Abstract. The homogeneous quaternionic Kähler structures on the Alekseevskian \( \mathcal{W} \)-spaces with their natural quaternionic structures, each of these spaces described as a solvable Lie group, and the type of such structures in Fino’s classification, are found.

1. Introduction. Quaternion-Kähler manifolds have attracted much attention since the classical papers by Wolf [W], Ishihara [I] and others to the present day: see for instance [J] and [V], among many papers.

A quaternion-Kähler manifold is said to be negative if it is complete and has negative scalar curvature. Homogeneous quaternion-Kähler spaces admitting a simply transitive completely solvable Lie group of isometries were classified by Alekseevsky [A] (see also de Wit and van Proeyen [WP] and Cortés [Co]). No other homogeneous negative quaternion-Kähler spaces are known. Alekseevsky conjectured in [A, p. 300] that the only homogeneous negative quaternion-Kähler manifolds are Alekseevskian spaces.

Homogeneous quaternionic Kähler structures, i.e., the \( \text{Sp}(n)\text{Sp}(1) \) case of Tricerri and Vanhecke [TV] homogeneous Riemannian structures, have been studied in [BGO1, BGO2, CGO1, CGO2, CGS, F]. Fino gave in [F, Lem. 5.1] a representation-theoretical classification of such structures into five basic geometric types \( \mathcal{QK}_1, \ldots, \mathcal{QK}_5 \). (We denote the type \( \mathcal{QK}_i \oplus \mathcal{QK}_j \) by \( \mathcal{QK}_{ij} \), and so on.) A classification by real tensors was given in [CGS, Th. 1.1], and it was also proved that a connected, simply-connected and complete homogeneous quaternion-Kähler manifold of dim \( \geq 8 \), admitting a nonvanishing structure in \( \mathcal{QK}_{123} \) with nonzero projection to \( \mathcal{QK}_3 \), is isometric to the quaternionic hyperbolic space \( \mathbb{HH}(n) \). Furthermore, a structure of type \( \mathcal{QK}_{134} \) on \( \mathbb{HH}(n) \), corresponding to its description as a solvable Lie group,

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has been given in [CGS, Prop. 5.3]. Then, in [CGO1, Th. 3.4] and [CGO2, Th. 5] it has been shown that the quaternion-Kähler symmetric spaces of dimension 8 or 12 furnish proper realisations of the types $QK_{134}, QK_{135}, QK_{1345}, QK_{12345}$. Fino’s classification has been extended to any signature of the metric in [BGO1, Th. 4.4], and the structures on rank-three Alekseevskian spaces, $T(p), p \geq 0$, endowed with their natural structure as solvable Lie groups, have been found in [BGO2, Th. 3.1].

Negative quaternion-Kähler spaces appear in $N = 2$ supergravity. If gravity is considered as a dynamical field, the holonomy group of the manifold is a subgroup of $Sp(n)Sp(1)$ and $M$ is a negative quaternion-Kähler manifold (Bagger and Witten [BW]). Cecotti [Ce] proved that Alekseevskian spaces naturally appear in the context of the $c$-map and that nonsymmetric ones are related to Vinberg $T$-algebras as symmetric ones are related to Jordan algebras. De Wit and van Proeyen [WP] completed Alekseevsky’s classification by using supergravity considerations. That Alekseevskian spaces do appear in three series, $T$-, $W$-, $V$-spaces, was proved by Cortés [Co, Th. II.28] with geometric arguments.

Our aim is to give the expression of the homogeneous quaternionic Kähler structures carried by the rank-four Alekseevskian spaces $W(p,q)$, each of them described as a solvable Lie group, and then their type in Fino’s classification. To this end, we make calculations which are crucially based on the explicit description of the spaces $W(p,q)$ as completely solvable Lie groups with a left-invariant quaternionic Kähler structure, given by Cortés in [Co].

After some preliminaries in §2, we obtain Theorem 3.1 giving the homogeneous quaternionic Kähler structure corresponding to the description of each space $W(p,q)$ as a solvable Lie group. Theorem 4.1 gives the type of such structure, proving that it has nonzero components in each basic Fino type.

2. Preliminaries. Ambrose and Singer [AS] proved that a connected, simply-connected and complete Riemannian manifold $(M,g)$ is Riemannian homogeneous if and only if it admits a homogeneous Riemannian structure, i.e., a $(1,2)$ tensor field $S$ satisfying $\tilde{\nabla}g = 0, \tilde{\nabla}R = 0, \tilde{\nabla}S = 0$, where $\tilde{\nabla} = \nabla - S$, $\nabla$ denotes the Levi-Civita connection and $R$ the curvature tensor of $\nabla$. We write as usual $S_{XYZ} = g(S_XY, Z)$. From $\nabla g = 0$ it follows that the condition $\tilde{\nabla}g = 0$ is equivalent to $S_{XYZ} = -S_{XZY}$.

Let $(M, g, v^3)$ be an almost quaternion-Hermitian manifold. Let $J_1, J_2, J_3$ be a standard local basis of $v^3$ and let $\omega_a(X,Y) = g(J_aX, Y)$, $a = 1, 2, 3$. The differential 4-form $\Omega = \sum_{a=1}^{3} \omega_a \wedge \omega_a$ is known to be globally defined. The manifold is said to be quaternion-Kähler if locally (cf. Ishihara [I])
Homogeneous quaternionic Kähler structures

\[ \nabla_X J_1 = \tau^3(X) J_2 - \tau^2(X) J_3, \quad \text{etc.,} \]

for certain differential 1-forms \( \tau^1, \tau^2, \tau^3 \) (‘etc.’ denoting the equations obtained by cyclically permuting 1, 2, 3); or, equivalently, if \( \nabla \Omega = 0 \).

We shall consider negative quaternion-Kähler manifolds of dimension \( \geq 8 \). A quaternion-Kähler manifold \((M, g, v^3)\) of dimension \( \geq 8 \) is said to be a homogeneous quaternion-Kähler manifold if \([\text{AC}, \text{p. } 218]\), cf. \([\text{CGS, Rem. } 2.2]\) it admits a transitive group of isometries. As a corollary to Kiričenko’s Theorem \([\text{K}]\), a connected, simply-connected and complete quaternion-Kähler manifold \((M, g, v^3)\) is homogeneous if and only if there exists a tensor field \( S \) of type \((1, 2)\) on \( M \) satisfying

\[ \tilde{\nabla} g = 0, \quad \tilde{\nabla} R = 0, \quad \tilde{\nabla} S = 0, \quad \tilde{\nabla} \Omega = 0, \]

where \( \tilde{\nabla} = \nabla - S \). Such a tensor \( S \) is called a homogeneous quaternionic Kähler structure on \( M \). The equation \( \tilde{\nabla} \Omega = 0 \) is equivalent to conditions similar to (2.1).

