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Homogeneous quaternionic Kähler structures on Alekseevskian $\mathcal{W}$-spaces

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Abstract. The homogeneous quaternionic Kähler structures on the Alekseevskian $\mathcal{W}$-spaces with their natural quaternionic structures, each of these spaces described as a solvable Lie group, and the type of such structures in Fino's classification, are found.

1. Introduction. Quaternion-Kähler manifolds have attracted much attention since the classical papers by Wolf [W], Ishihara [I] and others to the present day: see for instance [J] and [V], among many papers.

A quaternion-Kähler manifold is said to be negative if it is complete and has negative scalar curvature. Homogeneous quaternion-Kähler spaces admitting a simply transitive completely solvable Lie group of isometries were classified by Alekseevsky [A] (see also de Wit and van Proeyen [WP] and Cortés [Co]). No other homogeneous negative quaternion-Kähler spaces are known. Alekseevsky conjectured in [A, p. 300] that the only homogeneous negative quaternion-Kähler manifolds are Alekseevskian spaces.

Homogeneous quaternionic Kähler structures, i.e., the $\mathrm{Sp}(n)\mathrm{Sp}(1)$ case of Tricerri and Vanhecke [TV] homogeneous Riemannian structures, have been studied in [BGO1, BGO2, CGO1, CGO2, CGS, F]. Fino gave in [F, Lem. 5.1] a representation-theoretical classification of such structures into five basic geometric types $\mathcal{QK}_1, \dots, \mathcal{QK}_5$. (We denote the type $\mathcal{QK}_i \oplus \mathcal{QK}_j$ by $\mathcal{QK}_{ij}$, and so on.) A classification by real tensors was given in [CGS, Th. 1.1], and it was also proved that a connected, simply-connected and complete homogeneous quaternion-Kähler manifold of dim ≥ 8, admitting a nonvanishing structure in $\mathcal{QK}_{123}$ with nonzero projection to $\mathcal{QK}_3$, is isometric to the quaternionic hyperbolic space $\mathbb{H}\mathrm{H}(n)$. Furthermore, a structure of type $\mathcal{QK}_{134}$ on $\mathbb{H}\mathrm{H}(n)$, corresponding to its description as a solvable Lie group,

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has been given in [CGS, Prop. 5.3]. Then, in [CGO1, Th. 3.4] and [CGO2, Th. 5] it has been shown that the quaternion-Kähler symmetric spaces of dimension 8 or 12 furnish proper realisations of the types $\mathcal{QK}_{134}, \mathcal{QK}_{135}$, $\mathcal{QK}_{1345}$, $\mathcal{QK}_{12345}$. Fino's classification has been extended to any signature of the metric in [BGO1, Th. 4.4], and the structures on rank-three Alekseevskian spaces, $\mathcal{T}(p)$, $p \geq 0$, endowed with their natural structure as solvable Lie groups, have been found in [BGO2, Th. 3.1].

Negative quaternion-Kähler spaces appear in $N = 2$ supergravity. If gravity is considered as a dynamical field, the holonomy group of the manifold is a subgroup of $\mathrm{Sp}(n)\mathrm{Sp}(1)$ and M is a negative quaternion-Kähler manifold (Bagger and Witten [BW]). Cecotti [Ce] proved that Alekseevskian spaces naturally appear in the context of the c-map and that nonsymmetric ones are related to Vinberg T-algebras as symmetric ones are related to Jordan algebras. De Wit and van Proeyen [WP] completed Alekseevsky's classification by using supergravity considerations. That Alekseevskian spaces do appear in three series, $\mathcal{T}$-, $\mathcal{W}$-, $\mathcal{V}$-spaces, was proved by Cortés [Co, Th. II.28] with geometric arguments.

Our aim is to give the expression of the homogeneous quaternionic Kähler structures carried by the rank-four Alekseevskian spaces $\mathcal{W}(p,q)$, each of them described as a solvable Lie group, and then their type in Fino's classification. To this end, we make calculations which are crucially based on the explicit description of the spaces $\mathcal{W}(p,q)$ as completely solvable Lie groups with a left-invariant quaternionic Kähler structure, given by Cortés in [Co].

After some preliminaries in §2, we obtain Theorem 3.1, giving the homogeneous quaternionic Kähler structure corresponding to the description of each space $\mathcal{W}(p,q)$ as a solvable Lie group. Theorem 4.1 gives the type of such structure, proving that it has nonzero components in each basic Fino type.

2. Preliminaries. Ambrose and Singer [AS] proved that a connected, simply-connected and complete Riemannian manifold (M,g) is Riemannian homogeneous if and only if it admits a homogeneous Riemannian structure, i.e., a $(1,2)$ tensor field S satisfying $\widetilde{\nabla} g = 0$, $\widetilde{\nabla} R = 0$, $\widetilde{\nabla} S = 0$, where $\widetilde{\nabla} = \nabla - S$, ∇ denotes the Levi-Civita connection and R the curvature tensor of ∇. We write as usual $S_{XYZ} = g(S_X Y, Z)$. From $\nabla g = 0$ it follows that the condition $\widetilde{\nabla} g = 0$ is equivalent to $S_{XZY} = -S_{XYZ}$.

Let (M, g, υ^3) be an almost quaternion-Hermitian manifold. Let J_1, J_2, J_3 be a standard local basis of υ^3 and let $\omega_a(X,Y) = g(J_a X, Y)$, $a = 1,2,3$. The differential 4-form $\Omega = \sum_{a=1}^{3} \omega_a \wedge \omega_a$ is known to be globally defined. The manifold is said to be *quaternion-Kähler* if locally (cf. Ishihara [I])

$$\nabla_X J_1 = \tau^3(X)J_2 - \tau^2(X)J_3, \quad \text{etc.}, \tag{2.1}$$

for certain differential 1-forms τ^1, τ^2, τ^3 ('etc.' denoting the equations obtained by cyclically permuting $1, 2, 3$); or, equivalently, if $\nabla\Omega = 0$.

We shall consider negative quaternion-Kähler manifolds of dimension ≥ 8. A quaternion-Kähler manifold (M, g, υ^3) of dimension ≥ 8 is said to be a *homogeneous quaternion-Kähler manifold* if ([AC, p. 218], cf. [CGS, Rem. 2.2]) it admits a transitive group of isometries. As a corollary to Kiričenko's Theorem [K], a connected, simply-connected and complete quaternion-Kähler manifold (M, g, υ^3) is homogeneous if and only if there exists a tensor field S of type $(1, 2)$ on M satisfying

$$\widetilde{\nabla} g = 0, \quad \widetilde{\nabla} R = 0, \quad \widetilde{\nabla} S = 0, \quad \widetilde{\nabla}\Omega = 0, \tag{2.2}$$

where $\widetilde{\nabla} = \nabla - S$. Such a tensor S is called a *homogeneous quaternionic Kähler structure* on M. The equation $\widetilde{\nabla}\Omega = 0$ is equivalent to conditions similar to (2.1).

Fino [F, Lem. 5.1] gave a representation-theoretical classification of homogeneous quaternionic Kähler structures into five basic geometric types, which we denote by $\mathcal{QK}_1, \ldots, \mathcal{QK}_5$.

Let $(V, \langle\,,\rangle, \mathrm{q})$ be a quaternion-Hermitian vector space, i.e., a $4n$-dimensional real vector space endowed with an inner product $\langle\,,\rangle$ and a quaternionic structure q generated by suitable operators J_1, J_2, J_3. Consider the space of tensors $\mathcal{T}(V) = \{S \in \otimes^3 V^* : S_{XYZ} = -S_{XZY}\}$ and its vector subspace

$$\begin{aligned}\mathcal{QK}(V) = \{S \in \otimes^3 V^* : S_{XYZ} = -S_{XZY},\ \exists\,\theta^a \in V^* \text{ such that } S \text{ satisfies}\\ S_{XJ_1YJ_1Z} - S_{XYZ} = \theta^3(X)g(J_2Y, J_1Z) - \theta^2(X)g(J_3Y, J_1Z), \text{ etc.}\}.\end{aligned}$$

Any homogeneous Riemannian structure on M belongs to $\mathcal{T}(T_pM)$ pointwise, but homogeneous quaternionic Kähler structures are pointwise in the space $\mathcal{QK}(T_pM)$.

