotangent bundle and π the natural projection $T^*M^n \to M^n$. A system of local coordinates (U, x^i), $i = 1, \ldots, n$ on M^n induces on T^*M^n a system of local coordinates $(\pi^{-1}(U), x^i, x^{\bar{\imath}} = p_i)$, $\bar{\imath} := n+i$ ($\bar{\imath} = 1, \ldots, 2n$), where $x^{\bar{\imath}} = p_i$ are the components of the covector p in each cotangent space $T^*_x M^n$, $x \in U$, with respect to the natural coframe $\{dx^i\}$, $i = 1, \ldots, n$.

We denote by $\Im^r_s(M^n)$ (resp. $\Im^r_s(T^*M^n)$) the module over $F(M^n)$ (resp. $F(T^*M^n)$) of C^∞ tensor fields of type (r,s), where $F(M^n)$ (resp. $F(T^*M^n)$) is the ring of real-valued C^∞ functions on M^n (resp. T^*M^n).

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be the local expressions in $U \subset M^n$ of a vector and a covector (1-form) field $X \in \Im^1_0(M^n)$ and $\omega \in \Im^0_1(M^n)$, respectively. Then the complete and horizontal lifts ${}^CX, {}^HX \in \Im^1_0(T^*M^n)$ of $X \in \Im^1_0(M^n)$ and the vertical lift ${}^V\omega \in \Im^1_0(T^*M^n)$ of $\omega \in \Im^0_1(M^n)$ are given, respectively, by

$$ {}^CX = X^i \frac{\partial}{\partial x^i} - \sum_i p_h \partial_i X^h \frac{\partial}{\partial x^{\bar{\imath}}}, \tag{1.1} $$

$$ {}^HX = X^i \frac{\partial}{\partial x^i} + \sum_i p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial x^{\bar{\imath}}}, \tag{1.2} $$

$$ {}^V\omega = \sum_i \omega_i \frac{\partial}{\partial x^{\bar{\imath}}} \tag{1.3} $$

with respect to the natural frame $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{\imath}}}\right\}$, where Γ^h_{ij} are the components of the Levi-Civita connection ∇_g on M^n (see [11] for more details).

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For each $x \in M^n$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $\pi^{-1}(x) = T_x^*(M^n)$ by

$$g^{-1}(\omega, \theta) = g^{ij}\omega_i\theta_j$$

for all $\omega, \theta \in \Im_1^0(M^n)$.

A *Sasakian metric* ${}^S g$ is defined on T^*M^n by the following three equations

$$(1.4)\qquad {}^S g({}^V\omega, {}^V\theta) = {}^V(g^{-1}(\omega, \theta)) = g^{-1}(\omega, \theta) \circ \pi,$$

$$(1.5)\qquad {}^S g({}^V\omega, {}^H Y) = 0,$$

$$(1.6)\qquad {}^S g({}^H X, {}^H Y) = {}^V(g(X, Y)) = g(X, Y) \circ \pi$$

for any $X, Y \in \Im_0^1(M^n)$ and $\omega, \theta \in \Im_1^0(M^n)$. Since any tensor field of type $(0, 2)$ on T^*M^n is completely determined by its action on vector fields of type ${}^H X$ and ${}^V\omega$ (see [11, p. 280]), it follows that ${}^S g$ is completely determined by (1.4)–(1.6).

From (1.1) and (1.2) we notice that the complete lift ${}^C X$ of $X \in \Im_0^1(M^n)$ is expressed by

$$(1.7)\qquad {}^C X = {}^H X - {}^V(p(\nabla X)),$$

where $p(\nabla X) = p_i(\nabla_h X^i)dx^h$.

Using (1.4)–(1.7), we have

$$(1.8)\qquad {}^S g({}^C X, {}^C Y) = {}^V(g(X, Y)) + {}^V(g^{-1}(p(\nabla X), p(\nabla Y))),$$

where $g^{-1}(p(\nabla X), p(\nabla Y)) = g^{ij}(p_l\nabla_i X^l)(p_k\nabla_j Y^k)$.

Since the tensor field ${}^S g \in \Im_2^0(T^*M^n)$ is also completely determined by its action on vector fields of type ${}^C X$ and ${}^C Y$ (see [11, p. 237]), we have an alternative characterization of ${}^S g$: a Sasakian metric ${}^S g$ on T^*M^n is completely determined by the condition (1.8).