Fino \([\text{F}, \text{Lem. } 5.1]\) gave a representation-theoretical classification of homogeneous quaternionic Kähler structures into five basic geometric types, which we denote by \( \mathcal{Q} \mathcal{K}_1, \ldots, \mathcal{Q} \mathcal{K}_5 \).

Let \( (V, \langle , \rangle, q) \) be a quaternion-Hermitian vector space, i.e., a 4\( n \)-dimensional real vector space endowed with an inner product \( \langle , \rangle \) and a quaternionic structure \( q \) generated by suitable operators \( J_1, J_2, J_3 \). Consider the space of tensors \( T(V) = \{ S \in \otimes^3 V^* : S_{XYZ} = -S_{XYZ} \} \) and its vector subspace

\[ \mathcal{Q} \mathcal{K}(V) = \{ S \in \otimes^3 V^* : S_{XYZ} = -S_{XYZ}, \ \exists \theta^a \in V^* \text{ such that } S \text{ satisfies } S_{XJ_1 J_1 Z} - S_{XYZ} = \theta^3(X) g(J_2 Y, J_1 Z) - \theta^2(X) g(J_3 Y, J_1 Z), \ \text{etc.} \}. \]

Any homogeneous Riemannian structure on \( M \) belongs to \( T(T_p M) \) pointwise, but homogeneous quaternionic Kähler structures are pointwise in the space \( \mathcal{Q} \mathcal{K}(T_p M) \).

Consider the subspaces \( \hat{\mathcal{V}} \) and \( \hat{\mathcal{V}} \) of \( \mathcal{Q} \mathcal{K}(V) \) consisting of elements \( \Theta \) and \( \mathcal{T} \), respectively, such that \( \Theta_{XYZ} = \sum_{a=1}^3 \alpha^a(X) \langle J_a Y, Z \rangle, \ \alpha^a \in V^* \), and \( \mathcal{T}_{XJ_a J_a Z} = \mathcal{T}_{XYZ}, a = 1, 2, 3 \). Then one has \( \mathcal{Q} \mathcal{K}(V) = \hat{\mathcal{V}} \oplus \hat{\mathcal{V}} \), and each element \( S \in \mathcal{Q} \mathcal{K}(V) \) decomposes as \( S_{XYZ} = \Theta_{XYZ} + \mathcal{T}_{XYZ} \), where

\[ \Theta_{XYZ} = \frac{1}{2} \sum_{a=1}^3 \alpha^a(X) \langle J_a Y, Z \rangle. \]

The classification by real tensors is \([\text{CGS, Th. } 3.15]\) as follows: If \( n \geq 2 \), the space \( \mathcal{Q} \mathcal{K}(V) \) decomposes into the direct sum of the following \( \text{Sp}(n)\text{Sp}(1) \)-invariant and irreducible subspaces:
an elementary Kählerian Lie algebra

structure inherited from that of the manifold is a quaternion-Hermitian vector
metric. The corresponding metric Lie algebra with the quaternionic struc-
ture can be regarded as a completely solvable Lie group with a left-invariant
Lie group of isometries. An Alekseevskian space is simply-connected and

Alekseevskian space

zero scalar curvature is said to be

type of the form

hyperbolic plane

Any Alekseevskian algebra \((\mathfrak{s}, \langle \cdot, \cdot \rangle, q)\), which is called a
quaternionic or Alekseevskian Lie algebra. A metric Lie algebra \(\mathfrak{f}\) with an orthonormal basis \(\{G, H\}\) and a complex
structure \(J\) is said to be a key algebra with root \(\mu\) if \(G = JH, [H, G] = \mu G, \mu > 0\). A metric Lie algebra \(\mathfrak{f} + \mathfrak{r}\) with a complex structure \(J\) is said to be an
elementary Kählerian Lie algebra with root \(\mu\) if \(\mathfrak{f} = \text{Span}\{G, H\}\) is a key
subalgebra with root \(\mu\) and \(\text{ad}_H|_{\mathfrak{r}} = \frac{1}{2} \mu I, \text{ad}_G|_{\mathfrak{r}} = 0, [X, Y] = \mu \langle JX, Y \rangle G, X, Y \in r\). A representation \(U \to T_U\) of a Lie algebra \(u\) with a complex structure \(J\) on a Euclidean space \((u, \langle \cdot, \cdot \rangle)\) with a complex structure \(J_1\) is said to be
symplectic if it satisfies the two conditions given in [A] Def. 6.3. If \(T_u|_{\mathfrak{r}} = \mathfrak{r}, T\) is called nondegenerate. If \(T\) is a nondegenerate symplectic representation of a key algebra \(\mathfrak{f} = \text{Span}\{G, H\}\) with root \(\mu\) on a space \((\mathfrak{r}, \langle \cdot, \cdot \rangle, J_1, T)\), then \(\mathfrak{r}\) admits a weight decomposition \(\mathfrak{r} = \mathfrak{r}_+ + \mathfrak{r}_-\) such that

\[
(2.5) \quad \mathfrak{r}_- = J_1 \mathfrak{r}_+, \quad T_G|_{\mathfrak{r}_+} = 0, \quad T_G|_{\mathfrak{r}_-} = -\mu J_1, \quad T_H|_{\mathfrak{r}_\pm} = \pm \frac{1}{2} \mu I.
\]

