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# A proof of the valuation property and preparation theorem

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**Abstract.** The purpose of this article is to present a short model-theoretic proof of the valuation property for a polynomially bounded o-minimal theory $T$. The valuation property was conjectured by van den Dries, and proved for the polynomially bounded case by van den Dries–Speissegger and for the power bounded case by Tyne. Our proof uses the transfer principle for the theory $T_{\mathrm{conv}}$ (i.e. $T$ with an extra unary symbol denoting a proper convex subring), which—together with quantifier elimination—is due to van den Dries–Lewenberg. The main tools applied here are saturation, the Marker–Steinhorn theorem on parameter reduction and heir-coheir amalgams.

The significance of the valuation property lies to a great extent in its geometric content: it is equivalent to the preparation theorem which says, roughly speaking, that every definable function of several variables depends piecewise on any fixed variable in a certain simple fashion. The latter originates in the work of Parusiński for subanalytic functions, and of Lion–Rolin for logarithmic-exponential functions. Van den Dries–Speissegger have proved the preparation theorem in the o-minimal setting (for functions definable in a polynomially bounded structure or logarithmic-exponential over such a structure). Also, the valuation property makes it possible to establish quantifier elimination for polynomially bounded expansions of the real field $\mathbb{R}$ with exponential function and logarithm.

**1. Preliminaries.** Throughout this article we deal with a polynomially bounded o-minimal theory $T$ in a first-order language $\mathcal{L}$ with field of exponents $K$ (which is a subfield of the field $\mathbb{R}$ of reals). The word "definable" in a structure $\mathcal{R}$ always means "definable with parameters from $\mathcal{R}$"; "definable with no parameters" is called "0-definable". It is well known that one can always extend $\mathcal{L}$ and $T$ by definitions to:

$\mathcal{L}^{\mathrm{df}} :=$ $\mathcal{L}$ augmented by a new function symbol $f_\varphi(\overline{x})$ for each $\mathcal{L}$-formula $\varphi(\overline{x}, y)$ such that $T \vdash \forall \overline{x}\ \exists y\ \varphi(\overline{x}, y)$,

$T^{\mathrm{df}} :=$ $T$ extended by the new defining axioms $\varphi(\overline{x}, f_\varphi(\overline{x}))$.

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Every model $\mathcal{R}$ of $T$ expands uniquely to a model $\mathcal{R}^{\rm df}$ of $T^{\rm df}$. Since the theory $T$ has definable cell decomposition, $T^{\rm df}$ has quantifier elimination, and since $T$ has definable Skolem functions (which follows from cell decomposition as well), $T^{\rm df}$ has universal axiomatization. Consequently, $T$ has a prime model $\mathcal{P}$ which has a unique elementary embedding into every model $\mathcal{R}$ of $T$; the image of $\mathcal{P}$ in $\mathcal{R}$ consists of the interpretations of all constant symbols of the language $\mathcal{L}^{\rm df}$.

If $\mathcal{R}$ is an elementary substructure of a model $\mathcal{S}$ of $T$: $\mathcal{R} \prec \mathcal{S}$, then $\mathcal{R}^{\rm df}$ is a substructure of $\mathcal{S}^{\rm df}$ in the language $\mathcal{L}^{\rm df}$: $\mathcal{R}^{\rm df} \subset \mathcal{S}^{\rm df}$. For a subset $A \subset \mathcal{S}$, $\mathcal{R}\langle A\rangle$ denotes the definable closure of $A$ over $\mathcal{R}$ in $\mathcal{S}$, i.e. the substructure of $\mathcal{S}^{\rm df}$ generated by $\mathcal{R} \cup A$ in the language $\mathcal{L}^{\rm df}$; of course $\mathcal{R}\langle A\rangle$ is an elementary substructure of $\mathcal{S}$. The operation of definable closure satisfies the ordinary axioms for the span operation (in particular the Steinitz exchange property), whence one can define rank, $\operatorname{rk}\mathcal{R}$, or relative rank, $\operatorname{rk}(\mathcal{S}/\mathcal{R})$, in an ordinary fashion.

Consider two ordered fields $\mathcal{R} \subset \mathcal{S}$. We say that $\mathcal{R}$ is *Dedekind complete* or *tame* in $\mathcal{S}$ if one of the three equivalent conditions is satisfied:

(i) the trace on $\mathcal{R}$ of every interval in $\mathcal{S}$ is an interval in $\mathcal{R}$;
(ii) the cut made in $\mathcal{R}$ by every element $s \in \mathcal{S}$ is rational;
(iii) for each $\mathcal{R}$-bounded element $s \in \mathcal{S}$, there is a unique element $r \in \mathcal{R}$ such that $s - r$ is an $\mathcal{R}$-infinitesimal; we call $r := \operatorname{st}(s)$ the *standard part* of the element $s$.