Consider the subspaces $\check{\mathcal{V}}$ and $\hat{\mathcal{V}}$ of $\mathcal{QK}(V)$ consisting of elements Θ and $\mathfrak{T}$, respectively, such that $\Theta_{XYZ} = \sum_{a=1}^{3}\alpha^a(X)\langle J_aY, Z\rangle$, $\alpha^a \in V^*$, and $\mathfrak{T}_{XJ_aYJ_aZ} = \mathfrak{T}_{XYZ}$, $a = 1, 2, 3$. Then one has $\mathcal{QK}(V) = \check{\mathcal{V}} \oplus \hat{\mathcal{V}}$, and each element $S \in \mathcal{QK}(V)$ decomposes as $S_{XYZ} = \Theta_{XYZ} + \mathfrak{T}_{XYZ}$, where

$$\Theta_{XYZ} = \tfrac{1}{2}\sum_{a=1}^{3}\alpha^a(X)\langle J_aY, Z\rangle. \tag{2.3}$$

The classification by real tensors is ([CGS, Th. 3.15]) as follows: If $n \geq 2$, the space $\mathcal{QK}(V)$ decomposes into the direct sum of the following $\mathrm{Sp}(n)\mathrm{Sp}(1)$-invariant and irreducible subspaces:

$$
\begin{aligned}
&\mathcal{QK}_1 = \{\Theta \in \check{\mathcal{V}} : \Theta_{XYZ} = \textstyle\sum_{a=1}^{3} \alpha(J_a X)\langle J_a Y, Z\rangle,\ \alpha \in V^*\},\\
&\mathcal{QK}_2 = \{\Theta \in \check{\mathcal{V}} : \Theta_{XYZ} = \textstyle\sum_{a=1}^{3} \alpha^a(X)\langle J_a Y, Z\rangle,\\
&\qquad\qquad \textstyle\sum_{a=1}^{3} \alpha^a \circ J_a = 0,\ \alpha^a \in V^*\},\\
&\mathcal{QK}_3 = \big\{\mathfrak{T} \in \hat{\mathcal{V}} : \mathfrak{T}_{XYZ} = \langle X, Y\rangle\beta(Z) - \langle X, Z\rangle\beta(Y)\\
&\qquad\qquad + \textstyle\sum_{a=1}^{3}\big(\langle X, J_a Y\rangle\beta(J_a Z) - \langle X, J_a Z\rangle\beta(J_a Y)\big),\ \beta \in V^*\big\},\\
&\mathcal{QK}_4 = \big\{\mathfrak{T} \in \hat{\mathcal{V}} : \mathfrak{T}_{XYZ} = \tfrac{1}{6}\big(\underset{XYZ}{\mathfrak{S}} \mathfrak{T}_{XYZ} + \underset{XJ_aYJ_aZ}{\mathfrak{S}} \textstyle\sum_a \mathfrak{T}_{XJ_aYJ_aZ}\big),\\
&\qquad\qquad c_{12}(\mathfrak{T}) = 0\big\},\\
&\mathcal{QK}_5 = \big\{\mathfrak{T} \in \hat{\mathcal{V}} : \underset{XYZ}{\mathfrak{S}} \mathfrak{T}_{XYZ} = 0\big\},
\end{aligned} \tag{2.4}
$$

where $c_{12}(\mathfrak{T})(Z) = \sum_{i=1}^{4n} \mathfrak{T}_{e_i e_i Z}$ for any local orthonormal basis $\{e_i\}$ of V.

We now recall some definitions and results by Alekseevsky [A] (cf. [AC], [Co]). A quaternion-Kähler manifold of nonzero scalar curvature is said to be an *Alekseevskian space* if it admits a simply transitive, completely solvable Lie group of isometries. An Alekseevskian space is simply-connected and it can be regarded as a completely solvable Lie group with a left-invariant metric. The corresponding metric Lie algebra with the quaternionic structure inherited from that of the manifold is a quaternion-Hermitian vector space $(\mathfrak{s}, \langle\,,\rangle, \mathrm{q})$, which is called a *quaternionic* or *Alekseevskian Lie algebra.* A metric Lie algebra $\mathfrak{f}$ with an orthonormal basis $\{G, H\}$ and a complex structure J is said to be a *key algebra* with root μ if $G = JH$, $[H, G] = \mu G$, $\mu > 0$. A metric Lie algebra $\mathfrak{f} + \mathfrak{x}$ with a complex structure J is said to be an *elementary Kählerian Lie algebra* with root μ if $\mathfrak{f} = \mathrm{Span}\{G, H\}$ is a key subalgebra with root μ and $\mathrm{ad}_H|_{\mathfrak{x}} = \frac{1}{2}\mu I$, $\mathrm{ad}_G|_{\mathfrak{x}} = 0$, $[X, Y] = \mu\langle JX, Y\rangle G$, $X, Y \in \mathfrak{x}$. A representation $U \mapsto T_U$ of a Lie algebra $\mathfrak{u}$ with a complex structure J on a Euclidean space $(\mathfrak{x}, \langle\,,\rangle)$ with a complex structure J_1 is said to be *symplectic* if it satisfies the two conditions given in [A, Def. 6.3]. If $T_{\mathfrak{u}}\mathfrak{x} = \mathfrak{x}$, T is called *nondegenerate.* If T is a nondegenerate symplectic representation of a key algebra $\mathfrak{f} = \mathrm{Span}\{G, H\}$ with root μ on a space $(\mathfrak{x}, \langle\,,\rangle, J_1)$, then $\mathfrak{x}$ admits a weight decomposition $\mathfrak{x} = \mathfrak{x}_+ + \mathfrak{x}_-$ such that

$$
\mathfrak{x}_- = J_1\mathfrak{x}_+, \quad T_G|_{\mathfrak{x}_+} = 0, \quad T_G|_{\mathfrak{x}_-} = -\mu J_1, \quad T_H|_{\mathfrak{x}_\pm} = \pm\tfrac{1}{2}\mu I. \tag{2.5}
$$

Any Alekseevskian algebra $(\mathfrak{s}, \langle\,,\rangle, \mathrm{q})$, with $\mathrm{q} = \mathrm{Span}\{J_a : a = 1, 2, 3\}$, contains a unique (up to scaling) one-dimensional quaternionic subalgebra $\mathfrak{s}'$ (i.e., a subalgebra $\mathfrak{s}'$ such that $\mathrm{q}\mathfrak{s}' \subset \mathfrak{s}'$), corresponding either to the complex hyperbolic plane $\mathbb{C}\mathrm{H}(2)$ or to the quaternionic hyperbolic line $\mathbb{H}\mathrm{H}(1)$. In the former case it is of the form $\mathfrak{s} = \mathfrak{u} + J_2\mathfrak{u}$ (orthogonal sum), and $(\mathfrak{u}, J_{1|\mathfrak{u}})$ is the so-called principal Kählerian subalgebra of $\mathfrak{s}$. The Lie algebra $\mathfrak{u}$ contains a key subalgebra $\mathfrak{f}_0 = \mathrm{Span}\{G_0, H_0\}$ with root 1 such that $\mathfrak{f}_0 + J_2\mathfrak{f}_0$ is the canonical one-dimensional quaternionic subalgebra of $\mathfrak{s}$, and the adjoint representation of $\mathfrak{s}$ induces a representation of $\mathfrak{u}$ on $\mathfrak{u}^\perp = J_2\mathfrak{u}$. A Kählerian Lie algebra $(\mathfrak{u}, J)$, that is, a metric Lie algebra which corresponds to a Kählerian homogeneous space, is said to be *admissible* if $\mathfrak{u} = \mathfrak{f}_0 + \mathfrak{u}_0$ is a direct orthogonal sum of a key algebra $\mathfrak{f}_0 = \mathrm{Span}\{G_0, H_0\}$ with root 1 and a completely solvable Kählerian Lie algebra $\mathfrak{u}_0$. Write $\widetilde{U} = \varphi(U)$ for each $U \in \mathfrak{u}$, and denote by J_1 and $\hat{J}$ the complex structures on $\widetilde{\mathfrak{u}}$ given by

$$J_1 = -\varphi J \varphi^{-1}, \quad \hat{J}|_{\widetilde{\mathfrak{f}_0}} = -J_1|_{\widetilde{\mathfrak{f}_0}}, \quad \hat{J}|_{\widetilde{\mathfrak{u}_0}} = J_1|_{\widetilde{\mathfrak{u}_0}}. \tag{2.6}$$

Then a representation $U \mapsto T_U$ of such a Lie algebra $\mathfrak{u}$ on a Euclidean space $\widetilde{\mathfrak{u}}$ together with a vector space isometry $\varphi\colon \mathfrak{u} \to \widetilde{\mathfrak{u}}$ is said to be a *Q-representation* if it satisfies the eight conditions (Q1–8) given in [A, Lem. 5.5 and Def. 5.3] (cf. also Cortés [Co, Def. 1.8]).