Any Alekseevskian algebra \((\mathfrak{s}, \langle \cdot, \cdot \rangle, q)\), with \(q = \text{Span}\{J_a : a = 1, 2, 3\}\), contains a unique (up to scaling) one-dimensional quaternionic subalgebra \(\mathfrak{s}'\) (i.e., a subalgebra \(\mathfrak{s}'\) such that \(q\mathfrak{s}' \subset \mathfrak{s}'\)), corresponding either to the complex hyperbolic plane \(\mathbb{CH}(2)\) or to the quaternionic hyperbolic line \(\mathbb{HH}(1)\). In the former case it is of the form \(\mathfrak{s} = u + J_2 u\) (orthogonal sum), and \((u, J_1|_u)\) is the so-called principal Kählerian subalgebra of \(\mathfrak{s}\). The Lie algebra \(u\) contains a key subalgebra \(\mathfrak{f}_0 = \text{Span}\{G_0, H_0\}\) with root 1 such that \(\mathfrak{f}_0 + J_2 \mathfrak{f}_0\) is the canonical one-dimensional quaternionic subalgebra of \(\mathfrak{s}\), and the adjoint
representation of $s$ induces a representation of $u$ on $u^\perp = J_2u$. A Kählerian Lie algebra $(u, J)$, that is, a metric Lie algebra which corresponds to a Kählerian homogeneous space, is said to be admissible if $u = f_0 + u_0$ is a direct orthogonal sum of a key algebra $f_0 = \text{Span}\{G_0, H_0\}$ with root 1 and a completely solvable Kählerian Lie algebra $u_0$. Write $\tilde{U} = \varphi(U)$ for each $U \in u$, and denote by $J_1$ and $\hat{J}$ the complex structures on $\tilde{u}$ given by

$$(2.6) \quad J_1 = -\varphi J \varphi^{-1}, \quad \hat{J}|_{f_0} = -J_1|_{f_0}, \quad \hat{J}|_{u_0} = J_1|_{u_0}.$$  

Then a representation $U \mapsto T_U$ of such a Lie algebra $u$ on a Euclidean space $\tilde{u}$ together with a vector space isometry $\varphi: u \to \tilde{u}$ is said to be a $Q$-representation if it satisfies the eight conditions (Q1–8) given in [A] Lem. 5.5 and Def. 5.3 (cf. also Cortés [Co, Def. 1.8]).

If $s$ is an Alekseevskian Lie algebra with principal Kählerian subalgebra $(u, J)$, then the representation of $u$ on $J_0u$ induced by the adjoint representation of $s$ is a $Q$-representation with $\varphi = J_2|_u: u \to u^\perp$. Conversely, let $(T, \varphi)$ be a $Q$-representation of an admissible Kählerian Lie algebra $(u, J)$ on the Euclidean vector space $\tilde{u} = \varphi(u) = f_0 + \tilde{u}_0$. Then a quaternionic structure $q = \text{Span}\{J_a : a = 1, 2, 3\}$ on the Euclidean vector space $s = u + \tilde{u}$ (orthogonal sum) is defined by

$$(2.7) \quad J_1|_u = J, \quad J_1|_{\tilde{u}} = -\varphi J \varphi^{-1}, \quad J_2|_u = \varphi, \quad J_2|_{\tilde{u}} = -\varphi^{-1}, \quad J_3 = J_1J_2.$$  

Let $\hat{J}$ be the complex structure on $\tilde{u}$ defined as in (2.6), and let $\hat{\omega}$ denote the Kähler form on $\tilde{u}$ given by $\hat{\omega}(\tilde{U}, \tilde{V}) = \langle \hat{J}\tilde{U}, \tilde{V} \rangle$. Then the following conditions define the structure of Lie algebra of $s$:

$$(2.8) \quad u \text{ is a subalgebra of } s, \quad \text{ad}_U|_{\tilde{u}} = T_U, \quad [\tilde{U}, \tilde{V}] = \hat{\omega}(\tilde{U}, \tilde{V})G_0,$$

for all $U, V \in u$.

The rank of a solvable Lie algebra $s$ is the dimension of a Cartan subalgebra of $s$. The rank of an Alekseevskian space $S$ is the rank of its Alekseevskian Lie algebra $s$, which is proved to be at most 4. An admissible Kählerian Lie algebra $u = f_0 + u_0$ which admits a $Q$-representation decomposes as a semidirect sum of elementary Kählerian Lie algebras, with $u_0 = \sum_{i \geq 1}(f_{i} + r_{i})$, that is, $[f_{i} + r_{i}, f_{j} + r_{j}] \subset f_{j} + f_{j}$, $i \geq j$, with symplectic representation $\text{ad}_{f_i}|_{r_j}$ for $i > j$ and commuting key algebras, $[f_i, f_j] = 0$, for $i \neq j$ (see [Co, p. 134]). The rank of $u = f_0 + \sum_{i \geq 1}(f_{i} + r_{i})$ coincides with the number of key algebras of $u$. There are three types of admissible Kählerian Lie algebras, the first type corresponding to the case with smallest root 1.

3. Homogeneous quaternionic Kähler structures on $\mathcal{W}(p, q)$.

Now we focus on the rank-four $\mathcal{W}$-case. We shall make calculations essentially based on the explicit description, found by Cortés [Co], of the spaces
$W(p, q)$, $0 \leq p \leq q$, as completely solvable Lie groups with a left-invariant quaternionic Kählerian structure.

We recall that given Euclidean spaces $x, y, z$, a bilinear map $\psi: x \times z \to y$ is said to be isometric if $\langle \psi(X, Z), \psi(X, Z) \rangle = \langle X, X \rangle \langle Z, Z \rangle$, $X \in x$, $Z \in z$. Let $\mathfrak{r}_-, \mathfrak{z}_-, \mathfrak{n}_-$ be Euclidean vector spaces. Every isometric map $\psi: \mathfrak{r}_- \times \mathfrak{z}_- \to \mathfrak{n}_-$ defines a Kählerian Lie algebra $\mathfrak{u}(\psi) = (f_0 + u_0, J)$ of type 1 and rank 4 by means of a recipe given in [A, Prop. 9.3]. According to [A, Props. 9.2–9.4], there are two possibilities for Kählerian Lie algebras $\mathfrak{u} = \mathfrak{u}(\psi)$ of type 1 and rank $> 2$ which admit a Q-representation. These two possibilities originate the series of Alekseevskian $\mathcal{W}$- and $\mathcal{V}$-spaces, respectively. The $\mathcal{W}$-spaces correspond to the case $\mathfrak{r}_- = 0$ (hence $\psi = 0$), and $\mathfrak{u} = \mathfrak{u}(q, p) \cong \mathfrak{u}(q, p)$ is completely determined by the parameters $p = \dim \mathfrak{n}_- \geq 0$ and $q = \dim \mathfrak{z}_- \geq 0$. Any such Lie algebra $\mathfrak{u}$ has a unique Q-representation $T$ and the corresponding Alekseevskian spaces are denoted by $\mathcal{W}(p, q)$. In this case the set of rules of the aforementioned recipe reduces to:

1. The space $\mathfrak{u}_0$ is a semidirect sum $\mathfrak{u}_0 = (\mathfrak{f}_1 + \mathfrak{r}_1) + \mathfrak{f}_2 + \mathfrak{f}_3$ of elementary Kählerian key algebras with commuting Lie algebras with root 1.