Marker–Steinhorn Theorem on Parameter Reduction. *Consider o-minimal structures $\mathcal{R} \prec \mathcal{S}$ with $\mathcal{R}$ Dedekind complete in $\mathcal{S}$. Then the trace $X \cap \mathcal{R}^n$ of every definable set $X \subset \mathcal{S}^n$ in $\mathcal{S}^n$ is definable in $\mathcal{R}^n$. In other words, the trace $X \cap \mathcal{R}^n$ can be defined by means of parameters from $\mathcal{R}$ only.*

In this section we shall make use of the following well-known

Corollary (cf. [8, 1]). *Under the previous assumptions, if $f : \mathcal{S}^n \to \mathcal{S}$ is a definable function, then the sets*

$$E_{-\infty} := \{\overline{x} \in \mathcal{R}^n : f(\overline{x}) < \mathcal{R}\}, \qquad E_{+\infty} := \{\overline{x} \in \mathcal{R}^n : f(\overline{x}) > \mathcal{R}\}$$

*are definable, and the function*

$$\operatorname{st}(f(\overline{x})) : \mathcal{R}^n \setminus (E_{-\infty} \cup E_{+\infty}) \to \mathcal{R}$$

*is definable.*

Consider now the theory $T_{\rm conv}$ of pairs $(\mathcal{R}, V)$, where $V$ is a convex subring of a model $\mathcal{R}$ of $T$, in the language $\mathcal{L}$ with an extra unary relation symbol to denote $V$ (cf. [2]). Every maximal elementary substructure $\mathcal{R}'$ of $\mathcal{R}$ contained in $V$ is Dedekind complete and cofinal in $\mathcal{R}$, and isomorphic to the residue field $\overline{V}$ of $V$; furthermore, $V$ is the convex hull of $\mathcal{R}'$ in $\mathcal{R}$. Van

den Dries–Lewenberg [2] proved a relative version of quantifier elimination for $T_{\mathrm{conv}}$:

*If $T$ has quantifier elimination and universal axiomatization, then $T_{\mathrm{conv}}$ has quantifier elimination.*

Hence $T_{\mathrm{conv}}$ is a complete theory and we have at our disposal the transfer principle: in order to prove a theorem expressible in the first-order language $\mathcal{L}_{\mathrm{conv}}$ for all models of $T_{\mathrm{conv}}$, it suffices to prove it for one particular model.

The convex subring $V$ is a valuation ring in $\mathcal{R}$ with maximal ideal $\mathfrak{m} = \mathfrak{m}_V$ and valuation group $\Gamma = \Gamma_V$; let $v$ denote the induced valuation of the field $\mathcal{R}$. We now give a simple proof for the following proposition on stabilization of valuation due to van den Dries [1]. Our proof makes use of the transfer principle and piecewise uniform asymptotics only.

PROPOSITION 1 (on stabilization of valuation). *If $f : \mathcal{R} \to \mathcal{R}$ is a definable function, then there exists $u \in V$ such that for all $x \in V$, $x \geq u$, we have*

$$v(f(x)) = v(f(u)) \quad \textit{or equivalently} \quad \frac{f(x)}{f(u)} \in V \setminus \mathfrak{m}.$$

*Proof.* The function $f$ is of the form

$$f(x) = g(x, r_1, \ldots, r_m),$$

where $g$ is a 0-definable function and $\overline{r} \in \mathcal{R}^m$ are parameters. So we shall prove the statement for all parameters $\overline{r}$ from the model under consideration, $g$ being fixed. Since the assertion is expressible in the first-order language of the theory $T_{\mathrm{conv}}$, we may assume that $\mathcal{R} = \mathcal{P}\langle a \rangle$, where $|\mathcal{P}| < a$ and $V = \widehat{\mathcal{P}}$ is the convex hull of $\mathcal{P}$ in $\mathcal{R}$. Then every parameter $r_i$ equals $h_i(a)$ for a 0-definable function $h_i : \mathcal{P} \to \mathcal{P}$. Putting

$$k(x, y) := g(x, h_1(y), \ldots, h_m(y)),$$

we are thus reduced to considering 0-definable functions $f(x) = k(x, a)$, $k : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$.

