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On some types of slant curves in contact pseudo-Hermitian 3-manifolds

by Cihan Özgür and Şaban Güvenç (Balıkesir)

Abstract. We study slant curves in contact Riemannian 3-manifolds with pseudo-Hermitian proper mean curvature vector field and pseudo-Hermitian harmonic mean curvature vector field for the Tanaka–Webster connection in the tangent and normal bundles, respectively. We also study slant curves of pseudo-Hermitian $AW(k)$-type.

1. Introduction. A Riemannian submanifold with vanishing Laplacian ΔH of the mean curvature vector is called a *biharmonic submanifold* (see B.-Y. Chen [Chen]). In [Dim], Dimitrić proved that the only biharmonic curves in a Euclidean space are straight lines. In [BG], curves satisfying $\Delta^{\perp} H = \lambda H$ in a Euclidean space were classified, where $\Delta^{\perp}$ denotes the Laplacian of the curve in the normal bundle and λ is a real valued function. In [ABG], a classification of curves satisfying $\Delta H = \lambda H$ and $\Delta^{\perp} H = \lambda H$ in a real space form was given by J. Arroyo, M. Barros and O. J. Garay. In [KA], B. Kılıç and K. Arslan studied connected submanifolds satisfying $\Delta^{\perp} H = 0$ in a Euclidean space.

A curve in a contact 3-manifold is said to be *slant* if its tangent vector field has a constant angle with the Reeb vector field. In particular, if the contact angle is equal to $\pi/2$, then the curve is called a *Legendre curve*. Slant curves appear naturally in differential geometry of Sasakian manifolds. In [CL], J. T. Cho and J. E. Lee studied contact pseudo-Hermitian geometry in a 3-dimensional Sasakian space form whose holomorphic sectional curvature with respect to the Tanaka–Webster connection $\hat{\nabla}$ is $2c$. They proved that if a non-geodesic curve for $\hat{\nabla}$ in a 3-dimensional contact Riemannian manifold is a slant curve, then the ratio of $\widehat{\kappa}$ and $\widehat{\tau}$ is a constant, where $\widehat{\kappa}$ and $\widehat{\tau}$ denote the curvature and torsion of the curve with respect to the

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connection $\hat{\nabla}$. Furthermore, in [Lee], J. E. Lee studied Legendre curves in contact pseudo-Hermitian 3-manifolds. She considered Legendre curves with pseudo-Hermitian parallel mean curvature vector field, pseudo-Hermitian proper mean curvature vector field and pseudo-Hermitian proper mean curvature vector field in the normal bundle.

In [AO], K. Arslan and the first author studied curves of $AW(k)$-type. In [OT], the first author and M. M. Tripathi considered $AW(k)$-type Legendre curves in α-Sasakian manifolds. J. E. Lee [Lee] defined and studied Legendre curves of pseudo-Hermitian $AW(k)$-type in a 3-dimensional Sasakian manifold.

In the present paper, we study slant curves with pseudo-Hermitian parallel mean curvature vector field, pseudo-Hermitian proper mean curvature vector field and pseudo-Hermitian proper mean curvature vector field in the normal bundle. We also study slant curves of pseudo-Hermitian $AW(k)$-type in contact pseudo-Hermitian 3-manifolds. Since a Legendre curve is a special type of a slant curve, our results generalize the results of [Lee].

2. Preliminaries. A $(2n+1)$-dimensional manifold M is called a *contact manifold* if there exists a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Given a contact form η, there exists a unique vector field ξ, the *characteristic vector field*, which satisfies $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$ for any vector field X on M. There exists an associated Riemannian metric g and a $(1,1)$-type tensor field φ satisfying

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \varphi Y), \tag{2.1}$$

for all $X, Y \in \chi(M)$. From (2.1), it follows that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

A Riemannian manifold equipped with the structure tensors (φ, ξ, η, g) satisfying (2.1) is called a *contact Riemannian manifold*. It is denoted by $M = \{M, \varphi, \xi, \eta, g\}$. Using the Lie differentiation operator in the characteristic direction ξ, the operator h is defined by $h = \frac{1}{2} L_\xi \varphi$. From the definition, h is symmetric and satisfies the equations below (see [Blair]), where ∇ denotes the Levi-Civita connection:

$$h\xi = 0, \quad h\varphi = -\varphi h, \quad \nabla_X \xi = -\varphi X - \varphi h X. \tag{2.3}$$

For a $(2n+1)$-dimensional contact manifold $M = \{M, \varphi, \xi, \eta, g\}$, the almost complex structure J on $M \times \mathbb{R}$ is defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right), \tag{2.4}$$

where X is a vector field tangent to M, t is the coordinate function of $\mathbb{R}$

and f is a C^∞ function on $M \times \mathbb{R}$. The contact Riemannian manifold M is called a *Sasakian manifold* if J is integrable.

