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Gauge natural constructions on higher order principal prolongations

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Abstract. Let $W^r_m P$ be a principal prolongation of a principal bundle $P \to M$. We classify all gauge natural operators transforming principal connections on $P \to M$ and rth order linear connections on M into general connections on $W^r_m P \to M$. We also describe all geometric constructions of classical linear connections on $W^r_m P$ from principal connections on $P \to M$ and rth order linear connections on M.

Introduction. Let G be a Lie group and denote by $\mathcal{PB}_m(G)$ the category of principal G-bundles with m-dimensional bases and their local principal G-bundle isomorphisms with the identity isomorphisms of G. Given a principal bundle $P \to M$, we denote by $W^r_m P$ its principal prolongation (see Section 1 below). The aim of this paper is to study the prolongation of principal connections on $P \to M$ to general and classical linear connections on $W^r_m P$. In [2] and [16] it is clarified that in such geometric constructions the use of some additional geometric object cannot be avoided. Moreover, many geometric constructions on the prolongations of fibered manifolds use in an essential way an auxiliary linear connection on the base manifold M (see e.g. [6], [9] and [13]). So a linear connection on M is a useful tool, which enables a number of geometric constructions. Using that point of view, we have the following open problems:

Problem 1. Classify all $\mathcal{PB}_m(G)$-gauge natural operators transforming principal connections on $P \to M$ and rth order linear connections on M into *general connections* on $W^r_m P \to M$.

Problem 2. Classify all $\mathcal{PB}_m(G)$-gauge natural operators transforming principal connections on $P \to M$ and rth order linear connections on M into *classical linear connections* on $W^r_m P$.

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Up till now, Problem 1 has been solved only in some particular cases. For the first order differential group $G = G^1_m$, I. Kolář [10] has classified all $\mathcal{PB}_m(G)$-gauge natural operators transforming principal connections Γ on $P \to M$ and linear connections Λ on M into principal connections on $W^1_m P \to M$. Moreover, I. Kolář and G. Virsik [14] have solved a similar problem for an arbitrary Lie group G and for symmetric Λ.

We point out that gauge natural bundles and operators form the geometric background for field theories and many other areas of mathematical physics (see e.g. [4], [5], [7], [8], [15], [18]). We also underline that the principal prolongation $W^r_m P$ plays a fundamental role in the theory of gauge natural bundles and operators and this space is also a useful tool and powerful recurrence model for higher order geometry in general (see [1], [10], [13]). The most important result from this field is that every gauge bundle functor on $\mathcal{PB}_m(G)$ is associated to $W^r_m P$ (see [13]). Further, the jet prolongations of associated bundles are associated bundles to the principal prolongations of the corresponding principal bundles. Moreover, denoting by $P^r M$ the rth order frame bundle of M, we have the canonical inclusion $P^r M \subset W^1_m(P^{r-1}M)$. We also recall that the theory of prolongations of principal bundles and connections has its origins in the works of C. Ehresmann [3].

In Section 2 we determine all gauge natural operators transforming principal connections on $P \to M$ and rth order linear connections on M into maps $W^r_m P \to \mathbb{R}$. Section 3 is devoted to the solution of Problem 1. We show that all gauge natural operators in question are determined by the flow prolongation and by some natural difference tensor fields on $W^r_m P$. The solution of Problem 2 is described in Section 4. In what follows we use the notation and terminology from the book [13]. All manifolds and maps are assumed to be infinitely differentiable.

1. The foundations. We recall that a *general connection* on a fibered manifold $Y \to M$ is a smooth section $\Gamma : Y \to J^1 Y$ of the first jet prolongation of Y. If $P \to M$ is a principal G-bundle, then we have a canonical right action $b : J^1 P \times G \to J^1 P$, and a connection $\Gamma : P \to J^1 P$ is called *principal* if it is b-invariant. Moreover, an *rth order linear connection* on M means a linear splitting $\Delta : TM \to J^r TM$ of the projection $J^r TM \to TM$. Clearly, for $r = 1$ we obtain the classical linear connection on M.