If $\mathfrak{s}$ is an Alekseevskian Lie algebra with principal Kählerian subalgebra $(\mathfrak{u}, J)$, then the representation of $\mathfrak{u}$ on $J_2\mathfrak{u}$ induced by the adjoint representation of $\mathfrak{s}$ is a Q-representation with $\varphi = J_2|_{\mathfrak{u}}\colon \mathfrak{u} \to \mathfrak{u}^\perp$. Conversely, let (T, φ) be a Q-representation of an admissible Kählerian Lie algebra $(\mathfrak{u}, J)$ on the Euclidean vector space $\widetilde{\mathfrak{u}} = \varphi(\mathfrak{u}) = \widetilde{\mathfrak{f}}_0 + \widetilde{\mathfrak{u}}_0$. Then a quaternionic structure $\mathrm{q} = \mathrm{Span}\{J_a : a = 1, 2, 3\}$ on the Euclidean vector space $\mathfrak{s} = \mathfrak{u} + \widetilde{\mathfrak{u}}$ (orthogonal sum) is defined by

$$\begin{gathered} J_1|_{\mathfrak{u}} = J, \quad J_1|_{\widetilde{\mathfrak{u}}} = -\varphi J \varphi^{-1}, \\ J_2|_{\mathfrak{u}} = \varphi, \quad J_2|_{\widetilde{\mathfrak{u}}} = -\varphi^{-1}, \quad J_3 = J_1 J_2. \end{gathered} \tag{2.7}$$

Let $\hat{J}$ be the complex structure on $\widetilde{\mathfrak{u}}$ defined as in (2.6), and let $\hat{\omega}$ denote the Kähler form on $\widetilde{\mathfrak{u}}$ given by $\hat{\omega}(\widetilde{U}, \widetilde{V}) = \langle \hat{J}\widetilde{U}, \widetilde{V}\rangle$. Then the following conditions define the structure of Lie algebra of $\mathfrak{s}$:

$$\mathfrak{u} \text{ is a subalgebra of } \mathfrak{s}, \qquad \mathrm{ad}_U|_{\widetilde{\mathfrak{u}}} = T_U, \qquad [\widetilde{U}, \widetilde{V}] = \hat{\omega}(\widetilde{U}, \widetilde{V}) G_0, \tag{2.8}$$

for all $U, V \in \mathfrak{u}$.

The *rank* of a solvable Lie algebra $\mathfrak{s}$ is the dimension of a Cartan subalgebra of $\mathfrak{s}$. The rank of an Alekseevskian space $\mathcal{S}$ is the rank of its Alekseevskian Lie algebra $\mathfrak{s}$, which is proved to be at most 4. An admissible Kählerian Lie algebra $\mathfrak{u} = \mathfrak{f}_0 + \mathfrak{u}_0$ which admits a Q-representation decomposes as a semidirect sum of elementary Kählerian Lie algebras, with $\mathfrak{u}_0 = \sum_{i \geq 1}(\mathfrak{f}_i + \mathfrak{x}_i)$, that is, $[\mathfrak{f}_i + \mathfrak{x}_i, \mathfrak{f}_j + \mathfrak{x}_j] \subset \mathfrak{f}_j + \mathfrak{f}_j$, $i \geq j$, with symplectic representation $\mathrm{ad}_{\mathfrak{f}_i}|_{\mathfrak{x}_j}$ for $i > j$ and commuting key algebras, $[\mathfrak{f}_i, \mathfrak{f}_j] = 0$, for $i \neq j$ (see [Co, p. 134]). The rank of $\mathfrak{u} = \mathfrak{f}_0 + \sum_{i \geq 1}(\mathfrak{f}_i + \mathfrak{x}_i)$ coincides with the number of key algebras of $\mathfrak{u}$. There are three types of admissible Kählerian Lie algebras, the first type corresponding to the case with smallest root 1.

3. Homogeneous quaternionic Kähler structures on $\mathcal{W}(p, q)$. Now we focus on the rank-four $\mathcal{W}$-case. We shall make calculations essentially based on the explicit description, found by Cortés [Co], of the spaces

$\mathcal{W}(p,q)$, $0 \le p \le q$, as completely solvable Lie groups with a left-invariant quaternionic Kähler structure.

We recall that given Euclidean spaces x, y, z, a bilinear map $\psi\colon x \times z \to y$ is said to be *isometric* if $\langle \psi(X,Z), \psi(X,Z)\rangle = \langle X,X\rangle\langle Z,Z\rangle$, $X \in x$, $Z \in z$. Let $\mathfrak{x}_-, \mathfrak{z}_-, \mathfrak{y}_-$ be Euclidean vector spaces. Every isometric map $\psi\colon \mathfrak{x}_- \times \mathfrak{z}_- \to \mathfrak{y}_-$ defines a Kählerian Lie algebra $\mathfrak{u}(\psi) = (\mathfrak{f}_0 + \mathfrak{u}_0, J)$ of type 1 and rank 4 by means of a recipe given in [A, Prop. 9.3]. According to [A, Props. 9.2–9.4], there are two possibilities for Kählerian Lie algebras $\mathfrak{u} = \mathfrak{u}(\psi)$ of type 1 and rank > 2 which admit a Q-representation. These two possibilities originate the series of Alekseevskian $\mathcal{W}$- and $\mathcal{V}$-spaces, respectively. The $\mathcal{W}$-spaces correspond to the case $\mathfrak{x}_- = 0$ (hence $\psi = 0$), and $\mathfrak{u} = \mathfrak{u}(p,q) \cong \mathfrak{u}(q,p)$ is completely determined by the parameters $p = \dim \mathfrak{y}_- \ge 0$ and $q = \dim \mathfrak{z}_- \ge 0$. Any such Lie algebra $\mathfrak{u}$ has a unique Q-representation T and the corresponding Alekseevskian spaces are denoted by $\mathcal{W}(p,q)$. In this case the set of rules of the aforementioned recipe reduces to:

1. The space $\mathfrak{u}_0$ is a semidirect sum $\mathfrak{u}_0 = (\mathfrak{f}_1 + \mathfrak{x}_1) + \mathfrak{f}_2 + \mathfrak{f}_3$ of elementary Kählerian key algebras with commuting Lie algebras with root 1.

2. The space $\mathfrak{x}_1$ admits a J-invariant decomposition $\mathfrak{x}_1 = \mathfrak{y} + \mathfrak{z}$ such that $\mathrm{ad}_{\mathfrak{f}_3}|_{\mathfrak{y}}$ and $\mathrm{ad}_{\mathfrak{f}_2}|_{\mathfrak{z}}$ are nondegenerate symplectic representations with weight decompositions $\mathfrak{y} = \mathfrak{y}_+ + \mathfrak{y}_-$ and $\mathfrak{z} = \mathfrak{z}_+ + \mathfrak{z}_-$, where $\mathfrak{y}_+ = J\mathfrak{y}_-$ and $\mathfrak{z}_+ = J\mathfrak{z}_-$. Furthermore, $[\mathfrak{f}_2, \mathfrak{y}] = [\mathfrak{f}_3, \mathfrak{z}] = [\mathfrak{y}, \mathfrak{z}] = 0$.

Let $\{Y_{j+}\}$, $j = 1, \dots, p$, and $\{Z_{k+}\}$, $k = 1, \dots, q$, be orthonormal bases of $\mathfrak{y}_+$ and $\mathfrak{z}_+$, respectively, and let $Y_{j-} = JY_{j+}$, $Z_{k-} = JZ_{k+}$. Then, as $\mathrm{ad}_{G_0}\mathfrak{u}_0 = 0$ ([A, Lem. 4.6]) and $\mathrm{ad}_{H_0}\mathfrak{u}_0 = 0$ ([A, (5.2)]), we have the Lie brackets on $\mathfrak{u}$ given in Table 1.

Table 1. Lie brackets on $\mathfrak{u}$

	G_0	H_0	G_1	H_1	G_2	H_2	G_3	H_3	$Y_{j'+}$	$Y_{j'-}$	$Z_{k'+}$	$Z_{k'-}$
G_0	0	$-G_0$	0	0	0	0	0	0	0	0	0	0
H_0	G_0	0	0	0	0	0	0	0	0	0	0	0
G_1	0	0	0	$-G_1$	0	0	0	0	0	0	0	0
H_1	0	0	G_1	0	0	0	0	0	$\frac{1}{2}Y_{j'+}$	$\frac{1}{2}Y_{j'-}$	$\frac{1}{2}Z_{k'+}$	$\frac{1}{2}Z_{k'-}$
G_2	0	0	0	0	0	$-G_2$	0	0	0	0	0	$Z_{k'+}$
H_2	0	0	0	0	G_2	0	0	0	0	0	$\frac{1}{2}Z_{k'+}$	$-\frac{1}{2}Z_{k'-}$
G_3	0	0	0	0	0	0	0	$-G_3$	0	$Y_{j'+}$	0	0
H_3	0	0	0	0	0	0	G_3	0	$\frac{1}{2}Y_{j'+}$	$-\frac{1}{2}Y_{j'-}$	0	0
Y_{j+}	0	0	0	$-\frac{1}{2}Y_{j+}$	0	0	0	$-\frac{1}{2}Y_{j+}$	0	$\delta_{jj'}G_1$	0	0
Y_{j-}	0	0	0	$-\frac{1}{2}Y_{j-}$	0	0	$-Y_{j+}$	$\frac{1}{2}Y_{j-}$	$-\delta_{jj'}G_1$	0	0	0
Z_{k+}	0	0	0	$-\frac{1}{2}Z_{k+}$	0	$-\frac{1}{2}Z_{k+}$	0	0	0	0	0	$\delta_{kk'}G_1$
Z_{k-}	0	0	0	$-\frac{1}{2}Z_{k-}$	$-Z_{k+}$	$\frac{1}{2}Z_{k-}$	0	0	0	0	$-\delta_{kk'}G_1$	0

Furthermore, the Kählerian Lie algebra $(\mathfrak{u}, J)$ has a unique Q-representation on the Euclidean vector space $\widetilde{\mathfrak{u}} = \widetilde{\mathfrak{f}_0} + \widetilde{\mathfrak{u}}_0$, $T\colon \mathfrak{u} \to \mathrm{End}(\widetilde{\mathfrak{u}})$, where $\sim\colon \mathfrak{u} \to \widetilde{\mathfrak{u}}$ denotes the corresponding isometry of Euclidean vector spaces.