On a Sasakian manifold, the covariant derivative $\nabla\varphi$ is given by

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X, \qquad X,Y \in \chi(M). \tag{2.5}$$

Let γ be a non-geodesic curve in a 3-dimensional Riemannian manifold M and $\{T,N,B\}$ its Frenet frame field. Then the Frenet frame field satisfies the following *Frenet–Serret* equations:

$$\begin{aligned} \nabla_T T &= \kappa N, \\ \nabla_T N &= -\kappa T \ + \tau B, \\ \nabla_T B &= -\tau N, \end{aligned} \tag{2.6}$$

where $\kappa = \|\nabla_T T\|$ is the *geodesic curvature* of γ and τ its *geodesic torsion.*

Let $\{M,\varphi,\xi,\eta,g\}$ be a 3-dimensional contact Riemannian manifold. Then the tangent space T_pM of M at a point $p \in M$ decomposes as

$$T_pM = D_p \oplus \mathbb{R}\xi_p, \qquad D_p = \{v \in T_pM \mid \eta(v) = 0\}.$$

Here $D : p \to D_p$ defines a two-dimensional distribution orthogonal to ξ, which is called the *contact distribution.* The restriction of φ to D, $J = \varphi|_D$, defines an almost complex structure on D. The associated almost CR-structure of M is given by the holomorphic subbundle

$$H = \{X - iJX \mid X \in D\}$$

of the complexified tangent bundle $TM^{\mathbb{C}}$. Each fiber H_p is of complex dimension 1, $H \cap \overline{H} = \{0\}$, and $D \otimes \mathbb{C} = H \oplus \overline{H}$. Furthermore, denoting the space of all smooth sections of H by $\chi(H)$, the integrability condition

$$[\chi(H),\chi(H)] \subset \chi(H)$$

is satisfied, so the associated almost CR-structure is always integrable. For H the *Levi form* L is defined by

$$L : D \times D \to C^\infty(M,\mathbb{R}), \qquad L(X,Y) = -d\eta(X,JY),$$

where $C^\infty(M,\mathbb{R})$ denotes the algebra of smooth functions on M. The Levi form is Hermitian and positive definite. We call the pair (η, L) a *contact pseudo-convex pseudo-Hermitian structure* on M, and we call M a *contact strongly pseudo-convex pseudo-Hermitian* (or *almost CR-*) *manifold* [Blair].

The *Tanaka–Webster connection* ([Tan], [Web]) $\widehat{\nabla}$ (or the pseudo-Hermitian connection) on a contact pseudo-convex pseudo-Hermitian manifold $M = \{M,\eta,L\}$ is defined by

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X\eta)(Y)\xi - \eta(Y)\nabla_X\xi$$

for all $X,Y \in \chi(M)$. Using (2.3), $\widehat{\nabla}$ can be rewritten as

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + \eta(Y)(\varphi X + \varphi hX) - g(\varphi X + \varphi hX, Y)\xi. \tag{2.7}$$

The Tanaka–Webster connection $\widehat{\nabla}$ has the torsion

$$\widehat{T}(X,Y) = 2g(X,\varphi Y)\xi + \eta(Y)\varphi hX - \eta(X)\varphi hY. \tag{2.8}$$

In particular, since $h = 0$ for Sasakian manifolds (see [Blair]), equations (2.7) and (2.8) reduce to

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi,$$
$$\widehat{T}(X,Y) = 2g(X,\varphi Y)\xi.$$

Proposition 2.1 ([Tann]). *The Tanaka–Webster connection on a 3-dimensional contact Riemannian manifold $M = \{M,\varphi,\xi,\eta,g\}$ is the unique linear connection satisfying the following four conditions:*

(i) $\widehat{\nabla}\eta = 0,\ \widehat{\nabla}\xi = 0;$
(ii) $\widehat{\nabla}g = 0,\ \widehat{\nabla}\varphi = 0;$
(iii) $\widehat{T}(X,Y) = -\eta([X,Y])\xi,\ X,Y \in D;$
(iv) $\widehat{T}(\xi,\varphi Y) = -\varphi\widehat{T}(\xi,Y),\ Y \in D.$