Sasakian metrics on the tangent bundle were introduced in [9] by the Japanese geometer S. Sasaki. Sasakian metrics (diagonal lifts of metrics) on tangent bundles were also studied in [3], [11]. In the more general case of tensor bundles of type $(1, q)$, $(0, q)$ and (p, q), Sasakian metrics and their geodesics were considered in [1], [6], [7]. Sasakian metrics on the frame bundle were first considered by K. P. Mok [5] (see [2] for more details). This paper is concerned with para-Nordenian properties of the Sasakian metric on the cotangent bundle.

2. Levi-Civita connection of ${}^S g$. On $U \subset M^n$, we put

$$X_{(i)} = \frac{\partial}{\partial x^i}, \quad \theta^{(i)} = dx^i, \quad i = 1, \ldots, n.$$

Then from (1.2) and (1.3) we see that ${}^{H}X_{(i)}$ and ${}^{V}\theta^{(i)}$ have local expressions

$$\tilde{e}_{(i)} = {}^{H}X_{(i)} = \frac{\partial}{\partial x^{i}} + \sum_{h} p_a \Gamma^{a}_{hi} \frac{\partial}{\partial x^{\bar{h}}}, \tag{2.1}$$

$$\tilde{e}_{(\bar{\imath})} = {}^{V}\theta^{(i)} = \frac{\partial}{\partial x^{\bar{\imath}}} \, . \tag{2.2}$$

We call the set $\{\tilde{e}_{(\alpha)}\} = \{\tilde{e}_{(i)}, \tilde{e}_{(\bar{\imath})}\} = \{{}^{H}X_{(i)}, {}^{V}\theta^{(i)}\}$ the frame *adapted* to the Levi-Civita connection ∇_g. The indices $\alpha, \beta, \ldots = 1, \ldots, 2n$ indicate the indices with respect to the adapted frame.

From equations (1.2), (1.3), (2.1) and (2.2), we see that ${}^{H}X$ and ${}^{V}\omega$ have the components

$${}^{H}X = X^{i}\tilde{e}_{(i)}, \qquad {}^{H}X = ({}^{H}X^{\alpha}) = \begin{pmatrix} X^{i} \\ 0 \end{pmatrix}, \tag{2.3}$$

$${}^{V}\omega = \sum_{i} \omega_i \tilde{e}_{(\bar{\imath})}, \qquad {}^{V}\omega = ({}^{V}\omega^{\alpha}) = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix} \tag{2.4}$$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$, where X^{i} and ω_i are the local components of $X \in \Im^{1}_{0}(M^{n})$ and $\omega \in \Im^{0}_{1}(M^{n})$, respectively.

Let ${}^{S}\nabla$ be the Levi-Civita connection determined by the Sasakian metric ${}^{S}g$. The components of ${}^{S}\nabla$ are given by [7]

$$\begin{aligned}
&{}^{S}\Gamma^{h}_{j\,i} = \Gamma^{h}_{ji}, && {}^{S}\Gamma^{h}_{\bar{j}\,\bar{\imath}} = {}^{S}\Gamma^{\bar{h}}_{\bar{j}\,i} = {}^{S}\Gamma^{\bar{h}}_{\bar{j}\,\bar{\imath}} = 0, \\
&{}^{S}\Gamma^{h}_{j\,\bar{\imath}} = \tfrac{1}{2} p_m R^{h}{}_{.\,j.}{}^{im}, && {}^{S}\Gamma^{h}_{\bar{j}\,i} = \tfrac{1}{2} p_m R^{h}{}_{.\,i.}{}^{jm}, \\
&{}^{S}\Gamma^{\bar{h}}_{j\,i} = \tfrac{1}{2} p_m R_{jih}{}^{m}, && {}^{S}\Gamma^{\bar{h}}_{j\,\bar{\imath}} = -\Gamma^{i}_{j\,h}
\end{aligned} \tag{2.5}$$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$, where $R_{ijk}{}^{h}$ are the local components of the curvature tensor R of ∇_g.

Let now $\tilde{X}, \tilde{Y} \in \Im^{1}_{0}(T^{*}M^{n})$ and $\tilde{X} = \tilde{X}^{\alpha}\tilde{e}_{\alpha}$, $\tilde{Y} = \tilde{Y}^{\beta}\tilde{e}_{\beta}$. The covariant derivative ${}^{S}\nabla_{\tilde{Y}}\tilde{X}$ along $\tilde{Y}$ has components

$${}^{S}\nabla_{\tilde{Y}}\tilde{X}^{\alpha} = \tilde{Y}^{\gamma}\tilde{e}_{\gamma}\tilde{X}^{\alpha} + {}^{S}\Gamma^{\alpha}_{\gamma\,\beta}\tilde{X}^{\beta}\tilde{Y}^{\gamma} \tag{2.6}$$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$.