2. The space $\mathfrak{r}_1$ admits a $J$-invariant decomposition $\mathfrak{r}_1 = \mathfrak{n} + \mathfrak{z}$ such that $\text{ad}_{\mathfrak{f}_3}|_{\mathfrak{n}}$ and $\text{ad}_{\mathfrak{f}_2}|_{\mathfrak{z}}$ are nondegenerate symplectic representations with weight decompositions $\mathfrak{n} = \mathfrak{n} + \mathfrak{n}$ and $\mathfrak{z} = \mathfrak{z} + \mathfrak{z}$, where $\mathfrak{n} = J\mathfrak{n}$ and $\mathfrak{z} = J\mathfrak{z}$. Furthermore, $[\mathfrak{f}_2, \mathfrak{n}] = [\mathfrak{f}_3, \mathfrak{z}] = [\mathfrak{n}, \mathfrak{z}] = 0$.

Let $\{Y_{j'+}\}, j = 1, \ldots, p$, and $\{Z_{k'+}\}, k = 1, \ldots, q$, be orthonormal bases of $\mathfrak{n}$ and $\mathfrak{z}$, respectively, and let $Y_{j'-} = JY_{j'+}, Z_{k'-} = JZ_{k'+}$. Then, as $\text{ad}_{G_0}\mathfrak{u}_0 = 0$ ([A] Lem. 4.6]) and $\text{ad}_{H_0}\mathfrak{u}_0 = 0$ ([A] (5.2)), we have the Lie brackets on $\mathfrak{u}$ given in Table 1.

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<tr>
<th>$G_0$</th>
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<th>$G_1$</th>
<th>$H_1$</th>
<th>$G_2$</th>
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<th>$G_3$</th>
<th>$H_3$</th>
<th>$Y_{j'+}$</th>
<th>$Y_{j'-}$</th>
<th>$Z_{k'+}$</th>
<th>$Z_{k'-}$</th>
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<tr>
<td>$G_0$</td>
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<td>$-\delta_{kkG_1}$</td>
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</table>
Furthermore, the Kählerian Lie algebra \((u, J)\) has a unique Q-representation on the Euclidean vector space \(\tilde{u} = \tilde{f}_0 + \tilde{u}_0\), \(T: u \to \text{End}(\tilde{u})\), where \(\sim: u \to \tilde{u}\) denotes the corresponding isometry of Euclidean vector spaces.

Consider the quaternion-Hermitian vector space \((\mathfrak{w}(p, q), \langle \cdot, \cdot \rangle, q)\), where the space \(\mathfrak{w}(p, q) = u + \tilde{u}\) is a direct orthogonal sum, and \(q = \text{Span}\{J_a: a = 1, 2, 3\}\) is the quaternionic structure on \(\mathfrak{w}(p, q)\) defined by \((2.7)\). Then
\[
B = \{G_i, H_i, Y_{j+}, Y_{j-}, Z_{k+}, Z_{k-}, \tilde{G}_i, \tilde{H}_i, \tilde{Y}_{j+}, \tilde{Y}_{j-}, \tilde{Z}_{k+}, \tilde{Z}_{k-}\},
\]
for \(0 \leq i \leq 3, 1 \leq j \leq p, 1 \leq k \leq q\), is an orthonormal basis of \(\mathfrak{w}(p, q)\).

**Table 2.** The action of \(J_a, a = 1, 2, 3, \) on \(\mathfrak{w}(p, q)\)

<table>
<thead>
<tr>
<th></th>
<th>(G_i)</th>
<th>(H_i)</th>
<th>(Y_{j+})</th>
<th>(Y_{j-})</th>
<th>(Z_{k+})</th>
<th>(Z_{k-})</th>
<th>(\tilde{G}_i)</th>
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<th>(\tilde{Y}_{j+})</th>
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<th>(\tilde{Z}_{k+})</th>
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<td>(J_1)</td>
<td>(-H_i)</td>
<td>(G_i)</td>
<td>(-Y_{j+})</td>
<td>(-Y_{j-})</td>
<td>(Z_{k+})</td>
<td>(-Z_{k-})</td>
<td>(-\tilde{G}_i)</td>
<td>(-\tilde{H}_i)</td>
<td>(-\tilde{Y}_{j+})</td>
<td>(-\tilde{Y}_{j-})</td>
<td>(-\tilde{Z}_{k+})</td>
<td>(-\tilde{Z}_{k-})</td>
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<tr>
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<td>(\tilde{H}_i)</td>
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<td>(\tilde{Z}_{k+})</td>
<td>(\tilde{Z}_{k-})</td>
<td>(-G_i)</td>
<td>(-H_i)</td>
<td>(-Y_{j+})</td>
<td>(-Y_{j-})</td>
<td>(-Z_{k+})</td>
<td>(-Z_{k-})</td>
</tr>
<tr>
<td>(J_3)</td>
<td>(\tilde{H}_i)</td>
<td>(-G_i)</td>
<td>(-\tilde{Y}_{j-})</td>
<td>(\tilde{Y}_{j+})</td>
<td>(-\tilde{Z}_{k-})</td>
<td>(\tilde{Z}_{k+})</td>
<td>(-H_i)</td>
<td>(-G_i)</td>
<td>(-Y_{j-})</td>
<td>(Y_{j+})</td>
<td>(-Z_{k-})</td>
<td>(Z_{k+})</td>
</tr>
</tbody>
</table>

**Table 3.** The complex structure \(\hat{J}\) on \(\tilde{u}\)

<table>
<thead>
<tr>
<th></th>
<th>(\tilde{G}_0)</th>
<th>(\tilde{H}_0)</th>
<th>(\tilde{G}_i)</th>
<th>(\tilde{H}_i)</th>
<th>(\tilde{Y}_{j+})</th>
<th>(\tilde{Y}_{j-})</th>
<th>(\tilde{Z}_{k+})</th>
<th>(\tilde{Z}_{k-})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\hat{J})</td>
<td>(-\tilde{H}_0)</td>
<td>(\tilde{G}_0)</td>
<td>(\tilde{H}_i)</td>
<td>(-\tilde{G}_i)</td>
<td>(-\tilde{Y}_{j-})</td>
<td>(\tilde{Y}_{j+})</td>
<td>(-\tilde{Z}_{k-})</td>
<td>(\tilde{Z}_{k+})</td>
</tr>
</tbody>
</table>