In this case, in view of piecewise uniform asymptotics for polynomially bounded o-minimal theories, there exist an exponent $\mu \in K$, a $u \in \mathcal{P}$ and a 0-definable function $c(x)$ such that for every $x \in \mathcal{P}$, $x \geq u$, we have

$$\frac{k(x, y)}{c(x) y^{\mu}} \to 1 \quad \text{as } y \to \infty,\ y \in \mathcal{P}.$$

Hence for every $x, \varepsilon \in \mathcal{P}$, $x \geq u$, $\varepsilon > 0$, we have

$$\left| \frac{k(x, y)}{c(x) y^{\mu}} - 1 \right| < \varepsilon \quad \text{when } y \in \mathcal{P},\ y \gg 0.$$

It follows by overspill that

$$\left|\frac{k(x,a)}{c(x)a^{\mu}} - 1\right| < \varepsilon \quad \text{for every } x \in V,\ x \geq u.$$

Consequently, $v(f(x)) = v(k(x,a)) = \mu v(a)$ for all $x \in V$, $x \geq u$, as desired.

Let $f, g : \mathcal{R} \to \mathcal{R}$ be definable functions, and $\pi : V \to \overline{V}$ the canonical mapping onto the residue field $\overline{V}$ of $V$. We say that $f$ and $g$ are *asymptotic on* $V$, $f \underset{V}{\sim} g$, if either $f(x) = g(x) = 0$ for $x \in V$, $x \gg 0$, or

$$\frac{f(x)}{g(x)} \in V \quad \text{for } x \in V,\ x \gg 0, \quad \text{and} \quad \pi\left(\frac{f(x)}{g(x)}\right) \to 1 \quad \text{as } x \to \infty,\ x \in V.$$

Corollary. *There exist* $\lambda \in K$ *and* $c \in \mathcal{R}$ *such that* $f(x) \underset{V}{\sim} cx^{\lambda}$. *Consequently, for any* $\varepsilon = 1/v$ *with* $v \in V$, $v > 0$, *there exist* $u, w \in \mathcal{R}$, $u \in V$, $w > V$, *such that* $|f(x)/(cx^{\lambda}) - 1| < \varepsilon$ *for all* $x \in [u, w]$.

*Proof.* Indeed, the maximal elementary substructure $\mathcal{R}' \simeq \overline{V}$ of $\mathcal{R}$ contained in $V$ is Dedekind complete in $\mathcal{R}$. Taking $u \in \mathcal{R}'$ as in Proposition 1, it follows from the Corollary to the Marker–Steinhorn theorem that the function

$$\mathrm{st}\left(\frac{f(x)}{f(u)}\right) : \mathcal{R}' \to \mathcal{R}'$$

is $\mathcal{R}'$-definable. Then it is asymptotic to $c'x^{\lambda}$ for some $\lambda \in K$ and $c' \in \mathcal{R}'$:

$$\mathrm{st}\left(\frac{f(x)}{f(u)c'x^{\lambda}}\right) \to 1 \quad \text{as } x \to \infty,\ x \in \mathcal{R}'.$$

Putting $c := f(u)c'$ completes the proof.

Consider now a model $(\mathcal{R}, V)$ of the theory $T_{\mathrm{conv}}$ whose valuation $v$ has rank $d < \infty$. This means that the value group $\Gamma = \Gamma_V$ has $d+1$ isolated subgroups of the form

$$\Gamma_0 = (0) \subset \Gamma_1 \subset \Gamma_1 \oplus \Gamma_2 \subset \cdots \subset \Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_d,$$

where the subgroups $\Gamma_i$ are archimedian such that $\Gamma_1^+ < \Gamma_2^+ < \cdots < \Gamma_d^+$. We have, of course, a one-to-one correspondence between these isolated subgroups and the prime ideals $\mathfrak{p}_i$ in $V$, as well as the convex subrings $V_i$ such that $V \subset V_i \subset \mathcal{R}$:

$$\mathfrak{p}_i = \{x \in \mathcal{R} : v(x) > \Gamma_i\}, \quad V_i = \{x \in \mathcal{R} : v(x) > -\Gamma_{i+1}\} = V_{\mathfrak{p}_i},$$
$$\mathfrak{p}_0 = \mathfrak{m} \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_{d-1} \supset \mathfrak{p}_d = (0), \quad V_0 = V \subset V_1 \subset \cdots \subset V_{d-1} \subset V_d = \mathcal{R}.$$

The valuation group of $V_i$ is isomorphic to $\Gamma_{i+1} \oplus \cdots \oplus \Gamma_d$; the valuation group of $V_d = \mathcal{R}$ is $(0)$.

OBSERVATION. There exists an elementary extension $(\mathcal{R}^*, V^*)$ of $(\mathcal{R}, V)$ the value group $\Gamma^*$ of which is of the form

$$\Gamma^* = \underbrace{\mathbb{R} \oplus \cdots \oplus \mathbb{R}}_{d\,\text{times}}.$$

Such an elementary extension can be obtained by a successive adjunction of elements from an $\aleph_0$-saturation of $(\mathcal{R}, V)$. Here we sketch that procedure.