Consider the quaternion-Hermitian vector space $(\mathfrak{w}(p,q), \langle\,,\rangle, \mathrm{q})$, where the space $\mathfrak{w}(p,q) = \mathfrak{u} + \widetilde{\mathfrak{u}}$ is a direct orthogonal sum, and $\mathrm{q} = \mathrm{Span}\{J_a : a = 1,2,3\}$ is the quaternionic structure on $\mathfrak{w}(p,q)$ defined by (2.7). Then

$$\mathcal{B} = \{G_i, H_i, Y_{j+}, Y_{j-}, Z_{k+}, Z_{k-}, \widetilde{G}_i, \widetilde{H}_i, \widetilde{Y}_{j+}, \widetilde{Y}_{j-}, \widetilde{Z}_{k+}, \widetilde{Z}_{k-}\}, \tag{3.1}$$

for $0 \le i \le 3$, $1 \le j \le p$, $1 \le k \le q$, is an orthonormal basis of $\mathfrak{w}(p,q)$.

Table 2. The action of J_a, $a = 1,2,3$, on $\mathfrak{w}(p,q)$

	G_i	H_i	Y_{j+}	Y_{j-}	Z_{k+}	Z_{k-}	$\widetilde{G}_i$	$\widetilde{H}_i$	$\widetilde{Y}_{j+}$	$\widetilde{Y}_{j-}$	$\widetilde{Z}_{k+}$	$\widetilde{Z}_{k-}$
J_1	$-H_i$	G_i	Y_{j-}	$-Y_{j+}$	Z_{k-}	$-Z_{k+}$	$\widetilde{H}_i$	$-\widetilde{G}_i$	$-\widetilde{Y}_{j-}$	$\widetilde{Y}_{j+}$	$-\widetilde{Z}_{k-}$	$\widetilde{Z}_{k+}$
J_2	$\widetilde{G}_i$	$\widetilde{H}_i$	$\widetilde{Y}_{j+}$	$\widetilde{Y}_{j-}$	$\widetilde{Z}_{k+}$	$\widetilde{Z}_{k-}$	$-G_i$	$-H_i$	$-Y_{j+}$	$-Y_{j-}$	$-Z_{k+}$	$-Z_{k-}$
J_3	$\widetilde{H}_i$	$-\widetilde{G}_i$	$-\widetilde{Y}_{j-}$	$\widetilde{Y}_{j+}$	$-\widetilde{Z}_{k-}$	$\widetilde{Z}_{k+}$	H_i	$-G_i$	$-Y_{j-}$	Y_{j+}	$-Z_{k-}$	Z_{k+}

Table 3. The complex structure $\hat{J}$ on $\widetilde{\mathfrak{u}}$

	$\widetilde{G}_0$	$\widetilde{H}_0$	$\widetilde{G}_i$	$\widetilde{H}_i$	$\widetilde{Y}_{j+}$	$\widetilde{Y}_{j-}$	$\widetilde{Z}_{k+}$	$\widetilde{Z}_{k-}$
$\hat{J}$	$-\widetilde{H}_0$	$\widetilde{G}_0$	$\widetilde{H}_i$	$-\widetilde{G}_i$	$-\widetilde{Y}_{j-}$	$\widetilde{Y}_{j+}$	$-\widetilde{Z}_{k-}$	$\widetilde{Z}_{k+}$

The action of J_a, $a = 1,2,3$, on $\mathfrak{w}(p,q) = \mathfrak{u} + \widetilde{\mathfrak{u}}$ is given in Table 2. Moreover, the vector space $\mathfrak{w}(p,q)$ has a structure of Lie algebra given by (2.8), with $\mathfrak{s} = \mathfrak{w}(p,q)$, where the complex structure $\hat{J}$ on $\widetilde{\mathfrak{u}}$ is defined by Table 3. Hence, by the third condition in (2.8), the nonnull brackets of the elements of $\widetilde{\mathfrak{u}}$ are

$$\begin{aligned} [\widetilde{H}_0, \widetilde{G}_0] &= -[\widetilde{H}_i, \widetilde{G}_i] \\ &= -[\widetilde{Y}_{j+}, \widetilde{Y}_{j-}] = -[\widetilde{Z}_{k+}, \widetilde{Z}_{k-}] = G_0, \quad i = 1,2,3. \end{aligned} \tag{3.2}$$

If $U \in \mathfrak{u}$ and $\widetilde{V} \in \widetilde{\mathfrak{u}}$, then by the second condition in (2.8), one has $[U, \widetilde{V}] = T_U \widetilde{V}$, and the values of $T_U \widetilde{V}$ are given in Tables 4–7, where $T\colon \mathfrak{u} \to \mathrm{End}(\widetilde{\mathfrak{u}})$ is expressed in terms of the orthonormal basis $\{\widetilde{G}_i, \widetilde{H}_i, \widetilde{Y}_{j+}, \widetilde{Y}_{j-}, \widetilde{Z}_{k+}, \widetilde{Z}_{k-}\}$ of $\widetilde{\mathfrak{u}}$, from the conditions (Q1–8) of a Q-representation (cf. [Co, Prop. 2.1]). Table 6 follows from the properties of a weight decomposition with respect to a nondegenerate symplectic representation (2.5) and the properties in [A, Prop. 9.3].

The Lie algebra $\mathfrak{w}(p,q)$ is 4-step solvable with $\dim_{\mathbb{R}} \mathfrak{w}(p,q) = 4(4+p+q)$, and the corresponding simply-connected Lie group with left-invariant metric is the Alekseevskian space $\mathcal{W}(p,q)$.

We have $\mathfrak{w}(p,q)^* = \mathfrak{u}^* + \widetilde{\mathfrak{u}}^*$. Let

$$\mathcal{B}^* = \{\gamma^i, \eta^i, \xi^{j+}, \xi^{j-}, \zeta^{k+}, \zeta^{k-}, \widetilde{\gamma}^i, \widetilde{\eta}^i, \widetilde{\xi}^{j+}, \widetilde{\xi}^{j-}, \widetilde{\zeta}^{k+}, \widetilde{\zeta}^{k-}\}$$

be the basis of $\mathfrak{w}(p,q)^*$ dual to the basis (3.1) of $\mathfrak{w}(p,q)$, and denote by S_X the 2-form defined by $S_X(Y,Z) = S_{XYZ}$.

Theorem 3.1. *The homogeneous quaternionic Kähler structure S on each rank-four Alekseevskian space $\mathcal{W}(p,q)$, $0 \leq p \leq q$, which gives its description as the simply-connected solvable Lie group with Lie algebra $\mathfrak{w}(p,q)$, is given, in terms of the basis $\mathcal{B}^*$ of $\mathfrak{w}(p,q)^*$, by*

$$\begin{aligned}
S_{|\mathfrak{u}^*\otimes\bigwedge^2\mathfrak{u}^*} = {} & \textstyle\sum_{i=0}^{3}\gamma^i\otimes(\gamma^i\wedge\eta^i) - \frac{1}{2}\big(\gamma^1\otimes(\xi^{j+}\wedge\xi^{j-}+\zeta^{k+}\wedge\zeta^{k-}) \\
& \qquad + \gamma^2\otimes\zeta^{k+}\wedge\zeta^{k-} + \gamma^3\otimes\xi^{j+}\wedge\xi^{j-}\big) \\
& + \tfrac{1}{2}\textstyle\sum_{j=1}^{p}\xi^{j+}\otimes\big(\xi^{j+}\wedge(\eta^1+\eta^3)+\xi^{j-}\wedge(\gamma^1+\gamma^3)\big) \\
& + \tfrac{1}{2}\textstyle\sum_{j=1}^{p}\xi^{j-}\otimes\big(\xi^{j+}\wedge(\gamma^3-\gamma^1)+\xi^{j-}\wedge(\eta^1-\eta^3)\big) \\
& + \tfrac{1}{2}\textstyle\sum_{k=1}^{q}\zeta^{k+}\otimes\big(\zeta^{k+}\wedge(\eta^1+\eta^2)+\zeta^{k-}\wedge(\gamma^1+\gamma^2)\big) \\
& + \tfrac{1}{2}\textstyle\sum_{k=1}^{q}\zeta^{k-}\otimes\big(\zeta^{k+}\wedge(\gamma^2-\gamma^1)+\zeta^{k-}\wedge(\eta^1-\eta^2)\big),
\end{aligned}$$