3. Slant curves in contact pseudo-Hermitian geometry. Let M be a contact Riemannian 3-manifold and assume that $\gamma : I \to M$ is a curve parametrized by arc-length in M. In [CL], J. T. Cho and J. E. Lee defined the Frenet frame field $\{T,N,B\}$ along γ for the pseudo-Hermitian connection $\widehat{\nabla}$, which satisfies the following Frenet–Serret equations for $\widehat{\nabla}$:

$$\begin{aligned} \widehat{\nabla}_T T &= \widehat{\kappa} N,\\ \widehat{\nabla}_T N &= -\widehat{\kappa} T + \widehat{\tau} B,\\ \widehat{\nabla}_T B &= -\widehat{\tau} N, \end{aligned} \tag{3.1}$$

where $\widehat{\kappa} = \|\widehat{\nabla}_T T\|$ is the *pseudo-Hermitian curvature* of γ and $\widehat{\tau}$ its *pseudo-Hermitian torsion*. A *pseudo-Hermitian helix* is a curve whose pseudo-Hermitian curvature and pseudo-Hermitian torsion are non-zero constants. In particular, curves with constant non-zero pseudo-Hermitian curvature and zero pseudo-Hermitian torsion are called *pseudo-Hermitian circles*. *Pseudo-Hermitian geodesics* are pseudo-Hermitian helices whose pseudo-Hermitian curvature and pseudo-Hermitian torsion are zero [CL].

Let M be a contact metric 3-manifold and $\gamma(s)$ a Frenet curve in M parametrized by arc-length. The *contact angle* $\alpha(s)$ is defined by $\cos[\alpha(s)] = g(T(s),\xi)$. The curve γ is called a *slant curve* if its contact angle is constant. Slant curves with contact angle $\pi/2$ are traditionally called *Legendre curves*.

In the present paper, we assume that all curves are non-geodesic Frenet curves, that is, $\widehat{\kappa} \neq 0$.

Proposition 3.1 ([CL]). *A curve γ for $\widehat{\nabla}$ is a slant curve if and only if $\eta(N) = 0$.*

PROPOSITION 3.2 ([CL]). *Let γ be a slant curve for $\widehat{\nabla}$ in a 3-dimensional contact Riemannian manifold M. Then the ratio of $\widehat{\tau}$ and $\widehat{\kappa}$ is a constant.*

Note that

$$\widehat{\tau}/\widehat{\kappa} = \cot \alpha_0, \tag{3.2}$$

where α_0 is the contact angle of γ.

In [CL], J. T. Cho and J. E. Lee proved the following proposition:

PROPOSITION 3.3. *If a curve in a 3-dimensional contact Riemannian manifold is a Legendre curve for the Tanaka–Webster connection $\widehat{\nabla}$, then $\widehat{\tau} = 0$.*

We have the following corollary:

COROLLARY 3.4. *Let γ be a slant curve for the Tanaka–Webster connection $\widehat{\nabla}$ with contact angle α_0 in a 3-dimensional contact Riemannian manifold M. Then γ is a Legendre curve if and only if $\widehat{\tau} = 0$.*

4. Pseudo-Hermitian mean curvature vector field. The *pseudo-Hermitian mean curvature vector field* $\widehat{H}$ of a curve γ in a 3-dimensional contact Riemannian manifold is defined by

$$\widehat{H} = \widehat{\nabla}_T T = \widehat{\kappa} N \tag{4.1}$$

(see [Lee]).

In a 3-dimensional contact Riemannian manifold M with the Tanaka–Webster connection $\widehat{\nabla}$, a vector field X normal to the curve γ is called *pseudo-Hermitian parallel* [Lee] if $\widehat{\nabla}^{\perp}_T X = 0$.

Differentiating (4.1), we get

$$\widehat{\nabla}^{\perp}_T \widehat{H} = \widehat{\kappa}' N + \widehat{\kappa}\widehat{\tau} B. \tag{4.2}$$

PROPOSITION 4.1. *γ is a curve with pseudo-Hermitian parallel mean curvature vector field if and only if it is a pseudo-Hermitian circle.*

Proof. Let γ be a curve with $\widehat{\nabla}^{\perp}_T \widehat{H} = 0$. Using (4.2), we get

$$\widehat{\kappa}' N + \widehat{\kappa}\widehat{\tau} B = 0. \tag{4.3}$$

So $\widehat{\kappa}$ is a non-zero constant and $\widehat{\tau} = 0$. Hence γ is a pseudo-Hermitian circle.