Every archimedian group $\Gamma_i$ may be regarded as a subgroup of the additive group $\mathbb{R}$ of real numbers; we may assume that $1 \in \Gamma_i$. Take any number $\delta \in \mathbb{R} \setminus \Gamma_i$; then $\delta$ makes an irrational cut $C$ in $\Gamma_i$. One can lift $C$ to a unique cut $\widetilde{C} := \{x \in \mathcal{R} : x \leq 0 \text{ or } (x > 0,\, v(x) > C)\}$ in $\mathcal{R}$, and next adjoin to $\mathcal{R}$ an element $a$ which realizes the cut $\widetilde{C}$. We get an elementary extension $(\mathcal{R}\langle a\rangle, W)$ of $(\mathcal{R}, V)$; clearly, $w(a)$ realizes the cut $C$ in $\Gamma_i$ as well. One must show that the valuation $w$ with value group $\Gamma_W$ obtained in this fashion is also of rank $d$. This is due to the control over definable functions in the vicinity of the cuts made by the convex subrings $V_i$ in $\mathcal{R}$, as described in the Corollary to Proposition 1. Indeed, every definable function $f : \mathcal{R} \to \mathcal{R}$ is asymptotic in each $V_i$ to a function $cx^\lambda$. Hence $|f(x)/(cx^\lambda) - 1| < 1/2$ for all $x$ from an interval $[u, w]$ with $u \in V_i$, $w > V_i$, and this inequality extends to the cuts made by the subrings $V_i$ in $\mathcal{R}$. Therefore, for every $\mathcal{R}$-definable function $g$, the element $b = g(a)$ can realize none of those cuts, and consequently $w(b) = w(g(a))$ can realize no cut in $\Gamma$ made by the isolated subgroups $\Gamma_i$. Otherwise the element $w(b)$ would generate yet another isolated subgroup of $\Gamma_W$. Further, if $a = f(b)$, we would get $|f(a)/(ca^\lambda) - 1| < 1/2$, and thus

$$w(a) = v(c) + \lambda w(b) \in \Gamma + Kw(b),$$

whence $w(b) \in \Gamma + Kw(a)$. But this is impossible since the group $\Gamma + Kw(a)$, on a par with the group $\Gamma$, has exactly $d + 1$ isolated subgroups whenever $w(a)$ defines an irrational cut in $\Gamma$.

Repeating successively the above procedure for each subgroup $\Gamma_i$ and all real numbers, we obtain an increasing chain of elementary extensions (of cardinality $\leq$ power of the continuum). By the Tarski–Vaught lemma, the union of this chain is the desired elementary extension $(\mathcal{R}^*, V^*)$ of $(\mathcal{R}, V)$.

Applying the transfer principle and saturated models, we shall prove the following

PROPOSITION 2. *If a definable function $f : \mathcal{R} \to \mathcal{R}$ is constant on no interval in $\mathcal{R}$, then there exist $s \in \mathcal{R}$, $\lambda \in K \setminus \{0\}$ and $c \in \mathcal{R}$ such that*

$$f(x) - s \underset{V}{\sim} cx^\lambda.$$

*Consequently, for any $\varepsilon = 1/v$ with $v \in V$, $v > 0$, there exist $u, w \in \mathcal{R}$, $u \in V$, $w > V$, such that $|f(x) - s/(cx^\lambda) - 1| < \varepsilon$ for all $x \in [u, w]$.*

*Proof.* In view of the Corollary to Proposition 1, our statement is equivalent to the following first-order sentence:

$$\exists s \in \mathcal{R}\ \forall u \in V\ \exists x_1, x_2 \in V,\ x_1, x_2 > u \quad \left|\frac{f(x_1) - s}{f(x_2) - s}\right| > 2;$$

here the number 2 may be replaced by any real number $> 1$. Via the transfer principle, it suffices to consider one model of the theory $T_{\mathrm{conv}}$. Take a model $(\mathcal{R}, V)$ with $V = \widehat{\mathcal{P}}$ = convex hull of the prime model $\mathcal{P}$ in $\mathcal{R}$, where $\mathcal{R}$ is an $\alpha^+$-saturated model of the theory $T$ with $\alpha =$ cofinality of $\mathcal{P}$. For simplicity we confine ourselves to the case $\alpha = \aleph_0$; the general case goes the same way, but with transfinite induction instead of an ordinary induction argument.