Given a principal bundle $P \to M$ with m-dimensional basis, its rth *principal prolongation* $W^r_m P$ is the space of all r-jets $j^r_{(0,e)}\varphi$ of local principal bundle isomorphisms $\varphi : \mathbb{R}^m \times G \to P$, where $e \in G$ is the unit. By [13], $W^r_m P \to M$ is a principal bundle with the structure group

$$W^r_m G := J^r_{(0,e)}(\mathbb{R}^m \times G, \mathbb{R}^m \times G)_{(0,-)}.$$

Moreover, the fibered manifold $W^r_m P \to M$ coincides with the fibered product

$$W^r_m P = P^r M \times_M J^r P,$$

where $P^r M := \operatorname{inv} J^r_0(\mathbb{R}^m, M)$ is the rth order frame bundle of M. Using nonholonomic or semiholonomic jets, one can define the nonholonomic or semiholonomic principal prolongations $\widetilde{W}^r_m P$ or $\overline{W}^r_m P$, respectively.

Obviously, $W^r_m P$ is a gauge bundle functor on $\mathcal{PB}_m(G)$ in the following sense. Denote by $B : \mathcal{FM} \to \mathcal{M}f$ the base functor, where $\mathcal{FM}$ is the category of fibered manifolds and fiber respecting mappings and $\mathcal{M}f$ is the category of smooth manifolds and all smooth maps. A *gauge bundle functor* on $\mathcal{PB}_m(G)$ is a covariant functor $F : \mathcal{PB}_m(G) \to \mathcal{FM}$ such that

(a) every $\mathcal{PB}_m(G)$-object $\pi : P \to BP$ is transformed into a fibered manifold $q_P : FP \to BP$ over BP,
(b) every $\mathcal{PB}_m(G)$-morphism $f : P \to \overline{P}$ is transformed into a fibered morphism $Ff : FP \to F\overline{P}$ over Bf,
(c) for every open subset $U \subset BP$ the inclusion $i : \pi^{-1}(U) \to P$ is transformed into the inclusion $Fi : q_P^{-1}(U) \to FP$.

The general concept of gauge natural operators can be found in the book [13]. In particular, a $\mathcal{PB}_m(G)$-*gauge natural operator* D transforming principal connections Γ on $P \to M$ and rth order linear connections Λ on M into general connections $D(\Gamma, \Lambda)$ on $W^r_m P \to M$ is a system of $\mathcal{PB}_m(G)$-invariant regular operators (functions)

$$D_P : \mathrm{Con}_G(P) \times \mathrm{Con}^r(M) \to \mathrm{Con}(W^r_m P)$$

for any $\mathcal{PB}_m(G)$-object $P \to M$, where $\mathrm{Con}_G(P)$ is the set of all principal connections on $P \to M$, $\mathrm{Con}^r(M)$ is the set of all rth order linear connections on M and $\mathrm{Con}(W^r_m P)$ is the set of all general connections on $W^r_m P \to M$. The invariance means that if $(\Gamma, \Lambda) \in \mathrm{Con}_G(P) \times \mathrm{Con}^r(M)$ and $(\Gamma_1, \Lambda_1) \in \mathrm{Con}_G(P_1) \times \mathrm{Con}^r(M_1)$ are f-related by a $\mathcal{PB}_m(G)$-map $f : P \to P_1$ covering $\underline{f} : M \to M_1$, then $D_P(\Gamma, \Lambda)$ and $D_{P_1}(\Gamma_1, \Lambda_1)$ are $W^r_m f$-related. The regularity means that D_P transforms smoothly parametrized families of pairs of connections into smoothly parametrized families of connections. Quite similarly one can define $\mathcal{PB}_m(G)$-gauge natural operators D transforming principal connections Γ on $P \to M$ and rth order linear connections Λ into classical linear connections $D(\Gamma, \Lambda)$ on $W^r_m P$ (or functions $D(\Gamma, \Lambda) : W^r_m P \to \mathbb{R}$ or tensor fields $D(\Gamma, \Lambda)$ on $W^r_m P$ or into other geometric objects).

2. Construction of functions on $W^r_m P$. Write

$$\theta = j^1_0(\mathrm{id}_{\mathbb{R}^m}) \in (P^1\mathbb{R}^m)_0, \quad (P^r\mathbb{R}^m)_\theta = \{j^r_0\varphi \in (P^r\mathbb{R}^m)_0 | j^1_0\varphi = \theta\}$$

and

$$\Theta = j_0^\infty(\mathrm{id}_{\mathbb{R}^m}, e) \in J_0^\infty(\mathbb{R}^m \times G), \quad \text{where } e \text{ is the neutral element in } G.$$