Conversely, let γ be a pseudo-Hermitian circle. Then $\widehat{\kappa}$ is a non-zero constant and $\widehat{\tau} = 0$. This implies $\widehat{\nabla}^{\perp}_T \widehat{H} = (\widehat{\nabla}_T H)^{\perp} = \widehat{\kappa}' N + \widehat{\kappa}\widehat{\tau} B = 0$, as desired. ∎

In view of Corollary 3.4, we get the following corollary:

COROLLARY 4.2. *γ is a slant curve with pseudo-Hermitian parallel mean curvature vector field if and only if it is a pseudo-Hermitian Legendre circle.*

For a curve γ in a 3-dimensional contact Riemannian manifold M with the Tanaka–Webster connection $\widehat{\nabla}$,

$$\widehat{\Delta}\widehat{H} = -\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T, \tag{4.4}$$

where $\widehat{H}$ is the pseudo-Hermitian mean curvature vector field of γ [Lee]. The Laplacian of the pseudo-Hermitian mean curvature vector field *in the normal bundle* is defined by

$$\widehat{\Delta}^{\perp}\widehat{H} = -\widehat{\nabla}^{\perp}_T\widehat{\nabla}^{\perp}_T\widehat{\nabla}^{\perp}_T T, \tag{4.5}$$

where $\widehat{\nabla}^{\perp}$ denotes the the normal connection in the normal bundle [Lee].

A curve γ in a 3-dimensional contact Riemannian manifold M is called a *curve with pseudo-Hermitian proper mean curvature vector field* if $\widehat{\Delta}\widehat{H} = \lambda\widehat{H}$, where λ is a non-zero C^{∞} function. In particular if $\widehat{\Delta}\widehat{H} = 0$, then it is a *curve with pseudo-Hermitian harmonic mean curvature vector field* [Lee].

A curve γ in a 3-dimensional contact Riemannian manifold M is called a *curve with pseudo-Hermitian proper mean curvature vector field in the normal bundle* if $\widehat{\Delta}^{\perp}\widehat{H} = \lambda\widehat{H}$, where $\widehat{\Delta}^{\perp}$ is the Laplacian of the pseudo-Hermitian mean curvature vector field in the normal bundle, where λ is a non-zero C^{∞} function [Lee]. In particular if $\widehat{\Delta}^{\perp}\widehat{H} = 0$, then it is a *curve with pseudo-Hermitian harmonic mean curvature vector field in the normal bundle.*

Lemma 4.3. *Let γ be a curve in a 3-dimensional contact Riemannian manifold M. Then*

$$\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T = -3\widehat{\kappa}\widehat{\kappa}'T + (\widehat{\kappa}'' - \widehat{\kappa}^3 - \widehat{\kappa}\widehat{\tau}^2)N + (2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}')B, \tag{4.6}$$

$$\widehat{\nabla}^{\perp}_T\widehat{\nabla}^{\perp}_T\widehat{\nabla}^{\perp}_T T = (\widehat{\kappa}'' - \widehat{\kappa}\widehat{\tau}^2)N + (2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}')B, \tag{4.7}$$

$$\begin{aligned} \widehat{\Delta}\widehat{H} &= -\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T, \\ \widehat{\Delta}^{\perp}\widehat{H} &= -\widehat{\nabla}^{\perp}_T\widehat{\nabla}^{\perp}_T\widehat{\nabla}^{\perp}_T T. \end{aligned} \tag{4.8}$$

Proof. From (3.1),

$$\widehat{\nabla}_T T = \widehat{\kappa}N. \tag{4.9}$$

Differentiating (4.9) with respect to $\widehat{\nabla}$ and using (3.1), we find

$$\widehat{\nabla}_T\widehat{\nabla}_T T = -\widehat{\kappa}^2 T + \widehat{\kappa}'N + \widehat{\kappa}\widehat{\tau}B \tag{4.10}$$

and

$$\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T = -3\widehat{\kappa}\widehat{\kappa}'T + (\widehat{\kappa}'' - \widehat{\kappa}^3 - \widehat{\kappa}\widehat{\tau}^2)N + (2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}')B.$$

From (3.1), we obtain

$$\widehat{\nabla}^{\perp}_T T = \widehat{\kappa}N. \tag{4.11}$$

If we apply $\widehat{\nabla}^{\perp}$ to (4.11) and use (3.1), we get

$$\widehat{\nabla}^{\perp}_{T}\widehat{\nabla}^{\perp}_{T}T = \widehat{\kappa}'N + \widehat{\kappa}\widehat{\tau}B. \tag{4.12}$$