Using (2.3)–(2.6), we have

THEOREM 2.1. *Let M^{n} be a Riemannian manifold with metric g and ${}^{S}\nabla$ be the Levi-Civita connection of the cotangent bundle $T^{*}M^{n}$ equipped with the Sasakian metric ${}^{S}g$. Then ${}^{S}\nabla$ satisfies*

(i) ${}^{S}\nabla_{{}^{V}\omega}{}^{V}\theta = 0$,
(ii) ${}^{S}\nabla_{{}^{V}\omega}{}^{H}Y = \frac{1}{2}\,{}^{H}(p(g^{-1} \circ R(\ ,Y)\tilde{\omega}))$,
(iii) ${}^{S}\nabla_{{}^{H}X}{}^{V}\theta = {}^{V}(\nabla_X\theta) + \frac{1}{2}\,{}^{H}(p(g^{-1} \circ R(\ ,X)\tilde{\theta}))$,
(iv) ${}^{S}\nabla_{{}^{H}X}{}^{H}Y = {}^{H}(\nabla_X Y) + \frac{1}{2}\,{}^{V}(pR(X,Y))$

for all $X, Y \in \Im_0^1(M^n)$ *and* $\omega, \theta \in \Im_1^0(M^n)$*, where* $\tilde{\omega} = g^{-1} \circ \omega \in \Im_0^1(M^n)$*,* $R(\ ,X)\tilde{\omega} \in \Im_1^1(M^n)$*,* $g^{-1} \circ R(\ ,X)\tilde{\omega} \in \Im_0^2(M^n)$*.*

3. Para-Nordenian structures on $(T^*M^n, {}^S g)$**.** An *almost paracomplex* manifold is an almost product manifold (M^n, φ), $\varphi^2 = \mathrm{I}$, such that the two eigenbundles $T^+M^n$and T^-M^n associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even. Considering the paracomplex structure φ, we obtain the set $\{\mathrm{I}, \varphi\}$ on M^n, which is an isomorphic representation of the algebra of order 2, called the *algebra of paracomplex* (or *double*) *numbers* and denoted by $R(j)$, $j^2 = 1$.

A tensor field $\omega \in \Im_q^0(M^{2n})$ is said to be *pure* with respect to the paracomplex structure φ if

$$\omega(\varphi X_1, X_2, \ldots, X_q) = \omega(X_1, \varphi X_2, \ldots, X_q) = \cdots = \omega(X_1, X_2, \ldots, \varphi X_q)$$

for any $X_1, X_2, \ldots, X_q \in \Im_0^1(M^{2n})$.

We define the operator ϕ_φ associated with φ and applied to the pure tensor field ω by (see [10])

$$\begin{aligned}(\phi_\varphi \omega)(Y, X_1, \ldots, X_q) &= (\varphi Y)(\omega(X_1, \ldots, X_q)) - Y(\omega(\varphi X_1, X_2, \ldots, X_q)) \\ &\quad + \omega((\mathrm{L}_{X_1}\varphi)Y, X_2, \ldots, X_q) + \cdots + \omega(X_1, X_2, \ldots, (\mathrm{L}_{X_q}\varphi)Y),\end{aligned}$$

where L_X denotes the Lie derivative with respect to X. We note that $\phi_\varphi \omega \in \Im_{q+1}^0(M^{2n})$.

If $\phi_\varphi \omega = 0$, then ω is said to be *almost paraholomorphic* with respect to the paracomplex algebra $R(j)$ (see [4], [8]).

A Riemannian manifold (M^{2n}, g) with an almost paracomplex structure φ is said to be *almost para-Nordenian* if the Riemannian metric g is pure with respect to φ. It is well known that the almost para-Nordenian manifold is para-Kähler ($\nabla_g \varphi = 0$) if and only if g is paraholomorphic ($\phi_\varphi g = 0$) (see [8]).