For $s = 0, 1, \ldots, \infty$ let S^s be the space of all s-jets $j_0^s(\Lambda)$ at $0 \in \mathbb{R}^m$, where Λ is an rth order linear connection on $\mathbb{R}^m$ such that the underlying classical linear connection Λ_1 of Λ has the Christoffel symbols $(\Lambda_1)^i_{jk} : \mathbb{R}^m \to \mathbb{R}$ satisfying $\sum_{j,k=1}^m (\Lambda_1)^i_{jk}(x)x^j x^k = 0$ for $i = 1, \ldots, m$. Equivalently, S^s is the space of all s-jets $j_0^s(\Lambda)$ at 0, where Λ is an rth order linear connection on $\mathbb{R}^m$ such that the usual coordinate system $x^1, \ldots, x^m$ on $\mathbb{R}^m$ is a normal coordinate system with centre 0 for the underlying classical linear connection Λ_1 of Λ. Then S^s are manifolds diffeomorphic to some finite-dimensional vector spaces for $s = 0, 1, \ldots$.

For $s = 0, 1, \ldots, \infty$ let Z^s be the space of all s-jets $j_0^s(\Gamma)$ at $0 \in \mathbb{R}^m$, where Γ is a principal connection on $\mathbb{R}^m \times G \to \mathbb{R}^m$. Clearly, Z^s is an affine space (finite-dimensional if s is finite). Of course, Z^∞ has the inverse limit topology from $\cdots \to Z^s \to Z^{s-1} \to \cdots \to Z^1$ and Z^s has the usual topology for finite s. Consider a function

$$\mu : Z^\infty \times S^\infty \times (P^r\mathbb{R}^m)_\theta \to \mathbb{R} \tag{1}$$

with the following two properties **I** and **II**:

I. For any $\kappa \in Z^\infty$, $\varrho \in S^\infty$, $\sigma \in (P^r\mathbb{R}^m)_\theta$ and any $\mathcal{PB}_m(G)$-map $H : \mathbb{R}^m \times G \to \mathbb{R}^m \times G$ covering $\mathrm{id}_{\mathbb{R}^m}$ and preserving Θ we have

$$\mu(H_*\kappa, \varrho, \sigma) = \mu(\kappa, \varrho, \sigma), \tag{2}$$

where $H_*\kappa = j_0^\infty(H_*\Gamma)$, $\kappa = j_0^\infty\Gamma$.

II. For any $\kappa \in Z^\infty$, $\varrho \in S^\infty$ and $\sigma \in (P^r\mathbb{R}^m)_\theta$ we can find an open neighbourhood $W \subset Z^\infty$ of κ, an open neighbourhood $U \subset S^\infty$ of ϱ, an open neighbourhood $V \subset (P^r\mathbb{R}^m)_\theta$ of σ, a natural number s and a smooth map $f : \pi_s(W) \times \pi_s(U) \times V \to \mathbb{R}$ such that

$$\mu = f \circ (\pi_s \times \pi_s \times \mathrm{id}_V)$$

on $W \times U \times V$, where $\pi_s : J^\infty \to J^s$ is the jet projection.

A simple example of a μ satisfying **I** and **II** can be obtained as follows. Let $\widetilde{\mu} : S^s \to \mathbb{R}$ be a smooth map for some finite s. We can define $\mu : Z^\infty \times S^\infty \times (P^r\mathbb{R}^m)_\theta \to \mathbb{R}$ by $\mu(j_0^\infty\Gamma, j_0^\infty\Lambda, \sigma) = \widetilde{\mu}(j_0^s\Lambda)$.

Given a principal connection Γ on $P \to M$ and an rth order linear connection Λ on M with the underlying classical linear connection Λ_1, we define a smooth map $\mathcal{D}^{\langle\mu\rangle}(\Gamma, \Lambda) : W_m^r P = P^r M \times_M J^r P \to \mathbb{R}$ by

$$\mathcal{D}^{\langle\mu\rangle}(\Gamma, \Lambda)(\sigma, \eta) := \mu(j_0^\infty(\Phi_*\Gamma), j_0^\infty(\varphi_*\Lambda_1), P^r\varphi(\sigma)) \tag{3}$$

for $\sigma \in (P^r M)_x$, $\eta \in J_x^r(P)$, $x \in M$, where φ is a normal coordinate system on M for Λ_1 with centre x such that $\varphi(x) = 0$ and $P^r\varphi(\sigma) \in (P^r\mathbb{R}^m)_\theta$, and Φ is a principal coordinate system on P covering φ and sending η into Θ.