$$\begin{aligned}
S_{|\mathfrak{u}^*\otimes\bigwedge^2\widetilde{\mathfrak{u}}^*} = {} & \tfrac{1}{2}\gamma^0\otimes\big(\widetilde{\gamma}^0\wedge\widetilde{\eta}^0 - \textstyle\sum_{i=1}^{3}\widetilde{\gamma}^i\wedge\widetilde{\eta}^i \\
& + \textstyle\sum_{j=1}^{p}\widetilde{\xi}^{j+}\wedge\widetilde{\xi}^{j-} + \sum_{k=1}^{q}\widetilde{\zeta}^{k+}\wedge\widetilde{\zeta}^{k-}\big) \\
& - \tfrac{1}{2}\gamma^1\otimes(\widetilde{\gamma}^0\wedge\widetilde{\eta}^0 - \widetilde{\gamma}^1\wedge\widetilde{\eta}^1 + \widetilde{\gamma}^2\wedge\widetilde{\eta}^2 + \widetilde{\gamma}^3\wedge\widetilde{\eta}^3) \\
& - \tfrac{1}{2}\gamma^2\otimes\big(\widetilde{\gamma}^0\wedge\widetilde{\eta}^0 + \widetilde{\gamma}^1\wedge\widetilde{\eta}^1 - \widetilde{\gamma}^2\wedge\widetilde{\eta}^2 \\
& + \widetilde{\gamma}^3\wedge\widetilde{\eta}^3 - \textstyle\sum_{j=1}^{p}\widetilde{\xi}^{j+}\wedge\widetilde{\xi}^{j-}\big) \\
& - \tfrac{1}{2}\gamma^3\otimes\big(\widetilde{\gamma}^0\wedge\widetilde{\eta}^0 + \widetilde{\gamma}^1\wedge\widetilde{\eta}^1 + \widetilde{\gamma}^2\wedge\widetilde{\eta}^2 \\
& - \widetilde{\gamma}^3\wedge\widetilde{\eta}^3 - \textstyle\sum_{k=1}^{q}\widetilde{\zeta}^{k+}\wedge\widetilde{\zeta}^{k-}\big) \\
& + \tfrac{1}{2}\textstyle\sum_{j=1}^{p}\xi^{j+}\otimes\big(\widetilde{\xi}^{j+}\wedge(\widetilde{\eta}^1+\widetilde{\eta}^3)+\widetilde{\xi}^{j-}\wedge(\widetilde{\gamma}^1+\widetilde{\gamma}^3)\big) \\
& - \tfrac{1}{2}\textstyle\sum_{j=1}^{p}\xi^{j-}\otimes\big(\widetilde{\xi}^{j+}\wedge(\widetilde{\gamma}^1-\widetilde{\gamma}^3)-\widetilde{\xi}^{j-}\wedge(\widetilde{\eta}^1-\widetilde{\eta}^3)\big) \\
& + \tfrac{1}{2}\textstyle\sum_{k=1}^{q}\zeta^{k+}\otimes\big(\widetilde{\zeta}^{k+}\wedge(\widetilde{\eta}^1+\widetilde{\eta}^2)+\widetilde{\zeta}^{k-}\wedge(\widetilde{\gamma}^1+\widetilde{\gamma}^2)\big) \\
& - \tfrac{1}{2}\textstyle\sum_{k=1}^{q}\zeta^{k-}\otimes\big(\widetilde{\zeta}^{k+}\wedge(\widetilde{\gamma}^1-\widetilde{\gamma}^2)-\widetilde{\zeta}^{k-}\wedge(\widetilde{\eta}^1-\widetilde{\eta}^2)\big),
\end{aligned}$$

$$\begin{aligned}
S_{\widetilde{G}_0} = {} & \tfrac{1}{2}\big(\textstyle\sum_{i=0}^{3}(\widetilde{\gamma}^i\wedge\eta^i - \widetilde{\eta}^i\wedge\gamma^i) \\
& - \textstyle\sum_{j=1}^{p}(\widetilde{\xi}^{j+}\wedge\xi^{j-} - \widetilde{\xi}^{j-}\wedge\xi^{j+}) \\
& - \textstyle\sum_{k=1}^{q}(\widetilde{\zeta}^{k+}\wedge\zeta^{k-} - \widetilde{\zeta}^{k-}\wedge\zeta^{k-})\big),
\end{aligned}$$

$$\begin{aligned}
S_{\widetilde{H}_0} = {} & \tfrac{1}{2}\big(\textstyle\sum_{i=0}^{3}(\widetilde{\gamma}^i\wedge\gamma^i + \widetilde{\eta}^i\wedge\eta^i) \\
& + \textstyle\sum_{j=1}^{p}(\widetilde{\xi}^{j+}\wedge\xi^{j+} + \widetilde{\xi}^{j-}\wedge\xi^{j-}) \\
& + \textstyle\sum_{k=1}^{q}(\widetilde{\zeta}^{k+}\wedge\zeta^{k+} + \widetilde{\zeta}^{k-}\wedge\zeta^{k-})\big),
\end{aligned}$$

$$\begin{aligned}
S_{\widetilde{G}_1} &= \tfrac{1}{2}(\widetilde{\gamma}^0 \wedge \eta^1 + \widetilde{\eta}^0 \wedge \gamma^1 + \widetilde{\gamma}^1 \wedge \eta^0 + \widetilde{\eta}^1 \wedge \gamma^0 \\
&\qquad - \widetilde{\gamma}^2 \wedge \eta^3 - \widetilde{\eta}^2 \wedge \gamma^3 - \widetilde{\gamma}^3 \wedge \eta^2 - \widetilde{\eta}^3 \wedge \gamma^2), \\
S_{\widetilde{H}_1} &= -\tfrac{1}{2}(\widetilde{\gamma}^0 \wedge \gamma^1 - \widetilde{\eta}^0 \wedge \eta^1 + \widetilde{\gamma}^1 \wedge \gamma^0 - \widetilde{\eta}^1 \wedge \eta^0 \\
&\qquad + \widetilde{\gamma}^2 \wedge \gamma^3 - \widetilde{\eta}^2 \wedge \eta^3 + \widetilde{\gamma}^3 \wedge \gamma^2 - \widetilde{\eta}^3 \wedge \eta^2), \\
S_{\widetilde{G}_2} &= \tfrac{1}{2}\big(\widetilde{\gamma}^0 \wedge \eta^2 + \widetilde{\eta}^0 \wedge \gamma^2 - \widetilde{\gamma}^1 \wedge \eta^3 - \widetilde{\eta}^1 \wedge \gamma^3 \\
&\qquad + \widetilde{\gamma}^2 \wedge \eta^0 + \widetilde{\eta}^2 \wedge \gamma^0 - \widetilde{\gamma}^3 \wedge \eta^1 - \widetilde{\eta}^3 \wedge \gamma^1 \\
&\qquad - \textstyle\sum_{j=1}^{p}(\widetilde{\xi}^{j+} \wedge \xi^{j-} + \widetilde{\xi}^{j-} \wedge \xi^{j+})\big), \\
S_{\widetilde{H}_2} &= -\tfrac{1}{2}\big(\widetilde{\gamma}^0 \wedge \gamma^2 - \widetilde{\eta}^0 \wedge \eta^2 + \widetilde{\gamma}^1 \wedge \gamma^3 - \widetilde{\eta}^1 \wedge \eta^3 \\
&\qquad + \widetilde{\gamma}^2 \wedge \gamma^0 - \widetilde{\eta}^2 \wedge \eta^0 + \widetilde{\gamma}^3 \wedge \gamma^1 - \widetilde{\eta}^3 \wedge \eta^1 \\
&\qquad - \textstyle\sum_{j=1}^{p}(\widetilde{\xi}^{j+} \wedge \xi^{j+} - \widetilde{\xi}^{j-} \wedge \xi^{j-})\big), \\
S_{\widetilde{G}_3} &= \tfrac{1}{2}\big(\widetilde{\gamma}^0 \wedge \eta^3 + \widetilde{\eta}^0 \wedge \gamma^3 - \widetilde{\gamma}^1 \wedge \eta^2 - \widetilde{\eta}^1 \wedge \gamma^2 \\
&\qquad - \widetilde{\gamma}^2 \wedge \eta^1 - \widetilde{\eta}^2 \wedge \gamma^1 + \widetilde{\gamma}^3 \wedge \eta^0 + \widetilde{\eta}^3 \wedge \gamma^0 \\
&\qquad - \textstyle\sum_{k=1}^{q}(\widetilde{\zeta}^{k+} \wedge \zeta^{k-} + \widetilde{\zeta}^{k-} \wedge \zeta^{k+})\big), \\
S_{\widetilde{H}_3} &= -\tfrac{1}{2}\big(\widetilde{\gamma}^0 \wedge \gamma^3 - \widetilde{\eta}^0 \wedge \eta^3 + \widetilde{\gamma}^1 \wedge \gamma^2 - \widetilde{\eta}^1 \wedge \eta^2 \\
&\qquad + \widetilde{\gamma}^2 \wedge \gamma^1 - \widetilde{\eta}^2 \wedge \eta^1 + \widetilde{\gamma}^3 \wedge \gamma^0 - \widetilde{\eta}^3 \wedge \eta^0 \\
&\qquad - \textstyle\sum_{k=1}^{q}(\widetilde{\zeta}^{k+} \wedge \zeta^{k+} - \widetilde{\zeta}^{k-} \wedge \zeta^{k-})\big), \\
S_{\widetilde{Y}_{j+}} &= -\tfrac{1}{2}\textstyle\sum_{j=1}^{p}\big((\widetilde{\gamma}^0 + \widetilde{\gamma}^2) \wedge \xi^{j-} - (\widetilde{\eta}^0 + \widetilde{\eta}^2) \wedge \xi^{j+} \\
&\qquad + \widetilde{\xi}^{j-} \wedge (\gamma^0 + \gamma^2) - \widetilde{\xi}^{j+} \wedge (\eta^0 + \eta^2)\big), \\
S_{\widetilde{Y}_{j-}} &= \tfrac{1}{2}\textstyle\sum_{j=1}^{p}((\widetilde{\gamma}^0 - \widetilde{\gamma}^2) \wedge \xi^{j+} + (\widetilde{\eta}^0 - \widetilde{\eta}^2) \wedge \xi^{j-} \\
&\qquad + \widetilde{\xi}^{j+} \wedge (\gamma^0 - \gamma^2) + \widetilde{\xi}^{j-} \wedge (\eta^0 - \eta^2)\big), \\
S_{\widetilde{Z}_{k_+}} &= -\tfrac{1}{2}\textstyle\sum_{k=1}^{q}\big((\widetilde{\gamma}^0 + \widetilde{\gamma}^3) \wedge \zeta^{k-} - (\widetilde{\eta}^0 + \widetilde{\eta}^3) \wedge \zeta^{k+} \\
&\qquad + \widetilde{\zeta}^{k-} \wedge (\gamma^0 + \gamma^3) - \widetilde{\zeta}^{k+} \wedge (\eta^0 + \eta^3)\big), \\
S_{\widetilde{Z}_{k_-}} &= \tfrac{1}{2}\textstyle\sum_{k=1}^{q}\big((\widetilde{\gamma}^0 - \widetilde{\gamma}^3) \wedge \zeta^{k+} + (\widetilde{\eta}^0 - \widetilde{\eta}^3) \wedge \zeta^{k-} \\
&\qquad + \widetilde{\zeta}^{k+} \wedge (\gamma^0 - \gamma^3) + \widetilde{\zeta}^{k-} \wedge (\eta^0 - \eta^3)\big).
\end{aligned}$$