We prove Proposition 2 for the above model by *reductio ad absurdum*. Suppose the contrary, i.e. for any $s \in \mathcal{R}$ the function $f(x) - s \underset{V}{\sim} c$ is asymptotic to a constant function $c$, i.e. $\lambda = 0$; obviously, $c \neq 0$.

We then assert that $f(x) - s \in c(1 + \mathfrak{m})$ for all $x \in V$, $x \gg 0$. For otherwise, if $\mathcal{R}' \simeq \overline{V}$ is a maximal elementary substructure of $\mathcal{R}$ contained in $V$, the function

$$\mathrm{st}\left(\frac{f(x) - s}{c}\right) : \mathcal{R}' \to \mathcal{R}'$$

would be asymptotic to 1 but $\neq 1$ ultimately in $\mathcal{R}'$. So $(f(x) - s)/c - 1$ would be asymptotic to $dx^\lambda$ with $d \in \mathcal{R}'$, $\lambda \in K$, $\lambda < 0$, and consequently

$$f(x) - s - c \underset{V}{\sim} cdx^\lambda,$$

contrary to our hypothesis. Therefore $\mathrm{st}((f(x) - s)/c) = 1$ ultimately in $\mathcal{R}'$, and thus $f(x) - s \in c(1 + \mathfrak{m})$ for all $x \in V$, $x \gg 0$, as asserted.

Take any $a_0 \in V$; then there is some $c_0 \in \mathcal{R}$ such that

$$f(x) - f(a_0) \underset{V}{\sim} c_0, \qquad f(x) - f(a_0) \in c_0(1 + \mathfrak{m})$$

for all $x \in V$, $x \geq a_1$ with $a_1 \in V$. The last condition is equivalent to $|(f(x) - f(a_0) - c_0)/c_0| < \varepsilon_k$ for all $k$, where the $1/\varepsilon_k$ form a cofinal sequence in $V$. Each of these inequalities extends (by overspill) to an interval $[a_1, b_{1k}]$ with $b_{1k} > V$. Since $\mathcal{R}$ is $\alpha^+$-saturated, there is an element $b_1 \in \mathcal{R}$ such that $1/\varepsilon_k < b_1 < b_{1k}$ for all $k \in \mathbb{N}$. Clearly, $V < b_1$ and

$$f(x) - f(a_0) \in c_0(1 + \mathfrak{m}) \quad \text{for all } x \in [a_1, b_1].$$

Next we get $f(x) - f(a_1) \in c_0 \cdot \mathfrak{m}$ for $x \in [a_1, b_1]$, whence, as before,

$$f(x) - f(a_1) \underset{V}{\sim} c_1 \in c_0 \cdot \mathfrak{m} \quad \text{and} \quad f(x) - f(a_1) \in c_1(1 + \mathfrak{m})$$

for all $x \in [a_2, b_2]$ with $a_2, b_2 \in \mathcal{R}$, $a_2 \in V$, $b_2 > V$.

By induction we can construct three sequences $(a_n)$, $(b_n)$, $(c_n)$ of elements of $\mathcal{R}$ such that $a_n \in V$, $a_n > 1/\varepsilon_n$ are cofinal in $V$, $b_n > V$, the sequence

$(v(c_n))$ is strictly increasing, and

$$v(f(x) - f(a_n)) = v(c_n) \quad \text{for all } x \in [a_{n+1}, b_{n+1}].$$

Since $\mathcal{R}$ is $\alpha^+$-saturated, we can find an element $b \in \mathcal{R}$ such that $a_n < b < b_n$ for all $n \in \mathbb{N}$. Then $v(f(b) - f(a_n)) = v(c_n)$ is a strictly increasing sequence, and thus the valuation $v(g(x))$ of the function $g(x) := f(b) - f(x)$ does not stabilize. This contradiction with Proposition 1 completes the proof.

## 2. Valuation group of a simple extension

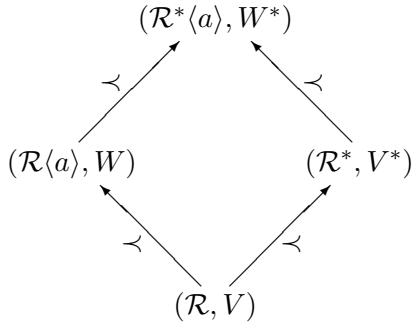
PROPOSITION 3. *Let $\mathcal{R}$ be a finitely generated and polynomially bounded structure, $V \subset \mathcal{R}$ a convex subring of $\mathcal{R}$, and $v : \mathcal{R} \setminus \{0\} \to \Gamma = \Gamma_V$ the induced valuation. Then*