The definition of $\mathcal{D}^{\langle\mu\rangle}$ is correct. Clearly, $\mathrm{germ}_x(\varphi)$ is uniquely determined. Moreover, if Φ_1 is another coordinate system with the properties of Φ, then we have locally $\Phi_1 = H \circ \Phi$ for some $H : \mathbb{R}^m \times G \to \mathbb{R}^m \times G$ covering the identity map $\mathrm{id}_{\mathbb{R}^m}$ and preserving Θ.

The correspondence $\mathcal{D}^{\langle\mu\rangle} : (\Gamma, \Lambda) \mapsto \mathcal{D}^{\langle\mu\rangle}(\Gamma, \Lambda)$ is a $\mathcal{PB}_m(G)$-gauge natural operator transforming principal connections Γ on $P \to M$ and rth order linear connections Λ on M into maps $\mathcal{D}^{\langle\mu\rangle}(\Gamma, \Lambda) : W^r_m P \to \mathbb{R}$.

PROPOSITION 1. *Any $\mathcal{PB}_m(G)$-gauge natural operator D transforming principal connections Γ on $P \to M$ and rth order linear connections Λ on M into maps $D(\Gamma, \Lambda) : W^r_m P \to \mathbb{R}$ is equal to $\mathcal{D}^{\langle\mu\rangle}$ for a unique function $\mu : Z^\infty \times S^\infty \times (P^r\mathbb{R}^m)_\theta \to \mathbb{R}$ satisfying* **I** *and* **II**. *Moreover, the space $\mathcal{N}$ of all such $\mathcal{PB}_m(G)$-gauge natural operators D is an algebra.*

Proof. Let D be an operator in question. Define $\mu : Z^\infty \times S^\infty \times (P^r\mathbb{R}^m)_\theta \to \mathbb{R}$ by

$$\mu(j^\infty_0 \Gamma, j^\infty_0(\Lambda), \sigma) = D(\Gamma, \Lambda)_{(\sigma,\Theta)}.$$

Using the naturality of D we can easily see that μ has property **I**. By the nonlinear Peetre theorem [13], μ also has property **II**. Finally, taking into account naturality, one directly verifies $D = \mathcal{D}^{\langle\mu\rangle}$. ∎

REMARK 1. One can construct the map μ with properties **I** and **II** such that $\mathcal{D}^{\langle\mu\rangle}$ is of strictly infinite order. For example, let $\mu : Z^\infty \times S^\infty \times (P^r\mathbb{R}^m)_\theta \to \mathbb{R}$ be given by $\mu(\kappa, \varrho, \sigma) = \widetilde{\mu}(\varrho)$ for some $\widetilde{\mu} : S^\infty \to \mathbb{R}$. Then condition **I** is trivially satisfied. Condition **II** and the strictly infinite order of $\mathcal{D}^{\langle\mu\rangle}$ can be obtained by choosing suitable $\widetilde{\mu} : S^\infty \to \mathbb{R}$ as follows. The system $\cdots \to S^s \to S^{s-1} \to \cdots \to S^1$ is diffeomorphic to $\cdots \to \mathbb{R}^{k_s} \to \mathbb{R}^{k_{s-1}} \to \cdots \to \mathbb{R}^{k_1}$. On $S^1 \cong \mathbb{R}^{k_1}$ we choose smooth maps $\lambda_s : \mathbb{R}^{k_1} \to \mathbb{R}$ which are equal to 1 in the ring $R_s = \{x \in \mathbb{R}^{k_1} \mid s - 1/4 \le |x| \le s + 1/4\}$ and to 0 outside the ring $R'_s = \{x \in \mathbb{R}^{k_1} \mid s - 1/3 \le |x| \le s + 1/3\}$. Let $\widetilde{\mu}_s : S^s \cong \mathbb{R}^{k_s} \to \mathbb{R}$ be a nonzero linear map which is zero on $\mathbb{R}^{k_{s-1}} \subset \mathbb{R}^{k_s}$. Then we put $\widetilde{\mu}(j^\infty_0 \Lambda) = \sum_{s\in\mathbb{N}} \lambda_s(j^1_0\Lambda)\widetilde{\mu}_s(j^s_0\Lambda)$. Clearly, condition **I** is satisfied. Moreover, $\mathcal{D}^{\langle\mu\rangle}$ is of strictly infinite order because $\widetilde{\mu}$ does not factorize (globally) through $S^s \to \mathbb{R}$ with finite s.