Proof. Consider the tensor field S on $\mathcal{W}(p,q)$, $0 \le p \le q$, given by

$$2\langle S_X Y, Z\rangle = \langle [X,Y], Z\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle \tag{3.3}$$

for $X, Y, Z \in \mathfrak{w}(p,q)$. Let ∇ be the Levi-Civita connection on $\mathcal{W}(p,q)$ with respect to the invariant metric defined by $\langle\,,\rangle$. Then $\widetilde{\nabla} = \nabla - S$ is the connection on the Lie group $\mathcal{W}(p,q)$ for which every left-invariant vector field is parallel. Thus, conditions (2.2) are satisfied and S is a homogeneous quaternionic Kähler structure. Moreover, the holonomy algebra of the connection $\widetilde{\nabla}$ is trivial, and then S provides the description of $\mathcal{W}(p,q)$ as a Lie group (see [TV, p. 32, Eqs. (1.79)]).

Table 4. The Q-representation $T\colon \mathfrak{u} \to \mathrm{End}(\widetilde{\mathfrak{u}})$

	$\widetilde{G}_0$	$\widetilde{H}_0$	$\widetilde{G}_1$	$\widetilde{H}_1$
G_0	0	0	0	0
H_0	$\frac{1}{2}\widetilde{G}_0$	$\frac{1}{2}\widetilde{H}_0$	$\frac{1}{2}\widetilde{G}_1$	$\frac{1}{2}\widetilde{H}_1$
G_1	$-\frac{1}{2}(\widetilde{H}_0+\widetilde{H}_1)$	$\frac{1}{2}(\widetilde{G}_0+\widetilde{G}_1)$	$\frac{1}{2}(\widetilde{H}_0+\widetilde{H}_1)$	$-\frac{1}{2}(\widetilde{G}_0+\widetilde{G}_1)$
H_1	$\frac{1}{2}\widetilde{G}_1$	$\frac{1}{2}\widetilde{H}_1$	$\frac{1}{2}\widetilde{G}_0$	$\frac{1}{2}\widetilde{H}_0$
G_2	$-\frac{1}{2}(\widetilde{H}_0+\widetilde{H}_2)$	$\frac{1}{2}(\widetilde{G}_0+\widetilde{G}_2)$	$-\frac{1}{2}(\widetilde{H}_3+\widetilde{H}_1)$	$-\frac{1}{2}(\widetilde{G}_3-\widetilde{G}_1)$
H_2	$\frac{1}{2}\widetilde{G}_2$	$\frac{1}{2}\widetilde{H}_2$	$-\frac{1}{2}\widetilde{G}_3$	$\frac{1}{2}\widetilde{H}_3$
G_3	$-\frac{1}{2}(\widetilde{H}_0+\widetilde{H}_3)$	$\frac{1}{2}(\widetilde{G}_0+\widetilde{G}_3)$	$-\frac{1}{2}(\widetilde{H}_1+\widetilde{H}_2)$	$\frac{1}{2}(\widetilde{G}_1-\widetilde{G}_2)$
H_3	$\frac{1}{2}\widetilde{G}_3$	$\frac{1}{2}\widetilde{H}_3$	$-\frac{1}{2}\widetilde{G}_2$	$\frac{1}{2}\widetilde{H}_2$
Y_{j+}	$\frac{1}{2}\widetilde{Y}_{j-}$	$\frac{1}{2}\widetilde{Y}_{j+}$	$-\frac{1}{2}\widetilde{Y}_{j-}$	$-\frac{1}{2}\widetilde{Y}_{j+}$
Y_{j-}	$-\frac{1}{2}\widetilde{Y}_{j+}$	$\frac{1}{2}\widetilde{Y}_{j-}$	$\frac{1}{2}\widetilde{Y}_{j+}$	$-\frac{1}{2}\widetilde{Y}_{j-}$
Z_{k+}	$\frac{1}{2}\widetilde{Z}_{k-}$	$\frac{1}{2}\widetilde{Z}_{k+}$	$-\frac{1}{2}\widetilde{Z}_{k-}$	$-\frac{1}{2}\widetilde{Z}_{k+}$
Z_{k-}	$-\frac{1}{2}\widetilde{Z}_{k+}$	$\frac{1}{2}\widetilde{Z}_{k-}$	$\frac{1}{2}\widetilde{Z}_{k+}$	$-\frac{1}{2}\widetilde{Z}_{k-}$

Table 5. The Q-representation $T\colon \mathfrak{u} \to \mathrm{End}(\widetilde{\mathfrak{u}})$