The action of \(J_a, a = 1, 2, 3, \) on \(\mathfrak{w}(p, q) = u + \tilde{u}\) is given in Table 2. Moreover, the vector space \(\mathfrak{w}(p, q)\) has a structure of Lie algebra given by \((2.8)\), with \(s = \mathfrak{w}(p, q)\), where the complex structure \(\hat{J}\) on \(\tilde{u}\) is defined by Table 3. Hence, by the third condition in \((2.8)\), the nonnull brackets of the elements of \(\tilde{u}\) are
\[
[\tilde{H}_0, \tilde{G}_0] = -[\tilde{H}_i, \tilde{G}_i] = -[\tilde{Y}_{j+}, \tilde{Y}_{j-}] = -[\tilde{Z}_{k+}, \tilde{Z}_{k-}] = G_0, \quad i = 1, 2, 3.
\]
If \(U \in u\) and \(\tilde{V} \in \tilde{u}\), then by the second condition in \((2.8)\), one has \([U, \tilde{V}] = T_U \tilde{V}\), and the values of \(T_U \tilde{V}\) are given in Tables \(4\) and \(7\), where \(T: u \to \text{End}(\tilde{u})\) is expressed in terms of the orthonormal basis \(\{G_i, H_i, \tilde{Y}_{j+}, \tilde{Y}_{j-}, \tilde{Z}_{k+}, \tilde{Z}_{k-}\}\) of \(\tilde{u}\), from the conditions \((Q1–8)\) of a Q-representation (cf. \([Co, Prop. 2.1]\)). Table \(6\) follows from the properties of a weight decomposition with respect to a nondegenerate symplectic representation \((2.5)\) and the properties in \([A\text{, Prop. 9.3}]\).

The Lie algebra \(\mathfrak{w}(p, q)\) is 4-step solvable with \(\dim \mathfrak{w}(p, q) = 4(p+q)\), and the corresponding simply-connected Lie group with left-invariant metric is the Alekseevskian space \(\mathcal{W}(p, q)\).

We have \(\mathfrak{w}(p, q)^* = u^* + \tilde{u}^*\). Let
\[
\mathcal{B}^* = \{\gamma^i, \eta^i, \xi^i, \bar{\xi}^i, \xi^k, \bar{\xi}^k, \gamma^i, \bar{\eta}^i, \bar{\xi}^j, \xi^j, \bar{\xi}^k, \gamma^k, \bar{\xi}^k\}
\]
be the basis of \( \mathfrak{w}(p,q)^* \) dual to the basis \([3.1]\) of \( \mathfrak{w}(p,q) \), and denote by \( S_X \) the 2-form defined by \( S_X(Y,Z) = S_{XYZ} \).

**Theorem 3.1.** The homogeneous quaternionic Kähler structure \( S \) on each rank-four Alekseevskian space \( \mathcal{W}(p,q) \), \( 0 \leq p \leq q \), which gives its description as the simply-connected solvable Lie group with Lie algebra \( \mathfrak{w}(p,q) \), is given, in terms of the basis \( \mathcal{B}^* \) of \( \mathfrak{w}(p,q)^* \), by

\[
S_{\mathfrak{u}^* \wedge \mathfrak{u}^*} = \sum_{i=0}^{3} \gamma^i \otimes (\gamma^i \wedge \eta^i) - \frac{1}{2} (\gamma^1 \otimes (\xi^1 + \eta^1 + \xi^2 + \eta^2) + \gamma^2 \otimes (\xi^1 \wedge \eta^1 + \xi^2 \wedge \eta^2) + \gamma^3 \otimes (\xi^1 \wedge \eta^1 + \xi^2 \wedge \eta^2))
\]

\[
+ \frac{1}{2} \sum_{j=1}^{p} \xi^j \otimes \left( (\xi^j + (\eta^1 + \xi^j) + \eta^3) + \xi^j \wedge (\gamma^1 + \gamma^3) \right)
\]

\[
+ \frac{1}{2} \sum_{j=1}^{p} \xi^j \otimes \left( (\xi^j + (\eta^3 - \gamma^1) + \xi^j \wedge (\gamma^1 - \gamma^3) \right)
\]

\[
+ \frac{1}{2} \sum_{k=1}^{q} \xi^k \otimes \left( (\xi^k + (\eta^1 + \xi^k) + \xi^k \wedge (\gamma^1 + \gamma^2) \right)
\]

\[
+ \frac{1}{2} \sum_{k=1}^{q} \xi^k \otimes \left( (\xi^k + (\eta^1 + \xi^k) + \gamma^k \wedge (\gamma^1 - \gamma^3)) \right)
\]

\[
S_{\mathfrak{g}^0} = \frac{1}{2} \left( \sum_{i=0}^{3} (\gamma^i \wedge \eta^i - \eta^i \wedge \gamma^i) \right)
- \sum_{j=1}^{p} (\xi^j \wedge \eta^j - \eta^j \wedge \xi^j)
- \sum_{k=1}^{q} (\xi^k \wedge \eta^k - \eta^k \wedge \xi^k)
\]

\[
S_{\mathfrak{h}^0} = \frac{1}{2} \left( \sum_{i=0}^{3} (\gamma^i \wedge \eta^i + \eta^i \wedge \gamma^i) \right)
+ \sum_{j=1}^{p} (\xi^j \wedge \eta^j + \eta^j \wedge \xi^j)
+ \sum_{k=1}^{q} (\xi^k \wedge \eta^k + \eta^k \wedge \xi^k)
\]
Proof. Consider the tensor field $S$ on $\mathcal{W}(p, q)$, $0 \leq p \leq q$, given by
\begin{equation}
(3.3) \quad 2\langle S_X Y, Z \rangle = \langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle
\end{equation}
for $X, Y, Z \in \mathfrak{w}(p, q)$. Let $\nabla$ be the Levi-Civita connection on $\mathcal{W}(p, q)$ with respect to the invariant metric defined by $\langle \cdot, \cdot \rangle$. Then $\tilde{\nabla} = \nabla - S$ is the connection on the Lie group $\mathcal{W}(p, q)$ for which every left-invariant vector field is
parallel. Thus, conditions (2.2) are satisfied and $S$ is a homogeneous quaternionic Kähler structure. Moreover, the holonomy algebra of the connection $\tilde{\nabla}$ is trivial, and then $S$ provides the description of $\mathcal{W}(p,q)$ as a Lie group (see [TV, p. 32, Eqs. (1.79)]).