3. Solution of Problem 1. The following assertion justifies the use of a linear connection Λ in the formulation of Problem 1.

PROPOSITION 2 ([2]). *Let F be any of the functors W^r_m, $\widetilde{W}^r_m$, $\overline{W}^r_m$. Then there is no $\mathcal{PB}_m(G)$-gauge natural operator $\mathcal{A}$ transforming principal connections Γ on $P \to M$ into general connections $\mathcal{A}(\Gamma)$ on $FP \to M$.*

EXAMPLE 1. Given a principal connection $\Gamma : P \to J^1P$ and an rth order linear connection $\Lambda : TM \to J^rTM$, one can construct a connection

$\mathcal{W}^r_m(\Gamma, \Lambda)$ on $W^r_m P$ as follows (see [13]). Take a vector field X on M and denote by $\Gamma X : P \to TP$ its Γ-lift to P. Then the flow prolongation $\mathcal{W}^r_m(\Gamma X)$ is a vector field on $W^r_m P$ depending on r-jets of X only. This can be interpreted as a bundle map $W^r_m P \times_M J^r TM \to TW^r_m P$. Then the composition with Λ is the lifting map $W^r_m P \times_M TM \to TW^r_m P$ of the required connection $\mathcal{W}^r_m(\Gamma, \Lambda)$ and $\mathcal{W}^r_m : (\Gamma, \Lambda) \mapsto \mathcal{W}^r_m(\Gamma, \Lambda)$ is a $\mathcal{PB}_m(G)$-gauge natural operator.

It is well known that $J^1 Y \to Y$ is an affine bundle with the associated vector bundle $VY \otimes T^* M$ (see [13]). Taking into account the operator $\mathcal{W}^r_m$ from Example 1, we have

THEOREM 1. *Any $\mathcal{PB}_m(G)$-gauge natural operator $\mathcal{A}$ transforming principal connections Γ on $P \to M$ and rth order linear connections Λ on M into general connections $\mathcal{A}(\Gamma, \Lambda)$ on $W^r_m P \to M$ is of the form*

$$\mathcal{A}(\Gamma, \Lambda) = \mathcal{W}^r_m(\Gamma, \Lambda) + \mathcal{C}(\Gamma, \Lambda)$$

for a unique $\mathcal{PB}_m(G)$-gauge natural operator $\mathcal{C}$ transforming Γ and Λ into tensor fields $C(\Gamma, \Lambda)$ of the type $T^* M \otimes VW^r_m P$ on $W^r_m P$.

In the rest of this section we describe all gauge natural operators $\mathcal{C}$ from Theorem 1. First we introduce some canonical (more precisely, natural in the sense of [13]) tensor fields on $W^r_m P$. Let

$$\varphi \in (T_0 \mathbb{R}^m)^* \otimes \mathcal{L}ie(W^r_m G).$$

Define a natural tensor field $\mathcal{C}^\varphi \in T^* M \otimes VW^r_m P$ as follows. Take an element

$$(\sigma, \eta) \in (W^r_m P)_x = (P^r M)_x \times J^r_x P,$$

$x \in M$, $v \in T_x M$ and let $\widetilde{\sigma} \in (P^1 M)_x$ be the element underlying σ. Choose a chart ψ on M near x such that $P^1 \psi(\widetilde{\sigma}) = \theta$. Then $T_x \psi$ is uniquely determined and we have $\varphi(T_x \psi(v)) \in \mathcal{L}ie(W^r_m G)$. We put

$$\mathcal{C}^\varphi(v)_{(\sigma,\eta)} = (\varphi(T_x \psi(v)))^*(\sigma, \eta), \tag{4}$$

where A^* means the fundamental vertical vector field on the principal $W^r_m G$-bundle $W^r_m P \to M$ for any $A \in \mathcal{L}ie(W^r_m G)$. Let A_α, $\alpha \in T$, be a basis over $\mathbb{R}$ of the vector space $\mathcal{L}ie(W^r_m G)$. Let $d_0 x^i$, $i = 1, \ldots, m$, be the usual basis in $(T_0 \mathbb{R}^m)^*$. Then $d_0 x^i \otimes A_\alpha \in (T_0 \mathbb{R}^m)^* \otimes \mathcal{L}ie(W^r_m G)$ and we easily obtain

LEMMA 1. *The natural tensor fields*

$$\mathcal{C}^{\alpha,i} = \mathcal{C}^{d_0 x^i \otimes A_\alpha} \tag{5}$$

for $\alpha \in T$ and $i = 1, \ldots, m$ (defined above for $\varphi := d_0 x^i \otimes A_\alpha$) form a basis of the $C^\infty(W^r_m P, \mathbb{R})$-module of tensor fields of the type $T^* M \otimes VW^r_m P$ on $W^r_m P$ over the algebra $C^\infty(W^r_m P, \mathbb{R})$ of smooth maps $W^r_m P \to \mathbb{R}$.