Finally (3.1) and (4.12) give

$$\widehat{\nabla}^{\perp}_{T}\widehat{\nabla}^{\perp}_{T}\widehat{\nabla}^{\perp}_{T}T = (\widehat{\kappa}'' - \widehat{\kappa}\widehat{\tau}^2)N + (2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}')B.$$

By the use of (4.1), (4.4) and (4.5), we get (4.8). ∎

Using Lemma 4.3, we have the following theorem:

THEOREM 4.4. *A curve γ has pseudo-Hermitian proper mean curvature vector field if and only if it is a pseudo-Hermitian circle satisfying $\lambda = \widehat{\kappa}^2$ or a pseudo-Hermitian helix satisfying $\lambda = \widehat{\kappa}^2 + \widehat{\tau}^2$.*

Proof. Assume that γ has pseudo-Hermitian proper mean curvature vector field. Then from (4.8), the condition $\widehat{\Delta}\widehat{H} = \lambda\widehat{H}$ gives

$$3\widehat{\kappa}\widehat{\kappa}'T + (\widehat{\kappa}^3 + \widehat{\kappa}\widehat{\tau}^2 - \widehat{\kappa}'')N - (2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}')B = \lambda\widehat{\kappa}N.$$

Hence

$$3\widehat{\kappa}\widehat{\kappa}' = 0, \quad \widehat{\kappa}^3 + \widehat{\kappa}\widehat{\tau}^2 - \widehat{\kappa}'' = \lambda\widehat{\kappa}, \quad -(2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}') = 0. \tag{4.13}$$

Since γ is a non-geodesic curve, $\widehat{\kappa} \neq 0$. Then $\widehat{\kappa}$ is a non-zero constant and $\widehat{\tau}$ is a constant. From the second equation of (4.13), we find $\lambda = \widehat{\kappa}^2 + \widehat{\tau}^2$. Hence γ is a pseudo-Hermitian circle satisfying $\lambda = \widehat{\kappa}^2$ or a pseudo-Hermitian helix satisfying $\lambda = \widehat{\kappa}^2 + \widehat{\tau}^2$.

The converse is trivial. ∎

COROLLARY 4.5. *A slant curve γ has pseudo-Hermitian proper mean curvature vector field if and only if it is a pseudo-Hermitian Legendre circle satisfying $\lambda = \widehat{\kappa}^2$ or a pseudo-Hermitian slant helix satisfying $\lambda = \widehat{\kappa}^2 + \widehat{\tau}^2$.*

Proof. Let γ be a non-geodesic slant curve in a 3-dimensional contact Riemannian manifold M. Then from Corollary 3.4, γ is a Legendre curve if and only if $\widehat{\tau} = 0$. Substituting $\widehat{\tau} = 0$ in (4.13) we obtain the result. ∎

COROLLARY 4.6. *There does not exist a slant curve with pseudo-Hermitian harmonic mean curvature vector field.*

Proof. Assume that γ is a non-geodesic curve in a 3-dimensional contact Riemannian manifold M. From (4.8), if $\widehat{\Delta}\widehat{H} = 0$, then

$$3\widehat{\kappa}\widehat{\kappa}'T + (\widehat{\kappa}^3 + \widehat{\kappa}\widehat{\tau}^2 - \widehat{\kappa}'')N - (2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}')B = 0,$$

which gives $\widehat{\kappa}^2 + \widehat{\tau}^2 = 0$. Hence $\widehat{\kappa} = 0$ and γ is a geodesic, a contradiction. ∎

THEOREM 4.7. *γ is a slant curve with pseudo-Hermitian proper mean curvature vector field in the normal bundle if and only if it is either a Legendre curve satisfying $\lambda = -\widehat{\kappa}''/\widehat{\kappa}$, $\widehat{\kappa}(s) \neq as + b$ (where a and b are constants), or a pseudo-Hermitian slant helix satisfying $\lambda = \widehat{\tau}^2$.*

Proof. Assume that γ is a non-geodesic slant curve with contact angle α_0 and has pseudo-Hermitian proper mean curvature vector field in the normal bundle. Then by definition, $\widehat{\Delta}^{\perp}\widehat{H} = \lambda\widehat{H}$. Using (4.8), we get

$$(\widehat{\kappa}\widehat{\tau}^2 - \widehat{\kappa}'')N - (2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}')B = \lambda\widehat{\kappa}N, \tag{4.14}$$

which gives

$$\widehat{\kappa}\widehat{\tau}^2 - \widehat{\kappa}'' = \lambda\widehat{\kappa}, \qquad -(2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}') = 0. \tag{4.15}$$