### Table 4. The Q-representation $T: u \rightarrow \text{End}(\bar{u})$

<table>
<thead>
<tr>
<th></th>
<th>$\tilde{G}_0$</th>
<th>$\tilde{H}_0$</th>
<th>$\tilde{G}_1$</th>
<th>$\tilde{H}_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$H_0$</td>
<td>$\frac{1}{2}\tilde{G}_0$</td>
<td>$\frac{1}{2}\tilde{H}_0$</td>
<td>$\frac{1}{2}\tilde{G}_1$</td>
<td>$\frac{1}{2}\tilde{H}_1$</td>
</tr>
<tr>
<td>$G_1$</td>
<td>$-\frac{1}{2}(\tilde{H}_0 + \tilde{H}_1)$</td>
<td>$\frac{1}{2}(\tilde{G}_0 + \tilde{G}_1)$</td>
<td>$\frac{1}{2}(\tilde{H}_0 + \tilde{H}_1)$</td>
<td>$-\frac{1}{2}(\tilde{G}_0 + \tilde{G}_1)$</td>
</tr>
<tr>
<td>$H_1$</td>
<td>$\frac{1}{2}\tilde{G}_1$</td>
<td>$\frac{1}{2}\tilde{H}_1$</td>
<td>$-\frac{1}{2}\tilde{G}_0$</td>
<td>$\frac{1}{2}\tilde{H}_0$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$-\frac{1}{2}(\tilde{H}_0 + \tilde{H}_3)$</td>
<td>$-\frac{1}{2}(\tilde{G}_0 + \tilde{G}_2)$</td>
<td>$-\frac{1}{2}(\tilde{H}_3 + \tilde{H}_1)$</td>
<td>$-\frac{1}{2}(\tilde{G}_3 - \tilde{G}_1)$</td>
</tr>
<tr>
<td>$H_2$</td>
<td>$\frac{1}{2}\tilde{G}_2$</td>
<td>$\frac{1}{2}\tilde{H}_2$</td>
<td>$-\frac{1}{2}\tilde{G}_3$</td>
<td>$\frac{1}{2}\tilde{H}_3$</td>
</tr>
<tr>
<td>$G_3$</td>
<td>$-\frac{1}{2}(\tilde{H}_3 + \tilde{H}_1)$</td>
<td>$\frac{1}{2}(\tilde{G}_3 + \tilde{G}_2)$</td>
<td>$\frac{1}{2}(\tilde{H}_1 + \tilde{H}_2)$</td>
<td>$\frac{1}{2}(\tilde{G}_1 - \tilde{G}_2)$</td>
</tr>
<tr>
<td>$H_3$</td>
<td>$\frac{1}{2}\tilde{G}_3$</td>
<td>$\frac{1}{2}\tilde{H}_3$</td>
<td>$-\frac{1}{2}\tilde{G}_2$</td>
<td>$\frac{1}{2}\tilde{H}_2$</td>
</tr>
</tbody>
</table>

### Table 5. The Q-representation $T: u \rightarrow \text{End}(\bar{u})$

<table>
<thead>
<tr>
<th></th>
<th>$\tilde{C}_2$</th>
<th>$\tilde{H}_2$</th>
<th>$\tilde{C}_3$</th>
<th>$\tilde{H}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$H_0$</td>
<td>$\frac{1}{2}\tilde{C}_2$</td>
<td>$\frac{1}{2}\tilde{H}_2$</td>
<td>$\frac{1}{2}\tilde{C}_3$</td>
<td>$\frac{1}{2}\tilde{H}_3$</td>
</tr>
<tr>
<td>$G_1$</td>
<td>$-\frac{1}{2}(\tilde{H}_2 + \tilde{H}_3)$</td>
<td>$\frac{1}{2}(\tilde{C}_2 - \tilde{C}_3)$</td>
<td>$-\frac{1}{2}(\tilde{H}_2 + \tilde{H}_3)$</td>
<td>$-\frac{1}{2}(\tilde{C}_2 - \tilde{C}_3)$</td>
</tr>
<tr>
<td>$H_1$</td>
<td>$-\frac{1}{2}\tilde{C}_3$</td>
<td>$\frac{1}{2}\tilde{H}_3$</td>
<td>$-\frac{1}{2}\tilde{C}_2$</td>
<td>$\frac{1}{2}\tilde{H}_2$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$\frac{1}{2}(\tilde{H}_0 + \tilde{H}_2)$</td>
<td>$-\frac{1}{2}(\tilde{C}_0 + \tilde{C}_2)$</td>
<td>$-\frac{1}{2}(\tilde{H}_0 + \tilde{H}_1)$</td>
<td>$\frac{1}{2}(\tilde{C}_3 - \tilde{C}_1)$</td>
</tr>
<tr>
<td>$H_2$</td>
<td>$\frac{1}{2}\tilde{C}_0$</td>
<td>$\frac{1}{2}\tilde{H}_0$</td>
<td>$-\frac{1}{2}\tilde{C}_1$</td>
<td>$\frac{1}{2}\tilde{H}_1$</td>
</tr>
<tr>
<td>$G_3$</td>
<td>$-\frac{1}{2}(\tilde{H}_1 + \tilde{H}_2)$</td>
<td>$-\frac{1}{2}(\tilde{C}_1 - \tilde{C}_2)$</td>
<td>$\frac{1}{2}(\tilde{H}_0 + \tilde{H}_3)$</td>
<td>$-\frac{1}{2}(\tilde{C}_0 + \tilde{C}_3)$</td>
</tr>
<tr>
<td>$H_3$</td>
<td>$-\frac{1}{2}\tilde{C}_1$</td>
<td>$\frac{1}{2}\tilde{H}_1$</td>
<td>$\frac{1}{2}\tilde{C}_0$</td>
<td>$\frac{1}{2}\tilde{H}_0$</td>
</tr>
<tr>
<td>$Y_{j+}$</td>
<td>$-\frac{1}{2}\tilde{Y}_{j+}$</td>
<td>$\frac{1}{2}\tilde{Y}_{j+}$</td>
<td>$-\frac{1}{2}\tilde{Y}_{j+}$</td>
<td>$-\frac{1}{2}\tilde{Y}_{j+}$</td>
</tr>
<tr>
<td>$Y_{j-}$</td>
<td>$-\frac{1}{2}\tilde{Y}_{j-}$</td>
<td>$\frac{1}{2}\tilde{Y}_{j-}$</td>
<td>$-\frac{1}{2}\tilde{Y}_{j-}$</td>
<td>$-\frac{1}{2}\tilde{Y}_{j-}$</td>
</tr>
<tr>
<td>$Z_{k+}$</td>
<td>$-\frac{1}{2}\tilde{Z}_{k+}$</td>
<td>$\frac{1}{2}\tilde{Z}_{k+}$</td>
<td>$-\frac{1}{2}\tilde{Z}_{k+}$</td>
<td>$-\frac{1}{2}\tilde{Z}_{k+}$</td>
</tr>
<tr>
<td>$Z_{k-}$</td>
<td>$-\frac{1}{2}\tilde{Z}_{k-}$</td>
<td>$\frac{1}{2}\tilde{Z}_{k-}$</td>
<td>$-\frac{1}{2}\tilde{Z}_{k-}$</td>
<td>$-\frac{1}{2}\tilde{Z}_{k-}$</td>
</tr>
</tbody>
</table>
Since (see (2.8)) we have \([u, u] \subset u, [u, \tilde{u}] \subset \tilde{u}, [\tilde{u}, \tilde{u}] \subset u\), and \(u\) and \(\tilde{u}\) are orthogonal, from (3.3) we have