Proof. We make use of the fact that the $(A_\alpha)^*$ for $\alpha \in T$ form a basis over $C^\infty(W^r_m P, \mathbb{R})$ of the vertical vector fields on $W^r_m P$. ■

Obviously, the space $\mathcal{M}$ of all $\mathcal{PB}_m(G)$-gauge natural operators $\mathcal{C}$ transforming principal connections Γ on $P \to M$ and rth order linear connections Λ on M into tensor fields $\mathcal{C}(\Gamma, \Lambda)$ of the type $T^*M \otimes VW^r_mP$ on W^r_mP is a module over the algebra $\mathcal{N}$ from Proposition 1.

PROPOSITION 3. *The above $\mathcal{N}$-module $\mathcal{M}$ is free and finite-dimensional. The natural tensor fields $\mathcal{C}^{\alpha,i}$ for $\alpha \in T$ and $i = 1, \dots, m$ form a basis of this module over $\mathcal{N}$.*

Proof. Let $\mathcal{C} \in \mathcal{M}$ be a natural operator in question. By Lemma 1, for any principal connection Γ on $P \to M$ and an rth order linear connection Λ on M we can write

$$\mathcal{C}(\Gamma, \Lambda) = \sum D_{\alpha,i}(\Gamma, \Lambda)\mathcal{C}^{\alpha,i},$$

where $D_{\alpha,i}(\Gamma, \Lambda) : W^r_mP \to \mathbb{R}$ are some uniquely determined maps. Because of the invariance of $\mathcal{C}$ with respect to $\mathcal{PB}_m(G)$-maps and the naturality of $\mathcal{C}^{\alpha,i}$ we get $D_{\alpha,i} \in \mathcal{N}$. ∎

EXAMPLE 2. Clearly, if $G = \{e\}$ is a singleton, then we have $P = M \times \{e\}$, $W^r_mP = P^rM$, $W^r_mG = G^r_m := \operatorname{inv} J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$ is the differential group of order r and the connection $\mathcal{W}^r_m(\Gamma, \Lambda)$ from Example 1 is nothing but Λ. Moreover, the vector fields A_α from Lemma 1 are basis of the vector space $\mathcal{L}ie(G^r_m)$. By [12], there is a canonical bijection between rth order linear connections $\Lambda : TM \to J^rTM$ and principal connections on P^rM. So Theorem 1 for $G = \{e\}$ describes all natural operators transforming principal connections Λ on P^rM into general connections on P^rM. All such natural operators are of the form

$$\Lambda \mapsto \Lambda + \mathcal{C}(\Lambda)$$

for a unique natural operator $\mathcal{C}$ transforming Λ into tensor fields $\mathcal{C}(\Lambda)$ of the type $T^*M \otimes VP^rM$ on P^rM. Moreover, all tensor fields $\mathcal{C}(\Lambda)$ are classified in Propositions 1 and 3 for $G = \{e\}$.

REMARK 2. The rth order nonholonomic principal prolongation can also be defined by the iteration $\widetilde{W}^r_mP = W^1_m(\widetilde{W}^{r-1}_mP)$ and we have $\widetilde{W}^r_m(\widetilde{W}^s_m) = \widetilde{W}^{r+s}_m$. By [19], the same method can be used to construct connections on $\widetilde{W}^r_mP \to M$. Indeed, if $\mathcal{A}$ is a $\mathcal{PB}_m(G)$-gauge natural operator transforming principal connections Γ on $P \to M$ and classical linear connections Λ on M into connections on $W^1_mP \to M$, we can write $\mathcal{A}_1(\Gamma, \Lambda) = \mathcal{A}(\Gamma, \Lambda)$ and $\mathcal{A}_r(\Gamma, \Lambda) = \mathcal{A}(\mathcal{A}_{r-1}(\Gamma, \Lambda), \Lambda)$. Then $\mathcal{A}_r(\Gamma, \Lambda)$ is the connection on $\widetilde{W}^r_mP \to M$. Moreover, quite analogously to Example 1 we have the operator $\widetilde{\mathcal{W}}^r_m$. We remark that P. Vašík [19] has also introduced other constructions of connections on nonholonomic and semiholonomic principal prolongations.