In view of (3.2), using (4.15) we can write

$$\widehat{\kappa}^3 \cot^2\alpha_0 - \widehat{\kappa}'' = \lambda\widehat{\kappa}, \qquad -3\widehat{\kappa}\widehat{\kappa}'\cot\alpha_0 = 0. \tag{4.16}$$

Finally we solve (4.16) in two cases:

(i) If $\alpha_0 = \pi/2$, then γ is a Legendre curve and $\cot\alpha_0 = 0$. Hence $-\widehat{\kappa}'' = \lambda\widehat{\kappa}$. Since $\lambda \neq 0$ and $\widehat{\kappa} \neq 0$, we have $\widehat{\kappa}'' \neq 0$. In this case, γ is a Legendre curve satisfying $\lambda = -\widehat{\kappa}''/\widehat{\kappa}$, $\widehat{\kappa}(s) \neq as + b$ where a and b are constants.

(ii) If $\alpha_0 \neq \pi/2$, then $\cot\alpha_0 \neq 0$. Using the second equation of (4.16), we see that $\widehat{\kappa}$ is a constant. Then $\widehat{\kappa}'' = 0$, so the first equation of (4.16) turns into $\widehat{\kappa}^3\cot^2\alpha_0 = \lambda\widehat{\kappa}$. Hence $\lambda = (\widehat{\kappa}\cot\alpha_0)^2 = \widehat{\tau}^2$. So γ is a pseudo-Hermitian slant helix satisfying $\lambda = \widehat{\tau}^2$.

Conversely, let γ be a Legendre curve satisfying $\lambda = -\widehat{\kappa}''/\widehat{\kappa}$, $\widehat{\kappa}(s) \neq as+b$ where a and b are constants, or a pseudo-Hermitian slant helix satisfying $\lambda = \widehat{\tau}^2$. In both cases, (4.14) is satisfied. Hence γ is a curve with pseudo-Hermitian proper mean curvature vector field in the normal bundle. ∎

Remark 4.8. In [Lee, Theorem 3.9], Lee studied the same problem for a constant λ and $\alpha_0 = \pi/2$. So our theorem is a generalization of her result.

Corollary 4.9. *γ is a curve with pseudo-Hermitian harmonic mean curvature vector field in the normal bundle if and only if it is a Legendre curve satisfying $\widehat{\kappa}(s) = as + b$, where a and b are constants.*

Corollary 4.10. *There does not exist a pseudo-Hermitian slant helix with pseudo-Hermitian harmonic mean curvature vector field in the normal bundle.*

5. Slant curves of pseudo-Hermitian $AW(k)$-type. In [AO], K. Arslan and the first author studied curves of $AW(k)$-type. Lee defined curves of pseudo-Hermitian $AW(k)$-type in [Lee]. In this section, we study non-geodesic slant curves of pseudo-Hermitian $AW(k)$-type in 3-dimensional contact Riemannian manifolds.

Definition 5.1 ([Lee]). Let M be a 3-dimensional contact Riemannian manifold with the Tanaka–Webster connection $\widehat{\nabla}$. Curves *of pseudo-*

Hermitian AW(1)-type satisfy

$$(\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T)^{\perp} = 0,$$

of pseudo-Hermitian AW(2)-type satisfy

$$(\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T)^{\perp} \wedge (\widehat{\nabla}_T\widehat{\nabla}_T T)^{\perp} = 0,$$

and *of pseudo-Hermitian AW(3)-type* satisfy

$$(\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T)^{\perp} \wedge (\widehat{\nabla}_T T)^{\perp} = 0,$$

where $(\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T)^{\perp}$, $(\widehat{\nabla}_T\widehat{\nabla}_T T)^{\perp}$ and $(\widehat{\nabla}_T T)^{\perp}$ are the normal parts of (4.6), (4.10) and (4.1), respectively.

Let γ be a curve in a 3-dimensional contact Riemannian manifold M and $\{T, N, B\}$ its Frenet frame field. Using (4.9), (4.10) and (4.6), we find

$$\begin{aligned}(\widehat{\nabla}_T T)^{\perp} &= \widehat{\kappa}N,\\ (\widehat{\nabla}_T\widehat{\nabla}_T T)^{\perp} &= \widehat{\kappa}'N + \widehat{\kappa}\widehat{\tau}B,\\ (\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T)^{\perp} &= (\widehat{\kappa}'' - \widehat{\kappa}^3 - \widehat{\kappa}\widehat{\tau}^2)N + (2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}')B.\end{aligned} \tag{5.1}$$

Firstly, we give the following theorem:

THEOREM 5.2. *Let γ be a slant curve with contact angle α_0. Then γ is of pseudo-Hermitian AW(1)-type if and only if it is a Legendre curve with pseudo-Hermitian curvature $\widehat{\kappa}$, which satisfies the differential equation $\widehat{\kappa}'' - \widehat{\kappa}^3 = 0$.*

Proof. Assume that γ is of pseudo-Hermitian $AW(1)$-type. By definition $(\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T)^{\perp} = 0$. Hence, we use the third equation of (5.1) with (3.2) to find

$$\widehat{\kappa}'' - \widehat{\kappa}^3 - \widehat{\kappa}^3\cot^2\alpha_0 = 0, \qquad 3\widehat{\kappa}\widehat{\kappa}'\cot\alpha_0 = 0. \tag{5.2}$$

The second equation implies that $\widehat{\kappa}$ is a constant or $\alpha_0 = \pi/2$. If $\widehat{\kappa}$ is a constant, then $\widehat{\kappa}'' = 0$. So the first equation turns into $(-\widehat{\kappa}^3)\cdot(1+\cot^2\alpha_0) = 0$. Thus $\widehat{\kappa} = 0$, which is a contradiction. If $\alpha_0 = \pi/2$, then γ is a Legendre curve and $\cot\alpha_0 = 0$, so the first equation becomes $\widehat{\kappa}'' - \widehat{\kappa}^3 = 0$. Hence γ is a Legendre curve with pseudo-Hermitian curvature $\widehat{\kappa}$, which satisfies the differential equation $\widehat{\kappa}'' - \widehat{\kappa}^3 = 0$.

Conversely, let γ be a Legendre curve with pseudo-Hermitian curvature $\widehat{\kappa}$, which satisfies the differential equation $\widehat{\kappa}'' - \widehat{\kappa}^3 = 0$. Since γ is a Legendre curve, $\alpha_0 = \pi/2$ and $\cot\alpha_0 = 0$. Then the equations of (5.2) are satisfied. Thus $(\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T)^{\perp} = 0$, so by definition, γ is of pseudo-Hermitian $AW(1)$-type. ∎

For Legendre curves, we can state the following corollary which was given in [Lee].

Corollary 5.3 ([Lee]). *Let γ be a Legendre curve with pseudo-Hermitian curvature $\widehat{\kappa}(s) = \pm\sqrt{2}/(s+c)$. Then γ is of pseudo-Hermitian $AW(1)$-type.*

Theorem 5.4. *Let γ be a slant curve with contact angle α_0. Then γ is of pseudo-Hermitian $AW(2)$-type if and only if it has pseudo-Hermitian torsion of the form $\widehat{\tau} = \pm\cos\alpha_0/\sqrt{-s^2+as+b}$, where a and b are arbitrary constants, $a^2+4b>0$ and $s \in I_{a,b} := ((a-\sqrt{a^2+4b})/2, (a+\sqrt{a^2+4b})/2)$.*

Proof. Assume that γ is of pseudo-Hermitian $AW(2)$-type. By definition, $(\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T)^\perp \wedge (\widehat{\nabla}_T\widehat{\nabla}_T T)^\perp = 0$, that is, $(\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T)^\perp$ and $(\widehat{\nabla}_T\widehat{\nabla}_T T)^\perp$ are linearly dependent. Using (5.1), we find

$$\begin{vmatrix} \widehat{\kappa}'' - \widehat{\kappa}^3 - \widehat{\kappa}\widehat{\tau}^2 & 2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}' \\ \widehat{\kappa}' & \widehat{\kappa}\widehat{\tau} \end{vmatrix} = 0. \tag{5.3}$$

So we get

$$\widehat{\kappa}\widehat{\tau}(\widehat{\kappa}'' - \widehat{\kappa}^3 - \widehat{\kappa}\widehat{\tau}^2) = \widehat{\kappa}'(2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}'). \tag{5.4}$$

Substituting (3.2) in (5.4), we get

$$\widehat{\kappa}^2\cot\alpha_0(\widehat{\kappa}'' - \widehat{\kappa}^3 - \widehat{\kappa}^3\cot^2\alpha_0) = 3\widehat{\kappa}(\widehat{\kappa}')^2\cot\alpha_0. \tag{5.5}$$