(3.4) \[ S_{U^VW} = 0, \quad S_{\tilde{U}^VW} = 0, \quad S_{UV^W} = 0, \quad S_{\tilde{U}^V\tilde{W}} = 0. \]

**Table 6.** The \(Q\)-representation \(T: u \to \text{End}(\tilde{u})\)

<table>
<thead>
<tr>
<th>(\tilde{Y}'_+-)</th>
<th>(\tilde{Y}'_-)</th>
<th>(\tilde{Z}_{k'+})</th>
<th>(\tilde{Z}_{k'-})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_0)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(H_0)</td>
<td>(\frac{1}{2}\tilde{Y}'_+)</td>
<td>(\frac{1}{2}\tilde{Y}'_-)</td>
<td>(\frac{1}{2}\tilde{Z}_{k'+})</td>
</tr>
<tr>
<td>(G_1)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(H_1)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(G_2)</td>
<td>0</td>
<td>(\tilde{Y}'_+)</td>
<td>0</td>
</tr>
<tr>
<td>(H_2)</td>
<td>(\frac{1}{2}\tilde{Y}'_+)</td>
<td>(-\frac{1}{2}\tilde{Y}'_-)</td>
<td>0</td>
</tr>
<tr>
<td>(G_3)</td>
<td>0</td>
<td>0</td>
<td>(-\tilde{Z}_{k'+})</td>
</tr>
<tr>
<td>(H_3)</td>
<td>0</td>
<td>(\frac{1}{2}\tilde{Z}_{k'+})</td>
<td>(-\frac{1}{2}\tilde{Z}_{k'-})</td>
</tr>
</tbody>
</table>

**Table 7.** The \(Q\)-representation \(T: u \to \text{End}(\tilde{u})\)

<table>
<thead>
<tr>
<th>(\tilde{Y}'_+)</th>
<th>(\tilde{Y}'_-)</th>
<th>(\tilde{Z}_{k'+})</th>
<th>(\tilde{Z}_{k'-})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Y_j^+)</td>
<td>(\frac{\delta_{jj'}}{2}(\tilde{H}_0 + \tilde{H}_1))</td>
<td>(\frac{\delta_{jj'}}{2}(\tilde{G}_0 + \tilde{G}_1))</td>
<td>0</td>
</tr>
<tr>
<td>(Y_j^-)</td>
<td>(-\frac{\delta_{jj'}}{2}(\tilde{G}_0 + \tilde{G}_1))</td>
<td>(-\frac{\delta_{jj'}}{2}(\tilde{H}_0 + \tilde{H}_1))</td>
<td>0</td>
</tr>
<tr>
<td>(Z_{k+})</td>
<td>0</td>
<td>0</td>
<td>(\frac{\delta_{kk'}}{2}(\tilde{H}_0 + \tilde{H}_1))</td>
</tr>
<tr>
<td>(Z_{k-})</td>
<td>0</td>
<td>0</td>
<td>(-\frac{\delta_{kk'}}{2}(\tilde{H}_0 + \tilde{H}_1))</td>
</tr>
</tbody>
</table>

On account of (3.3), Table 6 and the equation \(S_{U^VW} = 0\) in (3.4), one obtains the nonzero values of \(S_{U^VW}\) for \(U, V\) and \(W\) in the orthonormal basis \(\mathcal{B}\). In order to obtain \(S_{[u]^* \otimes [\tilde{u}]^*} = 0\), we use (3.2), (3.3), the equation \(S_{U^VW} = 0\) in (3.4) and Tables 4 to 7 since \([U, \tilde{V}] = T_U \tilde{V}\). From (3.3), by using (3.2), Tables 4 to 7 and the equations \(S_{[U]^* \otimes [\tilde{U}]^*} = S_{U\tilde{U}\tilde{V}W} = 0\) in (3.4), we obtain the values of \(S_{\tilde{U}}\) for each \(\tilde{U} = \tilde{G}_i, \tilde{H}_i, \tilde{Y}_j^+, \tilde{Y}_j^-, \tilde{Z}_{k+}, \tilde{Z}_{k-}\).

4. The type of the structure on \(W(p, q)\). We now determine the type of the previous structure \(S\).

**Theorem 4.1.** The homogeneous quaternionic Kähler structure on each rank-four Alekseevskian space \(W(p, q)\), given in Theorem 3.1 has a nonzero component in each basic Fino type.
Proof. From the expression of $S$ in Theorem 3.1 and from Table [1] we find that the forms $\alpha^a$, $a = 1, 2, 3$, in (2.3) corresponding to $S$ are given by

$$\alpha^1 = -\frac{1}{2} \sum_{i=0}^3 \gamma^i, \quad \alpha^2 = -\tilde{\eta}^0, \quad \alpha^3 = \tilde{\gamma}^0.$$  