(i) $\operatorname{rk} v \leq \dim_K \Gamma \leq \operatorname{rk} \mathcal{R}$.
(ii) *If $(\mathcal{R}, V) \subset (\mathcal{R}\langle a\rangle, W)$, then $\dim_K \Gamma_W \leq \dim_K \Gamma + 1$; more precisely, $\Gamma_W = \Gamma \oplus Kw(a)$ whenever $w(a) \notin \Gamma$.*

*Proof.* We proceed by induction on $\operatorname{rk} \mathcal{R}$ and prove point (ii), which is the induction step. By induction hypothesis, $\operatorname{rk} v =: d \leq \operatorname{rk} \mathcal{R} < \infty$. Then $\Gamma = \Gamma_V = \Gamma_1 \oplus \cdots \oplus \Gamma_d$, and we can find—as described in the observation of Section 1—an elementary extension $(\mathcal{R}^*, V^*)$ of $(\mathcal{R}, V)$ such that

$$\Gamma^* = \underbrace{\mathbb{R} \oplus \cdots \oplus \mathbb{R}}_{d \text{ times}}.$$

Consider now an heir-coheir amalgam of elementary extensions




Clearly, $w^*(a) \notin \Gamma^*$ if $w(a) \notin \Gamma$. Indeed, were $w^*(a) \in \Gamma^*$, then

$$\exists c^* \in \mathcal{R}^* \quad w^*(a) = w^*(c^*), \quad \text{i.e.} \quad \frac{a}{c^*} \in W^* \setminus \mathfrak{m}_{W^*},$$

and thus, by the heir-coheir property,

$$\exists c \in \mathcal{R} \quad w(a) = w(c), \quad \text{i.e.} \quad \frac{a}{c} \in W \setminus \mathfrak{m}_W,$$

which is a contradiction.

It suffices to establish (ii) for the elementary extension

$$(\mathcal{R}^*, V^*) \subset (\mathcal{R}^*\langle a\rangle, W^*).$$

Indeed, let $b = f(a)$, where $f : \mathcal{R} \to \mathcal{R}$ is a definable function. Suppose $w^*(b) \in \Gamma^* + \lambda w^*(a)$ for a $\lambda \in K$, whence

$$\exists c^* \in \mathcal{R}^* \quad w^*(b) = w^*(c^* a^\lambda), \quad \text{i.e.} \quad \frac{f(a)}{c^* a^\lambda} \in W^* \setminus \mathfrak{m}_{W^*}.$$

It follows from the heir-coheir property that

$$\exists c \in \mathcal{R} \quad w(b) = w(ca^\lambda), \quad \text{i.e.} \quad \frac{f(a)}{ca^\lambda} \in W \setminus \mathfrak{m}_W,$$

and thus $w(b) \in \Gamma + \lambda w(a)$, as required.

Now we shall show that if $w^*(a) \notin \Gamma^*$, then for every $b \in \mathcal{R}^*\langle a\rangle$ we have

$$w^*(b) \in \Gamma^* + K w^*(a).$$

Since $\Gamma^*$ is the direct sum of a finite number of copies of $\mathbb{R}$, and the ordered set $\mathbb{R}$ of real numbers is Dedekind complete, there exists an element $\gamma = v^*(r) \in \Gamma^*$, $r \in \mathcal{R}^*$, such that $w^*(a) - w^*(r)$ realizes the cut made by an isolated subgroup of $\Gamma^*$ in $\Gamma^*$. For, if the cut made by $w^*(a)$ lies inside of a $\Gamma_i^* \simeq \mathbb{R}$, we take the real number $\delta \in \Gamma_i^*$ which is closest to $w^*(a)$. Hence $|w^*(a) - \delta| < (\Gamma_i^*)^+$, and either we are done or we repeat the reasoning for a $\Gamma_j^*$ with $j < i$.

Therefore the element $a/r$ realizes the cut made by a convex subring of $\mathcal{R}^*$ in $\mathcal{R}^*$. Clearly, $b = f(a/r)$ for some definable function $f : \mathcal{R}^* \to \mathcal{R}^*$. It follows from the Corollary to Proposition 1 that

$$\begin{aligned} w^*(b) &= w^*(f(a/r)) = w^*(c(a/r)^\lambda) \\ &= w^*(c) - \lambda w^*(r) + \lambda w^*(a) \in \Gamma^* + K w^*(a), \end{aligned}$$

concluding the proof.

COROLLARY. *Consider a polynomially bounded, o-minimal theory $T$ and a simple extension $(\mathcal{R}, V) \subset (\mathcal{R}\langle a\rangle, W)$ of models of the theory $T_{\mathrm{conv}}$. Then* $\dim_K \Gamma_W \le \dim_K \Gamma_V + 1$; *more precisely,* $\Gamma_W = \Gamma_V \oplus K w(a)$ *whenever* $w(a) \notin \Gamma_V$.