Let $(T^*M^n, {}^S g)$ be the cotangent bundle with the Sasakian metric ${}^S g$. We define a tensor field F of type $(1,1)$ on T^*M^n by

$$\begin{cases} F\,{}^H X = {}^V \tilde{X}, \\ F\,{}^V \omega = {}^H \tilde{\omega} \end{cases} \tag{3.1}$$

for any $X \in \Im_0^1(M^n)$ and $\omega \in \Im_1^0(M^n)$, where $\tilde{X} = g \circ X \in \Im_1^0(M^n)$, $\tilde{\omega} = g^{-1} \circ \omega \in \Im_0^1(M^n)$. Then we obtain

$$F^2 = \mathrm{I}.$$

Indeed, by virtue of (3.1) we have

$$F^2({}^HX) = F(F\,{}^HX) = F({}^V\tilde{X}) = {}^H\tilde{\tilde{X}} = {}^HX,$$
$$F^2({}^V\omega) = F(F\,{}^V\omega) = F({}^H\tilde{\omega}) = {}^V\tilde{\tilde{\omega}} = {}^V\omega$$

for any $X \in \Im_0^1(M^n)$ and $\omega \in \Im_1^0(M^n)$, which implies $F^2 = \mathrm{I}$.

THEOREM 3.1. *The triple* $(T^*M^n, {}^Sg, F)$ *is an almost para-Nordenian manifold.*

Proof. We put

$$A(\tilde{X},\tilde{Y}) = {}^Sg(F\tilde{X},\tilde{Y}) - {}^Sg(\tilde{X},F\tilde{Y})$$

for any $\tilde{X},\tilde{Y} \in \Im_0^1(T^*M^n)$. From (1.4)–(1.6) and (3.1), we have

$$\begin{aligned}
A({}^HX,{}^HY) &= {}^Sg(F\,{}^HX,{}^HY) - {}^Sg({}^HX,F\,{}^HY)\\
&= {}^Sg({}^V\tilde{X},{}^HY) - {}^Sg({}^HX,{}^V\tilde{Y}) = 0,\\
A({}^HX,{}^V\omega) &= {}^Sg(F\,{}^HX,{}^V\omega) - {}^Sg({}^HX,F\,{}^V\omega) = {}^Sg({}^V\tilde{X},{}^V\omega) - {}^Sg({}^HX,{}^H\tilde{\omega})\\
&= g^{-1}(g\circ X,\omega) - g(X,g^{-1}\circ\omega) = 0,\\
A({}^V\omega,{}^HY) &= -A({}^HY,{}^V\omega) = 0,\\
A({}^V\omega,{}^V\theta) &= {}^Sg(F\,{}^V\omega,{}^V\theta) - {}^Sg({}^V\omega,F\,{}^V\theta) = g^{-1}({}^H\tilde{\omega},{}^V\theta) - {}^Sg({}^V\omega,{}^H\tilde{\theta}) = 0,
\end{aligned}$$

i.e. ${}^Sg$ is pure with respect to F. Thus Theorem 3.1 is proved. ∎

We now consider the covariant derivative of F. Taking into account (i)–(iv) of Theorem 2.1 and (3.1), we obtain

$$\begin{aligned}
(3.2)\qquad ({}^S\nabla_{{}^HX}F)({}^HY) &= {}^S\nabla_{{}^HX}(F\,{}^HY) - F({}^S\nabla_{{}^HX}{}^HY)\\
&= {}^S\nabla_{{}^HX}{}^V\tilde{Y} - F({}^S\nabla_{{}^HX}{}^HY)\\
&= {}^V(\nabla_X\tilde{Y}) + \tfrac{1}{2}\,{}^H(p(g^{-1}\circ R(,X)Y))\\
&\quad - F({}^H(\nabla_XY) + \tfrac{1}{2}\,{}^V(pR(X,Y)))\\
&= \tfrac{1}{2}\,{}^H(pg^{-1}\circ(R(\ ,X)Y - R(X,Y))),\\
(3.3)\qquad ({}^S\nabla_{{}^V\omega}F)({}^HY) &= {}^S\nabla_{{}^V\omega}(F\,{}^HY) - F({}^S\nabla_{{}^V\omega}{}^HY)\\
&= {}^S\nabla_{{}^V\omega}{}^V\tilde{Y} - \tfrac{1}{2}\,F\,{}^H(p(g^{-1}\circ R(\ ,Y)\tilde{\omega}))\\
&= -\tfrac{1}{2}\,{}^V(pR(\ ,Y)\tilde{\omega}),\\
(3.4)\qquad ({}^S\nabla_{{}^HX}F)({}^V\theta) &= {}^S\nabla_{{}^HX}(F\,{}^V\theta) - F({}^S\nabla_{{}^HX}{}^V\theta)\\
&= {}^S\nabla_{{}^HX}{}^H\tilde{\theta} - F({}^V(\nabla_X\theta) + \tfrac{1}{2}\,{}^H(p(g^{-1}\circ R(\ ,X)\tilde{\theta})))\\
&= {}^H(\nabla_X\tilde{\theta}) + \tfrac{1}{2}\,{}^V(pR(X,\tilde{\theta})) - {}^H(g^{-1}\circ(\nabla_X\theta))\\
&\quad - \tfrac{1}{2}\,{}^V(pg\circ(g^{-1}\circ R(\ ,X)\tilde{\theta}))\\
&= \tfrac{1}{2}\,{}^V(pR(X,\tilde{\theta}) - pR(\ ,X)\tilde{\theta}),
\end{aligned}$$