	$\widetilde{G}_2$	$\widetilde{H}_2$	$\widetilde{G}_3$	$\widetilde{H}_3$
G_0	0	0	0	0
H_0	$\frac{1}{2}\widetilde{G}_2$	$\frac{1}{2}\widetilde{H}_2$	$\frac{1}{2}\widetilde{G}_3$	$\frac{1}{2}\widetilde{H}_3$
G_1	$-\frac{1}{2}(\widetilde{H}_2+\widetilde{H}_3)$	$\frac{1}{2}(\widetilde{G}_2-\widetilde{G}_3)$	$-\frac{1}{2}(\widetilde{H}_2+\widetilde{H}_3)$	$-\frac{1}{2}(\widetilde{G}_2-\widetilde{G}_3)$
H_1	$-\frac{1}{2}\widetilde{G}_3$	$\frac{1}{2}\widetilde{H}_3$	$-\frac{1}{2}\widetilde{G}_2$	$\frac{1}{2}\widetilde{H}_2$
G_2	$\frac{1}{2}(\widetilde{H}_0+\widetilde{H}_2)$	$-\frac{1}{2}(\widetilde{G}_0+\widetilde{G}_2)$	$-\frac{1}{2}(\widetilde{H}_3+\widetilde{H}_1)$	$\frac{1}{2}(\widetilde{G}_3-\widetilde{G}_1)$
H_2	$\frac{1}{2}\widetilde{G}_0$	$\frac{1}{2}\widetilde{H}_0$	$-\frac{1}{2}\widetilde{G}_1$	$\frac{1}{2}\widetilde{H}_1$
G_3	$-\frac{1}{2}(\widetilde{H}_1+\widetilde{H}_2)$	$-\frac{1}{2}(\widetilde{G}_1-\widetilde{G}_2)$	$\frac{1}{2}(\widetilde{H}_0+\widetilde{H}_3)$	$-\frac{1}{2}(\widetilde{G}_0+\widetilde{G}_3)$
H_3	$-\frac{1}{2}\widetilde{G}_1$	$\frac{1}{2}\widetilde{H}_1$	$\frac{1}{2}\widetilde{G}_0$	$\frac{1}{2}\widetilde{H}_0$
Y_{j+}	$-\frac{1}{2}\widetilde{Y}_{j-}$	$\frac{1}{2}\widetilde{Y}_{j+}$	$-\frac{1}{2}\widetilde{Y}_{j-}$	$-\frac{1}{2}\widetilde{Y}_{j+}$
Y_{j-}	$-\frac{1}{2}\widetilde{Y}_{j+}$	$-\frac{1}{2}\widetilde{Y}_{j-}$	$-\frac{1}{2}\widetilde{Y}_{j+}$	$\frac{1}{2}\widetilde{Y}_{j-}$
Z_{k+}	$-\frac{1}{2}\widetilde{Z}_{k-}$	$-\frac{1}{2}\widetilde{Z}_{k+}$	$-\frac{1}{2}\widetilde{Z}_{k-}$	$\frac{1}{2}\widetilde{Z}_{k+}$
Z_{k-}	$-\frac{1}{2}\widetilde{Z}_{k+}$	$\frac{1}{2}\widetilde{Z}_{k-}$	$-\frac{1}{2}\widetilde{Z}_{k+}$	$-\frac{1}{2}\widetilde{Z}_{k-}$

Since (see (2.8)) we have $[\mathfrak{u},\mathfrak{u}]\subset\mathfrak{u}$, $[\mathfrak{u},\widetilde{\mathfrak{u}}]\subset\widetilde{\mathfrak{u}}$, $[\widetilde{\mathfrak{u}},\widetilde{\mathfrak{u}}]\subset\mathfrak{u}$, and $\mathfrak{u}$ and $\widetilde{\mathfrak{u}}$ are orthogonal, from (3.3) we have

$$S_{UV\widetilde{W}}=0,\quad S_{\widetilde{U}VW}=0,\quad S_{U\widetilde{V}W}=0,\quad S_{\widetilde{U}\widetilde{V}\widetilde{W}}=0. \tag{3.4}$$

Table 6. The Q-representation $T\colon\mathfrak{u}\to\mathrm{End}(\widetilde{\mathfrak{u}})$

	$\widetilde{Y}_{j'+}$	$\widetilde{Y}_{j'-}$	$\widetilde{Z}_{k'+}$	$\widetilde{Z}_{k'-}$
G_0	0	0	0	0
H_0	$\frac12\widetilde{Y}_{j'+}$	$\frac12\widetilde{Y}_{j'-}$	$\frac12\widetilde{Z}_{k'+}$	$\frac12\widetilde{Z}_{k'-}$
G_1	0	0	0	0
H_1	0	0	0	0
G_2	0	$-\widetilde{Y}_{j'+}$	0	0
H_2	$\frac12\widetilde{Y}_{j'+}$	$-\frac12\widetilde{Y}_{j'-}$	0	0
G_3	0	0	0	$-\widetilde{Z}_{k+}$
H_3	0	0	$\frac12\widetilde{Z}_{k'+}$	$-\frac12\widetilde{Z}_{k'-}$

Table 7. The Q-representation $T\colon\mathfrak{u}\to\mathrm{End}(\widetilde{\mathfrak{u}})$

	$\widetilde{Y}_{j'+}$	$\widetilde{Y}_{j'-}$	$\widetilde{Z}_{k'+}$	$\widetilde{Z}_{k'-}$
Y_{j+}	$\frac{\delta_{jj'}}{2}(\widetilde{H}_0+\widetilde{H}_1+\widetilde{H}_2+\widetilde{H}_3)$	$\frac{\delta_{jj'}}{2}(\widetilde{G}_0+\widetilde{G}_1-\widetilde{G}_2+\widetilde{G}_3)$	0	0
Y_{j-}	$-\frac{\delta_{jj'}}{2}(\widetilde{G}_0+\widetilde{G}_1+\widetilde{G}_2-\widetilde{G}_3)$	$\frac{\delta_{jj'}}{2}(\widetilde{H}_0+\widetilde{H}_1-\widetilde{H}_2-\widetilde{H}_3)$	0	0
Z_{k+}	0	0	$\frac{\delta_{kk'}}{2}(\widetilde{H}_0+\widetilde{H}_1+\widetilde{H}_2+\widetilde{H}_3)$	$\frac{\delta_{kk'}}{2}(\widetilde{G}_0+\widetilde{G}_1+\widetilde{G}_2-\widetilde{G}_3)$
Z_{k-}	0	0	$-\frac{\delta_{kk'}}{2}(\widetilde{G}_0+\widetilde{G}_1-\widetilde{G}_2+\widetilde{G}_3)$	$\frac{\delta_{kk'}}{2}(\widetilde{H}_0+\widetilde{H}_1-\widetilde{H}_2-\widetilde{H}_3)$

On account of (3.3), Table 1, and the equation $S_{UV\widetilde{W}}=0$ in (3.4), one obtains the nonzero values of S_{UVW} for U, V and W in the orthonormal basis $\mathcal{B}$. In order to obtain $S_{|\mathfrak{u}^*\otimes\bigwedge^2\widetilde{\mathfrak{u}}^*}$, we use (3.2), (3.3), the equation $S_{U\widetilde{V}W}=0$ in (3.4) and Tables 4 to 7, since $[U,\widetilde{V}]=T_U\widetilde{V}$. From (3.3), by using (3.2), Tables 4 to 7, and the equations $S_{\widetilde{U}VW}=S_{\widetilde{U}\widetilde{V}\widetilde{W}}=0$ in (3.4), we obtain the values of $S_{\widetilde{U}}$ for each $\widetilde{U}=\widetilde{G}_i,\widetilde{H}_i,\widetilde{Y}_{j+},\widetilde{Y}_{j-},\widetilde{Z}_{k+},\widetilde{Z}_{k-}$. ∎

4. The type of the structure on $\mathcal{W}(p,q)$. We now determine the type of the previous structure S.

THEOREM 4.1. *The homogeneous quaternionic Kähler structure on each rank-four Alekseevskian space $\mathcal{W}(p,q)$, given in Theorem 3.1, has a nonzero component in each basic Fino type.*

Proof. From the expression of S in Theorem 3.1 and from Table 1 we find that the forms α^a, $a = 1, 2, 3$, in (2.3) corresponding to S are given by

$$\alpha^1 = -\tfrac{1}{2}\sum_{i=0}^{3}\gamma^i, \quad \alpha^2 = -\widetilde{\eta}^0, \quad \alpha^3 = \widetilde{\gamma}^0. \tag{4.1}$$

Hence, since $S = \varTheta + \mathfrak{T}$, where $\varTheta$ is given by (2.3), from (4.1) and using Table 1, it follows that the tensor field $\varTheta$ on $\mathcal{W}(p,q)$ corresponding to S is given by

$$\begin{aligned} &\tfrac{1}{4}\textstyle\sum_{i=0}^{3}\gamma^i \otimes \{\sum_{l=0}^{3}(\gamma^l \wedge \eta^l - \widetilde{\gamma}^l \wedge \widetilde{\eta}^l) \\ &\quad - \textstyle\sum_{j=1}^{p}(\xi^{j+} \wedge \xi^{j-} - \widetilde{\xi}^{j+} \wedge \widetilde{\xi}^{j-}) - \sum_{k=1}^{q}(\zeta^{k+} \wedge \zeta^{k-} - \widetilde{\zeta}^{k+} \wedge \widetilde{\zeta}^{k-})\} \\ &+ \tfrac{1}{2}\widetilde{\gamma}^0 \otimes \{\textstyle\sum_{l=0}^{3}(\gamma^l \wedge \widetilde{\eta}^l + \widetilde{\gamma}^l \wedge \eta^l) \\ &\quad - \textstyle\sum_{j=1}^{p}(\xi^{j+} \wedge \widetilde{\xi}^{j-} + \widetilde{\xi}^{j+} \wedge \xi^{j-}) - \sum_{k=1}^{q}(\zeta^{k+} \wedge \widetilde{\zeta}^{k-} + \widetilde{\zeta}^{k+} \wedge \zeta^{k-})\} \\ &- \tfrac{1}{2}\widetilde{\eta}^0 \otimes \{\textstyle\sum_{l=0}^{3}(\gamma^l \wedge \widetilde{\gamma}^l + \eta^l \wedge \widetilde{\eta}^l) \\ &\quad + \textstyle\sum_{j=1}^{p}(\xi^{j+} \wedge \widetilde{\xi}^{j+} + \xi^{j-} \wedge \widetilde{\xi}^{j-}) + \sum_{k=1}^{q}(\zeta^{k+} \wedge \widetilde{\zeta}^{k+} + \zeta^{k-} \wedge \widetilde{\zeta}^{k-})\}. \end{aligned} \tag{4.2}$$

On the other hand, considering again that the structure decomposes as $S = \varTheta + \mathfrak{T}$, and the values of the 1-forms α^a are those in (4.1), we infer that as, for instance,

$$\sum_{a=1}^{3}\alpha^a(J_a H_2) = -1/2 \neq 0,$$

the component $\varTheta$ of the structure S does not belong to $\mathcal{QK}_2$.