From (5.5), $\cot\alpha_0 = 0$ or $\widehat{\kappa}\widehat{\kappa}'' - 3(\widehat{\kappa}')^2 = \widehat{\kappa}^4\operatorname{cosec}^2\alpha_0$. If $\cot\alpha_0 = 0$, then γ is slant and Legendrian, hence $\widehat{\tau} = 0$ identically and the condition (5.3) is also satisfied identically. On the other hand if $\cot\alpha_0 \neq 0$, then $\widehat{\kappa}\widehat{\kappa}'' - 3(\widehat{\kappa}')^2 = \widehat{\kappa}^4\operatorname{cosec}^2\alpha_0$. The general solution of this differential equation is $\widehat{\kappa} = \pm\sin\alpha_0/\sqrt{-s^2+as+b}$, where a and b are arbitrary constants, $a^2+4b>0$ and $s \in I_{a,b}$. So using (3.2), we obtain $\widehat{\tau} = \pm\cos\alpha_0/\sqrt{-s^2+as+b}$.

Conversely, suppose that the curve γ has pseudo-Hermitian torsion $\widehat{\tau} = \pm\cos\alpha_0/\sqrt{-s^2+as+b}$, where a and b are constants, $a^2+4b>0$ and $s \in I_{a,b}$. It is easy to show that (5.3) is satisfied. Hence $(\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T)^\perp \wedge (\widehat{\nabla}_T\widehat{\nabla}_T T)^\perp = 0$, that is, γ is of pseudo-Hermitian $AW(2)$-type. ∎

For Legendre curves, as a result of Theorem 5.4, we have the following corollary which was obtained in [Lee]:

Corollary 5.5 ([Lee]). *Every Legendre curve in a 3-dimensional contact Riemannian manifold is of pseudo-Hermitian $AW(2)$-type.*

In a 3-dimensional contact Riemannian manifold, it is obvious that there are pseudo-Hermitian circles of pseudo-Hermitian $AW(2)$-type. A simplest example is a pseudo-Hermitian Legendre circle.

From Theorem 5.4, we have the following corollary for pseudo-Hermitian slant helices:

Corollary 5.6. *There does not exist a pseudo-Hermitian slant helix of pseudo-Hermitian $AW(2)$-type.*

Finally, we have the following theorem for slant curves of pseudo-Hermitian $AW(3)$-type.

THEOREM 5.7. *Let γ be a slant curve with contact angle α_0. Then γ is of pseudo-Hermitian $AW(3)$-type if and only if it has constant pseudo-Hermitian torsion.*

Proof. Assume that γ is of pseudo-Hermitian $AW(3)$-type. By definition $(\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T)^{\perp} \wedge (\widehat{\nabla}_T T)^{\perp} = 0$, which implies

$$\begin{vmatrix} \widehat{\kappa}'' - \widehat{\kappa}^3 - \widehat{\kappa}\widehat{\tau}^2 & 2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}' \\ \widehat{\kappa} & 0 \end{vmatrix} = 0. \tag{5.6}$$

Using (3.2) in (5.6), since $\widehat{\kappa} \neq 0$, we obtain

$$3\widehat{\kappa}' \cot \alpha_0 = 0, \tag{5.7}$$

so $\alpha_0 = \pi/2$ or $\widehat{\kappa}$ is a constant. If $\alpha_0 = \pi/2$, then γ is a Legendre curve, so $\widehat{\tau} = 0$ identically. If $\alpha_0 \neq \pi/2$, then $\widehat{\kappa}$ is a constant. In this case, using (3.2), it is clear that $\widehat{\tau}$ is a constant.

Conversely, let γ have constant pseudo-Hermitian torsion $\widehat{\tau}$. If γ is a Legendre curve, then $\widehat{\tau} = 0$. Hence (5.6) is satisfied. If γ is not a Legendre curve, then using (3.2), we find that $\widehat{\kappa}$ is a constant. Since $\widehat{\kappa}' = 0$ and $\widehat{\tau}' = 0$, (5.6) is satisfied. In both cases, we obtain $(\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T)^{\perp} \wedge (\widehat{\nabla}_T T)^{\perp} = 0$, which completes the proof. ∎

Using Corollary 3.4, Theorem 5.7 and (3.2) we can state the following corollary:

COROLLARY 5.8. *γ is a slant curve of pseudo-Hermitian $AW(3)$-type if and only if it is a Legendre curve or a pseudo-Hermitian slant helix.*

REMARK 5.9. Theorems 5.2, 5.4 and 5.7 are generalizations of Lemma 3.13(i)&(ii) in [Lee].

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Cihan Özgür, Şaban Güvenç
Department of Mathematics
Balıkesir University
10145, Çağış, Balıkesir, Turkey
E-mail: cozgur@balikesir.edu.tr
sguvenc@balikesir.edu.tr

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