$$(3.5)\qquad ({}^S\nabla_{{}^V\omega}F)({}^V\theta) = {}^S\nabla_{{}^V\omega}(F\,{}^V\theta) - F({}^S\nabla_{{}^V\omega}{}^V\theta) = {}^S\nabla_{{}^V\omega}{}^H\tilde{\theta} = \tfrac{1}{2}\,{}^H(p(g^{-1}\circ R(\ ,\tilde{\theta})\tilde{\omega})).$$

From (3.2)–(3.5) we have

THEOREM 3.2. *The cotangent bundle of a Riemannian manifold is para-Kählerian (paraholomorphic Nordenian) with respect to the metric* ${}^Sg$ *and almost paracomplex structure* F *defined by* (3.1) *if and only if the Riemannian manifold is flat.*

4. A necessary and sufficient condition for the complete lift of a vector field to be paraholomorphic. A vector field $\tilde{X} \in \Im^1_0(T^*M^n)$ with respect to which the almost para-Nordenian structure F has a vanishing Lie derivative ($\mathrm{L}_{\tilde{X}}F = 0$) is said to be *almost paraholomorphic* (see [4]).

It is well known that [11, p. 277]

$$(4.1)\qquad \begin{cases} [{}^CX, {}^HY] = {}^H[X,Y] + {}^V(p(\mathrm{L}_X\nabla)Y), \\ [{}^CX, {}^V\omega] = {}^V(\mathrm{L}_X\omega), \end{cases}$$

where $(\mathrm{L}_X\nabla)Y = \nabla_Y\nabla X + R(X,Y)$ and $(\mathrm{L}_X\nabla)(Y,Z) = \mathrm{L}_X(\nabla_Y X) - \nabla_Y(\mathrm{L}_X Z) - \nabla_{[X,Y]}Z$.

A vector field $X \in \Im^1_0(M^n)$ is called a *Killing* vector field (or *infinitesimal isometry*) if $\mathrm{L}_Xg = 0$, and X is called an *infinitesimal affine transformation* if $\mathrm{L}_X\nabla_g = 0$. A Killing vector field is necessarily an infinitesimal affine transformation, i.e. we have $\mathrm{L}_X\nabla_g = 0$ as a consequence of $\mathrm{L}_Xg = 0$.

We now consider the Lie derivative of F with respect to the complete lift ${}^CX$. Taking account of (3.1) and (4.1), we obtain

$$\begin{aligned}(4.2)\qquad (\mathrm{L}_{{}^CX}F)^V\theta &= \mathrm{L}_{{}^CX}F\,{}^V\theta - F(\mathrm{L}_{{}^CX}{}^V\theta) = \mathrm{L}_{{}^CX}{}^H\tilde{\theta} - F({}^V(\mathrm{L}_X\theta)) \\ &= \mathrm{L}_{{}^CX}{}^H\tilde{\theta} - {}^H(g^{-1}\circ(\mathrm{L}_X\theta)) \\ &= {}^V[X,\tilde{\theta}] + {}^V(p(\mathrm{L}_X\nabla)\tilde{\theta}) - {}^H(g^{-1}\circ(\mathrm{L}_X\theta)) \\ &= {}^H(\mathrm{L}_X(g^{-1}\circ\theta) - g^{-1}\circ(\mathrm{L}_X\theta)) + {}^V(p(\mathrm{L}_X\nabla)\tilde{\theta}),\end{aligned}$$