*Proof.* Indeed, if $b \in \mathcal{R}\langle a\rangle$, then $b \in \mathcal{R}'\langle a\rangle$ for a finitely generated substructure $\mathcal{R}' \prec \mathcal{R}$, $\operatorname{rk}\mathcal{R}' < \infty$; let $V' := V \cap \mathcal{R}'$ and $\Gamma_{V'}$ be its valuation group. By Proposition 3, we get $w(b) \in \Gamma_{V'} + K w(a) \subset \Gamma_V + K w(a)$, as asserted.

REMARK. Proposition 3 immediately implies a stronger inequality (cf. [1, Section 5]):

*If* $\operatorname{rk}\mathcal{R} < \infty$ *and* $\overline{V}$ *is the residue field of the convex subring* $V$, *then*

$$\operatorname{rk}\overline{V} + \dim_K \Gamma_V \le \operatorname{rk}\mathcal{R}.$$

From the above one can derive the following Wilkie inequality (*loc. cit.*) through an argument of Wilkie (cf. [12]), based on saturated models and an iteration procedure:

*Suppose $T$ is a polynomially bounded theory and $(\mathcal{R}, V) \prec (\mathcal{S}, W)$ are models of $T_{\mathrm{conv}}$ with $\mathrm{rk}(\mathcal{S}/\mathcal{R}) \leq \infty$. Then*

$$\dim_K(\Gamma_W/\Gamma_V) + \mathrm{rk}(\overline{W}/\overline{V}) \leq \mathrm{rk}(\mathcal{S}/\mathcal{R}).$$

## 3. Valuation property and preparation theorem

Valuation Property. *Consider a polynomially bounded, o-minimal theory $T$ and a simple extension*

$$(\mathcal{R}, V) \subset (\mathcal{R}\langle a\rangle, W)$$

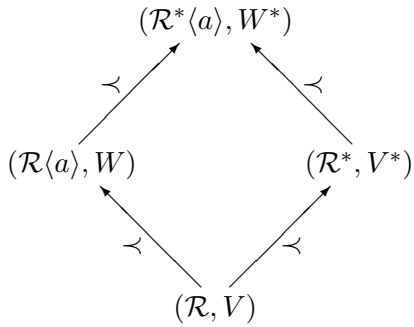
*of models of the theory $T_{\mathrm{conv}}$ with valuation groups $\Gamma_V$ and $\Gamma_W$, respectively. If $\Gamma_V \neq \Gamma_W$, then there is an $r \in \mathcal{R}$ for which $w(a-r) \notin \Gamma_V$.*

*Proof.* We may assume that $\mathrm{rk}\,\mathcal{R} < \infty$, because $w(b) \notin \Gamma_V$ for some $b \in \mathcal{R}'\langle a\rangle$, where $\mathcal{R}'$ is a finitely generated substructure of $\mathcal{R}$. Now, if for some $r \in \mathcal{R}'$ the valuation $w'(a-r)$ does not belong to the valuation group $\Gamma_{V'}$ of the restriction $v' := v|\mathcal{R}'$, it follows from Proposition 3 that

$$w'(a-r) = \gamma + \lambda w'(b)$$

for some $\lambda \in K$, $\lambda \neq 0$, and $\gamma \in \Gamma_{V'}$. Hence $w(a-r) \notin \Gamma_V$.

Consider, as in the proof of Proposition 3, an heir-coheir amalgam of elementary extensions




It suffices to establish the valuation property for the simple extension $(\mathcal{R}^*, V^*) \subset (\mathcal{R}^*\langle a\rangle, W^*)$. Indeed, $w^*(b) \notin \Gamma_{V^*}$, for otherwise

$$\exists r^* \in \mathcal{R}^* \quad w^*(b) = v^*(r^*),$$

and thus, by the heir-coheir property, we would get a contradiction

$$\exists r \in \mathcal{R} \quad w(b) = v(r) \in \Gamma_V.$$

Further, if $w^*(a-r^*) \notin \Gamma_{W^*}$, then $w^*(a-r^*) \in \Gamma_{V^*} + \lambda w^*(b)$ for some $\lambda \in K$, $\lambda \neq 0$, whence

$$\exists r^*, s^* \in \mathcal{R}^* \quad w^*(a - r^*) = v^*(s^*) + \lambda w^*(b).$$

Therefore, again by the heir-coheir property, we get the required result

$$\exists r, s \in \mathcal{R} \quad w(a - r) = v(s) + \lambda w(b) \notin \Gamma_V.$$