4. Solution of Problem 2. According to the following general result from [16], to obtain a classical linear connection on W^r_mP from a principal connection Γ on $P \to M$, the use of an auxiliary linear connection Λ on M is unavoidable.

Proposition 4. *Let F be a gauge bundle functor on $\mathcal{PB}_m(G)$. Then there is no $\mathcal{PB}_m(G)$-gauge natural operator $\mathcal{A}$ transforming principal connections Γ on $P \to M$ into classical linear connections $\mathcal{A}(\Gamma)$ on FP.*

Example 3. By [13], a principal connection Γ on $\pi : P \to M$ and a classical linear connection Λ_1 on M determine a classical linear connection $N_P(\Gamma, \Lambda_1)$ on P in the following way. Given a tangent vector $A \in T_yP$, denote by vA its vertical component and by bA its projection to M. Consider now a vector field X on M such that $j^1_xX = \Lambda_1(bA)$, $x = \pi(y)$. Further, let X^Γ be the Γ-lift of X and denote by $\varphi(vA)$ the fundamental vector field determined by vA. Then the formula

$$A \mapsto j^1_y(X^\Gamma + \varphi(vA))$$

determines a classical linear connection $N_P(\Gamma, \Lambda_1) : TP \to J^1(TP \to P)$. We remark that all $\mathcal{PB}_m(G)$-gauge natural operators of this type for symmetric Λ_1 are determined in [11] and a similar problem in the case of a vector bundle was solved in [6].

Example 4. Consider the principal connection $\mathcal{W}^r_m(\Gamma, \Lambda)$ on $W^r_mP \to M$ from Example 1 and denote by $\Lambda_1 : TM \to J^1TM$ the underlying classical linear connection of $\Lambda : TM \to J^rTM$. Using the operator N_P from Example 3, we have the classical linear connection $N^r_m(\Gamma, \Lambda)$ on W^r_mP determined by

$$N^r_m(\Gamma, \Lambda) := N_{W^r_mP}(\mathcal{W}^r_m(\Gamma, \Lambda), \Lambda_1).$$

Obviously, $N^r_m : (\Gamma, \Lambda) \mapsto N^r_m(\Gamma, \Lambda)$ is a $\mathcal{PB}_m(G)$-gauge natural operator.

The difference of two classical linear connections on M is a tensor of the type $TM \otimes T^*M \otimes T^*M$. So we have

Theorem 2. *Any $\mathcal{PB}_m(G)$-gauge natural operator $\mathcal{A}$ transforming principal connections Γ on $P \to M$ and rth order linear connections Λ on M into classical linear connections $\mathcal{A}(\Gamma, \Lambda)$ on W^r_mP is of the form*

$$\mathcal{A}(\Gamma, \Lambda) = N^r_m(\Gamma, \Lambda) + \mathcal{C}(\Gamma, \Lambda)$$

for a unique $\mathcal{PB}_m(G)$-gauge natural operator $\mathcal{C}$ transforming Γ and Λ into tensor fields $\mathcal{C}(\Gamma, \Lambda)$ of the type $(1, 2)$ on W^r_mP.

In the rest of this section we describe all gauge natural operators $\mathcal{C}$ from Theorem 2. First we show that Γ and Λ induce certain parallelism on W^r_mP. Let A_α, $\alpha \in T$, be a basis over $\mathbb{R}$ of $\mathcal{L}ie(W^r_mG)$. Then we have fundamental vertical vector fields A^*_α on W^r_mP. Moreover, we have vector fields $B_i(\Gamma,\Lambda)$, $i=1,\dots,m$, on W^r_mP given by

$$B_i(\Gamma,\Lambda)_{(j^r_0\varphi,j^r_x\sigma)} = \left(\varphi_*\frac{\partial}{\partial x^i}\right)^{\mathcal{W}^r_m(\Gamma,\Lambda)}(j^r_0\varphi,j^r_x\sigma),$$

$(j^r_0\varphi,j^r_x\sigma)\in P^rM\times_M J^rP = W^r_mP$, where $X^{\mathcal{W}^r_m(\Gamma,\Lambda)}$ means the horizontal lift of a vector field X on M to W^r_mP with respect to the principal connection $\mathcal{W}^r_m(\Gamma,\Lambda)$ from Example 1.