$$\begin{aligned}(4.3)\qquad (\mathrm{L}_{{}^CX}F)^HY &= \mathrm{L}_{{}^CX}F\,{}^HY - F(\mathrm{L}_{{}^CX}{}^HY) \\ &= \mathrm{L}_{{}^CX}{}^V\tilde{Y} - F({}^H[X,Y] + {}^V(p(\mathrm{L}_X\nabla)_Y)) \\ &= {}^V(\mathrm{L}_X(g\circ Y) - g\circ\mathrm{L}_XY) - {}^H(g^{-1}\circ p(\mathrm{L}_X\nabla)_Y).\end{aligned}$$

Let now X be a Killing vector field ($\mathrm{L}_Xg = 0$). Then by virtue of $\mathrm{L}_X\nabla = 0$, from (4.2) and (4.3) we have $\mathrm{L}_{{}^CX}F = 0$, i.e. ${}^CX$ is paraholomorphic with respect to F. If we assume that $\mathrm{L}_{{}^CX}F = 0$ and compute the equation (4.3) at $(x^i, 0)$, $p_i = 0$, then we get $\mathrm{L}_X(g\circ Y) = g\circ\mathrm{L}_XY$. It follows that $\mathrm{L}_Xg = 0$. Hence, we have

Theorem 4.1. *An infinitesimal transformation X of the Riemannian manifold (M^n, g) is a Killing vector field if and only if its complete lift ${}^C X$ to the cotangent bundle T^*M^n is an almost paraholomorphic vector field with respect to the almost para-Nordenian structure $(F, {}^S g)$.*

Remark. Let ${}^R\nabla \in \Im^0_2(T^*M^n)$ be a Riemannian extension of the connection ∇_g defined by (cf. [11, p. 268])

$$ {}^R\nabla({}^C X, {}^C Y) = -p(\nabla_X Y + \nabla_Y X), \qquad X, Y \in \Im^1_0(\mathrm{M}_n). $$

The metric ${}^R\nabla$ has components

$$ {}^R\nabla = \begin{pmatrix} -2p_a \Gamma^a_{ji} & \delta^j_i \\ \delta^i_j & 0 \end{pmatrix} \tag{4.4} $$

with respect to the natural frame $\{\partial_i, \partial_{\bar\imath}\}$. From (1.2), (1.3) and (4.4) we easily see that

$$ {}^R\nabla({}^H X, {}^H Y) = 0, \quad {}^R\nabla({}^V\omega, {}^V\theta) = 0, \quad {}^R\nabla({}^H X, {}^V\theta) = {}^V(\theta(X)), \tag{4.5} $$

i.e. the metric ${}^R\nabla$ is completely determined also by conditions (4.5). Using (1.6), (3.1) and (4.5), we have

$$ \begin{aligned} ({}^R\nabla \circ F)({}^H X, {}^H Y) &= {}^R\nabla(F\,{}^H X, {}^H Y) = {}^R\nabla({}^V\tilde{X}, {}^H Y) = {}^V(\tilde{X}(Y)) \\ &= {}^V(g(X,Y)) = {}^S g({}^H X, {}^H Y), \\ ({}^R\nabla \circ F)({}^H X, {}^V\theta) &= {}^R\nabla(F\,{}^H X, {}^V\theta) = {}^R\nabla({}^V\tilde{X}, {}^V\theta) = {}^S g({}^H X, {}^V\theta) = 0, \\ ({}^R\nabla \circ F)({}^V\omega, {}^H Y) &= {}^R\nabla(F\,{}^V\omega, {}^H Y) = {}^R\nabla({}^H\tilde{\omega}, {}^H Y) = {}^S g({}^V\omega, {}^H Y) = 0, \\ ({}^R\nabla \circ F)({}^V\omega, {}^V\theta) &= {}^R\nabla(F\,{}^V\omega, {}^V\theta) = {}^R\nabla({}^H\tilde{\omega}, {}^V\theta) = {}^V(\theta(\tilde{\omega})) \\ &= {}^V(g^{-1}(\omega,\theta)) = {}^S g({}^V\omega, {}^V\theta), \end{aligned} $$

i.e. ${}^R\nabla \circ F = {}^S g$. Thus the almost para-Nordenian structure F determined by the condition (3.1) has an expression of the form $F = ({}^R\nabla)^{-1} \circ {}^S g$.

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