From (4.2), the nonzero values of $\varTheta_{XYZ}$ are those with $X = G_0, G_1, G_2, G_3, \widetilde{G}_0, \widetilde{H}_0$. In particular one has the next nonzero values of type $\varTheta_{XXY}$:

$$\begin{aligned} &\varTheta_{G_0G_0H_0} = \varTheta_{G_1G_1H_1} = \varTheta_{G_2G_2H_2} = \varTheta_{G_3G_3H_3} = 1/4, \\ &\varTheta_{\widetilde{G}_0\widetilde{G}_0H_0} = \varTheta_{\widetilde{H}_0\widetilde{H}_0H_0} = 1/2. \end{aligned} \tag{4.3}$$

Suppose next that $\varTheta \in \mathcal{QK}_1$. Then there would be a 1-form α as that in expressions (2.4), and in particular we would have

$$1/4 = \varTheta_{G_0G_0H_0} = \alpha(H_0), \quad 1/2 = \varTheta_{\widetilde{G}_0\widetilde{G}_0H_0} = \alpha(H_0),$$

which is absurd. Hence $\varTheta \in \mathcal{QK}_{12} \setminus \{\mathcal{QK}_1 \cup \mathcal{QK}_2\}$.

Furthermore, as $\dim \mathcal{W}(p,q) = 4(4+p+q)$ and on account of (4.3), the form β defining the $\mathcal{QK}_3$-component (see expressions (2.4)), that is,

$$\beta = \frac{1}{2 + \dim \mathfrak{w}(p,q)}\, c_{12}(\mathfrak{T}) = \frac{1}{18 + 4(p+q)}\, c_{12}(\mathfrak{T}),$$

is given by

$$\beta = \frac{1}{18+4(p+q)}\left\{\left(\frac{15}{4}+p+q\right)\eta^0+\left(\frac{3}{4}+p+q\right)\eta^1+\frac{3}{4}(\eta^2+\eta^3)\right\}. \tag{4.4}$$

Hence S has a nonzero component in $\mathcal{QK}_3$ for all $0 \le p \le q$.

Consider now the operator $\Psi\colon \hat{\mathcal{V}} \to \hat{\mathcal{V}}$ defined by

$$\Psi(\mathcal{T})_{XYZ} = \mathcal{T}_{YZX} + \mathcal{T}_{ZXY} + \sum_{a=1}^{3}(\mathcal{T}_{J_aYJ_aZX} + \mathcal{T}_{J_aZXJ_aY}),$$

having eigenvalues 2 and -4, with corresponding eigenspaces $\mathcal{QK}_{34}$ and $\mathcal{QK}_5$, respectively (see expressions (2.4)). Consider $\mathcal{T}^\beta \in \mathcal{QK}_3$, given by

$$\begin{aligned}\mathcal{T}^\beta_{XYZ} &= \langle X,Y\rangle\beta(Z) - \langle X,Z\rangle\beta(Y)\\ &\quad + \sum_{a=1}^{3}\big(\langle X,J_aY\rangle\beta(J_aZ) - \langle X,J_aZ\rangle\beta(J_aY)\big),\end{aligned}$$

where β stands for the 1-form (4.4). Then $\mathcal{T} - \mathcal{T}^\beta \in \mathcal{QK}_{45}$, so that we have $\Psi(\mathcal{T}-\mathcal{T}^\beta)_{XYZ} = \Psi(\mathcal{T})_{XYZ} - 2\,\mathcal{T}^\beta_{XYZ}$. Taking then for instance the vectors $X = Y = G_0, Z = H_0$, we get

$$(\mathcal{T}-\mathcal{T}^\beta)_{G_0G_0H_0} = \frac{6+p+q}{2\big(9+2(p+q)\big)}, \quad \Psi(\mathcal{T}-\mathcal{T}^\beta)_{G_0G_0H_0} = -\frac{21+5(p+q)}{9+2(p+q)},$$

hence $\mathcal{T}-\mathcal{T}^\beta \in \mathcal{QK}_{45} \setminus \{\mathcal{QK}_4 \cup \mathcal{QK}_5\}$ for all $0 \le p \le q$. That is, S has, for all $0 \le p \le q$, a nonzero component in each basic type. ∎

As the simplest examples, consider the $4(4+q)$-dimensional spaces $\mathcal{W}(0,q) \cong \mathrm{SO}_0(4+q,4)/\big(\mathrm{SO}(4+q)\times\mathrm{SO}(4)\big)$, $q \ge 0$, which (cf. [Co, Table 1]) are the Alekseevskian $\mathcal{W}$-spaces which are symmetric. As such, they admit the structure $S = 0$. Moreover, being solvable Lie groups with Lie algebra $\mathfrak{w}(0,q)$, they admit the corresponding structure given by Theorem 3.1, when $p = 0$.

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References

[A] D. V. Alekseevsky, *Classification of quaternionic spaces with a transitive solvable group of motions*, Math. USSR Izv. 9 (1975), 297–339.

[AC] D. V. Alekseevsky and V. Cortés, *Homogeneous quaternionic Kähler manifolds of unimodular group*, Boll. Un. Mat. Ital. B (7) 11 (1997), no. 2, suppl., 217–219.

[AS] W. Ambrose and I. M. Singer, *On homogeneous Riemannian manifolds*, Duke Math. J. 25 (1958), 647–669.

[BW] J. Bagger and E. Witten, *Matter couplings in $N = 2$ supergravity*, Nucl. Phys. B222 (1983), 1–10.

[BGO1] W. Batat, P. M. Gadea and J. A. Oubiña, *Homogeneous pseudo-Riemannian structures of linear type*, J. Geom. Phys. 61 (2011), 745–764.

[BGO2] —, —, —, *Homogeneous quaternionic Kähler structures on rank-three Alekseevsky spaces*, Publ. Math. Debrecen 78 (2011), 691–707.

[CGO1] M. Castrillón López, P. M. Gadea and J. A. Oubiña, *Homogeneous quaternionic Kähler structures on eight-dimensional non-compact quaternion-Kähler symmetric spaces*, Math. Phys. Anal. Geom. 12 (2009), 47–74.

[CGO2] —, —, —, *Homogeneous quaternionic Kähler structures on* 12*-dimensional Alekseevsky spaces*, J. Geom. Phys. 57 (2007), 2098–2113.

[CGS] M. Castrillón López, P. M. Gadea and A. F. Swann, *Homogeneous quaternionic Kähler structures and quaternionic hyperbolic space*, Transform. Groups 11 (2006), 575–608.

[Ce] S. Cecotti, *Homogeneous Kähler manifolds and T-algebras in $N = 2$ supergravity and superstrings*, Comm. Math. Phys. 124 (1989), 23–55.

[Co] V. Cortés, *Alekseevskian spaces*, Differential Geom. Appl. 6 (1996), 129–168.

[F] A. Fino, *Intrinsic torsion and weak holonomy*, Math. J. Toyama Univ. 21 (1998), 1–22.

[I] S. Ishihara, *Quaternion Kählerian manifolds*, J. Differential Geom. 9 (1974), 483–500.

[J] W. Jelonek, *Quaternionic-Kähler geometry and almost Kähler $\mathcal{A}$-manifolds*, Ann. Polon. Math. 75 (2000), 111–124.

[K] V. F. Kiričenko, *On homogeneous Riemannian spaces with an invariant structure tensor*, Dokl. Akad. Nauk SSSR 252 (1980), 291–293 (in Russian).

[TV] F. Tricerri and L. Vanhecke, *Homogeneous Structures on Riemannian Manifolds*, London Math. Soc. Lecture Note Ser. 83, Cambridge Univ. Press, 1983.

[V] G. E. Vîlcu, *3-submersions from QR-hypersurfaces of quaternionic Kähler manifolds*, Ann. Polon. Math. 98 (2010), 301–309.

[WP] B. de Wit and A. van Proeyen, *Special geometry, cubic polynomials and homogeneous quaternionic spaces*, Comm. Math. Phys. 149 (1992), 307–333.

[W] J. A. Wolf, *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J. Math. Mech. 14 (1965), 1033–1047.

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