LEMMA 2. *The vector fields* A^*_α *and* $B_i(\Gamma,\Lambda)$ *form a basis of the* $C^\infty(W^r_mP,\mathbb{R})$*-module of vector fields on* W^r_mP *over the algebra* $C^\infty(W^r_mP,\mathbb{R})$ *of smooth maps* $W^r_mP\to\mathbb{R}$.

Proof. This is a simple observation. ∎

Using tensor products and dualization of base vector fields from Lemma 2, we have the corresponding basis $\mathcal{B}_\beta(\Gamma,\Lambda)$, $\beta\in B$, of the module of tensor fields of the type $(1,2)$ on W^r_mP over $C^\infty(W^r_mP,\mathbb{R})$. The space $\mathcal{K}$ of all $\mathcal{PB}_m(G)$-gauge natural operators $\mathcal{C}$ transforming principal connections Γ on $P\to M$ and rth order linear connections Λ on M into tensor fields $\mathcal{C}(\Gamma,\Lambda)$ of the type $(1,2)$ on W^r_mP is obviously the module over the algebra $\mathcal{N}$ described in Proposition 1.

PROPOSITION 5. *The above* $\mathcal{N}$*-module* $\mathcal{K}$ *is free and* $(\dim(W^r_mG)+m)^3$*-dimensional. Moreover*, $\mathcal{B}_\beta$ *for* $\beta\in B$ *is a basis of* $\mathcal{K}$.

Proof. Let $\mathcal{C}\in\mathcal{K}$ be a natural operator in question. For any principal connection Γ on $P\to M$ and an rth order linear connection Λ on M we can write

$$\mathcal{C}(\Gamma,\Lambda)=\sum D_\beta(\Gamma,\Lambda)\mathcal{B}_\beta(\Gamma,\Lambda),$$

where $D_\beta(\Gamma,\Lambda): W^r_mP\to\mathbb{R}$ are some uniquely determined maps. Because of the invariance of $\mathcal{C}$ with respect to $\mathcal{PB}_m(G)$-maps and the invariance of $s\mathcal{B}_\beta$ we get $D_\beta\in\mathcal{N}$. ∎

By Lemma 2 we have the basis A^*_α, $B_i(\Gamma,\Lambda)$ of the module of vector fields on W^r_mP. Using tensor products and dualization we also have the corresponding basis $\mathcal{B}^{p,q}_\beta(\Gamma,\Lambda)$, $\beta\in B_{p,q}$, of tensor fields of the type (p,q) on W^r_mP. Similarly to Proposition 5 we have

PROPOSITION 6. *The* $\mathcal{N}$*-module* $\mathcal{K}_{p,q}$ *of all* $\mathcal{PB}_m(G)$*-gauge natural operators* $\mathcal{C}$ *transforming principal connections* Γ *on* $P\to M$ *and* r*th order linear connections* Λ *on* M *into tensor fields* $\mathcal{C}(\Gamma,\Lambda)$ *of the type* (p,q) *on* W^r_mP *is free and* $(\dim(W^r_mG)+m)^{p+q}$*-dimensional. Moreover*, $\mathcal{B}^{r,q}_\beta$, $\beta\in B_{p,q}$, *is a basis of* $\mathcal{K}_{p,q}$ *over* $\mathcal{N}$.

EXAMPLE 5. Quite analogously to Example 2, Theorem 2 for $G=\{e\}$ describes all natural operators transforming principal connections Λ on P^rM

into classical linear connections on P^rM. If we denote by Λ_1 the underlying classical linear connection on M and by N_P the operator from Example 3, all such natural operators are of the form

$$\Lambda \mapsto N_{P^rM}(\Lambda, \Lambda_1) + \mathcal{C}(\Lambda)$$

for a unique natural operator $\mathcal{C}$ transforming Λ into tensor fields $\mathcal{C}(\Lambda)$ of the type $(1,2)$ on P^rM. Further, all natural tensor fields $\mathcal{C}(\Lambda)$ are described in Propositions 1 and 5 for $G = \{e\}$. We remark that the second author [17] has described in a similar way all natural operators transforming classical linear connections on M into classical linear connections on P^rM.

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