We thus have to show that if $w^*(b) \notin \Gamma_{V^*}$, then $w^*(a - r) \notin \Gamma_{V^*}$ for some $r \in \mathcal{R}^*$. Since the valuation group $\Gamma_{V^*}$ is the direct sum of a finite number of copies of $\mathbb{R}$ (which are Dedekind complete), we deduce—as in the proof of Proposition 3—that for some $v^*(r) \in \Gamma_{V^*}$, $r \in \mathcal{R}^*$, the element $w^*(b) - v^*(r)$ realizes the cut made by an isolated subgroup of $\Gamma_{V^*}$ in $\Gamma_{V^*}$. Replacing $b$ by $b/r$ we may, of course, assume that $b$ realizes the cut made by a convex subring of $\mathcal{R}^*$ in $\mathcal{R}^*$. Clearly, $a = f(b)$ for some definable function $f : \mathcal{R}^* \to \mathcal{R}^*$ which is constant on no interval in $\mathcal{R}^*$. Now it follows from Proposition 2 that there exist $s \in \mathcal{R}^*$, $\lambda \in K$, $\lambda \neq 0$, and $c \in \mathcal{R}^*$ such that

$$w^*(a - r) = w^*(f(b) - r) = w^*(c\, b^\lambda) = w^*(c) + \lambda w^*(b) \notin \Gamma^*_{V^*},$$

which completes the proof.

The valuation property yields, via a routine compactness argument, the preparation theorem for one variable:

Corollary. *Let $\mathcal{R}$ be a polynomially bounded, o-minimal structure, $f : \mathcal{R} \to \mathcal{R}$ be a definable function and $\varepsilon \in \mathbb{Q}$, $\varepsilon > 0$. Then there exist*

$$\lambda_1, \ldots, \lambda_k \in K, \quad r_1, \ldots, r_k, c_1, \ldots, c_k \in \mathcal{R}$$

*such that for all $x \in \mathcal{R}$ we have*

$$f(x) = |x - r_i|^{\lambda_i} \cdot c_i \cdot u$$

*for an $i = 1, \ldots, k$ and some $u \in \mathcal{R}$ with $|u - 1| < \varepsilon$.*

*Proof.* Indeed, by passing to the theory $T^{\mathrm{df}}$ in the extended language $\mathcal{L}^{\mathrm{df}}$, we shall deal with models $\mathcal{S}$ which are elementary extensions of $\mathcal{R}$, $\mathcal{R} \prec \mathcal{S}$. Through compactness, we must show that for each $a \in \mathcal{S}$ there exist $\lambda \in K$, $r, c \in \mathcal{R}$ and $u \in \mathcal{S}$, $|u - 1| < \varepsilon$, such that

$$f(a) = |a - r|^\lambda \cdot c \cdot u.$$

Consider now, as the convex subrings of $\mathcal{R}$, $\mathcal{R}\langle a \rangle$ and $\mathcal{S}$, respectively, the convex hulls of the field of real numbers $\mathbb{R}$ in these fields. Then it follows from the valuation property and the Corollary to Proposition 3 that there exist $\lambda \in K$, $r, c \in \mathcal{R}$ and $u \in \mathcal{S}$ such that

$$w(u) = 0, \quad f(a) = |a - r|^\lambda \cdot c \cdot u.$$

Since $w(u) = 0$, we see that $u$ is of the form $u_0 + \mathbb{R}$-infinitesimal, and thus $|u/u_0 - 1| < \varepsilon$. Replacing $c$ and $u$ with $u_0 c$ and $u/u_0$, respectively, we get the desired result.

The general version below can immediately be obtained through model-theoretic compactness and definable choice:

PREPARATION THEOREM. *Under the previous assumptions, consider a definable function* $f : \mathcal{R}^{n+1} \to \mathcal{R}$ *and an* $\varepsilon \in \mathbb{Q}$, $\varepsilon > 0$. *Then there exist* $\lambda_1, \dots, \lambda_k \in K$ *and definable functions*

$$r_1, \dots, r_k, c_1, \dots, c_k : \mathcal{R}^n \to \mathcal{R}, \qquad u_1, \dots, u_k : \mathcal{R}^{n+1} \to (1-\varepsilon, 1+\varepsilon) \subset \mathcal{R}$$

*such that for all* $\overline{t} \in \mathcal{R}^n$ *and* $x \in \mathcal{R}$ *we have*

$$f(\overline{t}, x) = |x - r_i(\overline{t})|^{\lambda_i} \cdot c_i(\overline{t}) \cdot u_i(\overline{t}, x) \quad \text{for an } i = 1, \dots, k.$$

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