Hence, since $S = \Theta + \mathcal{I}$, where $\Theta$ is given by (2.3), from (4.1) and using Table [1], it follows that the tensor field $\Theta$ on $\mathcal{W}(p, q)$ corresponding to $S$ is given by

$$\frac{1}{4} \sum_{i=0}^3 \gamma^i \otimes \{ \sum_{l=0}^3 (\gamma^l \wedge \eta^l - \tilde{\gamma}^l \wedge \tilde{\eta}^l) - \sum_{j=1}^p (\xi^j \wedge \xi^j - \tilde{\xi}^j \wedge \tilde{\xi}^j) - \sum_{k=1}^q (\zeta^k \wedge \zeta^k - \tilde{\zeta}^k \wedge \tilde{\zeta}^k) \}$$

$$+ \frac{1}{2} \tilde{\eta}^0 \otimes \{ \sum_{l=0}^3 (\gamma^l \wedge \tilde{\gamma}^l + \tilde{\gamma}^l \wedge \gamma^l) - \sum_{j=1}^p (\xi^j \wedge \tilde{\xi}^j + \tilde{\xi}^j \wedge \xi^j) - \sum_{k=1}^q (\zeta^k \wedge \tilde{\zeta}^k + \tilde{\zeta}^k \wedge \zeta^k) \}$$

$$+ \sum_{j=1}^p (\xi^j \wedge \tilde{\xi}^j + \xi^j \wedge \zeta^j - \tilde{\xi}^j \wedge \zeta^j) + \sum_{k=1}^q (\zeta^k \wedge \tilde{\zeta}^k + \zeta^k \wedge \zeta^k - \tilde{\zeta}^k \wedge \zeta^k) \}.$$  

On the other hand, considering again that the structure decomposes as $S = \Theta + \mathcal{I}$, and the values of the 1-forms $\alpha^a$ are those in (4.1), we infer that as, for instance,

$$\sum_{a=1}^3 \alpha^a(J_a H_2) = -1/2 \neq 0,$$

the component $\Theta$ of the structure $S$ does not belong to $\mathcal{QK}_2$.

From (4.2), the nonzero values of $\Theta_{XYZ}$ are those with $X = G_0, G_1, G_2, G_3$, $\tilde{G}_0, H_0$. In particular one has the next nonzero values of type $\Theta_{XXY}$:

$$\Theta_{G_0 G_0 H_0} = \Theta_{G_0 G_1 H_1} = \Theta_{G_0 G_2 H_2} = \Theta_{G_0 G_3 H_3} = 1/4,$$

$$\Theta_{\tilde{G}_0 G_0 H_0} = \Theta_{\tilde{G}_0 G_0 H_0} = 1/2.$$  

Suppose next that $\Theta \in \mathcal{QK}_1$. Then there would be a 1-form $\alpha$ as that in expressions (2.4), and in particular we would have

$$1/4 = \Theta_{G_0 G_0 H_0} = \alpha(H_0), \quad 1/2 = \Theta_{\tilde{G}_0 G_0 H_0} = \alpha(H_0),$$

which is absurd. Hence $\Theta \in \mathcal{QK}_1 \setminus \{ \mathcal{QK}_1 \cup \mathcal{QK}_2 \}$.

Furthermore, as $\dim \mathcal{W}(p, q) = 4(4 + p + q)$ and on account of (4.3), the form $\beta$ defining the $\mathcal{QK}_3$-component (see expressions (2.4)), that is,

$$\beta = \frac{1}{2 + \dim \mathcal{W}(p, q)} c_{12}(\mathcal{I}) = \frac{1}{18 + 4(p + q)} c_{12}(\mathcal{I}),$$

is given by
\[ \beta = \frac{1}{18 + 4(p+q)} \left\{ \left(\frac{15}{4} + p + q\right) \eta^0 + \left(\frac{3}{4} + p + q\right) \eta^1 + \frac{3}{4} (\eta^2 + \eta^3) \right\}. \]

Hence \( S \) has a nonzero component in \( \mathcal{Q} \mathcal{K}_3 \) for all \( 0 \leq p \leq q \).

Consider now the operator \( \Psi : \hat{V} \to \hat{V} \) defined by
\[
\Psi(\mathcal{T})_{XYZ} = \mathcal{T}_{YXZ} + \mathcal{T}_{ZXY} + \sum_{a=1}^{3} (\mathcal{T}_{J_aYJ_aZ} + \mathcal{T}_{J_aZXJ_aY}),
\]
having eigenvalues 2 and \(-4\), with corresponding eigenspaces \( \mathcal{Q} \mathcal{K}_3 \) and \( \mathcal{Q} \mathcal{K}_5 \), respectively (see expressions (2.4)). Consider \( \mathcal{T}^\beta \in \mathcal{Q} \mathcal{K}_3 \), given by
\[
\mathcal{T}^\beta_{XYZ} = \langle X, Y \rangle \beta(Z) - \langle X, Z \rangle \beta(Y)
+ \sum_{a=1}^{3} \left( \langle X, J_aY \rangle \beta(J_aZ) - \langle X, J_aZ \rangle \beta(J_aY) \right),
\]
where \( \beta \) stands for the 1-form (4.4). Then \( \mathcal{T} - \mathcal{T}^\beta \in \mathcal{Q} \mathcal{K}_4 \mathcal{K}_5 \), so that we have
\[
\Psi(\mathcal{T} - \mathcal{T}^\beta)_{XYZ} = \Psi(\mathcal{T})_{XYZ} - 2 \mathcal{T}^\beta_{XYZ}.
\]
Taking then for instance the vectors \( X = Y = G_0, Z = H_0 \), we get
\[
(\mathcal{T} - \mathcal{T}^\beta)_{G_0G_0H_0} = \frac{6 + p + q}{2(9 + 2(p+q))}, \quad \Psi(\mathcal{T} - \mathcal{T}^\beta)_{G_0G_0H_0} = -\frac{21 + 5(p+q)}{9 + 2(p+q)},
\]
hence \( \mathcal{T} - \mathcal{T}^\beta \in \mathcal{Q} \mathcal{K}_4 \mathcal{K}_5 \setminus \{ \mathcal{Q} \mathcal{K}_4 \cup \mathcal{Q} \mathcal{K}_5 \} \) for all \( 0 \leq p \leq q \). That is, \( S \) has, for all \( 0 \leq p \leq q \), a nonzero component in each basic type.

As the simplest examples, consider the \( 4(4+q) \)-dimensional spaces \( \mathcal{W}(0,q) \cong \text{SO}_0(4+q,4)/\left(\text{SO}(4+q) \times \text{SO}(4)\right) \), \( q \geq 0 \), which (cf. [C6, Table 1]) are the Alekseevskian \( \mathcal{W} \)-spaces which are symmetric. As such, they admit the structure \( S = 0 \). Moreover, being solvable Lie groups with Lie algebra \( \mathfrak{w}(0,q) \), they admit the corresponding structure given by Theorem 3.1 when \( p = 0 \).

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References


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