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Opial's type inequalities on time scales and some applications

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Abstract. We prove some new Opial type inequalities on time scales and employ them to prove several results related to the spacing between consecutive zeros of a solution or between a zero of a solution and a zero of its derivative for second order dynamic equations on time scales. We also apply these inequalities to obtain a lower bound for the smallest eigenvalue of a Sturm–Liouville eigenvalue problem on time scales. The results contain as special cases some results obtained for second order differential equations, give some new results for difference equations and yield conditions for disfocality for second order dynamic equations on time scales.

1. Introduction. During the past decade a number of dynamic inequalities have been established by several authors which are motivated by some applications (see [5], [14], [20] and [22] and the references cited therein). The general idea is to prove a result for a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$, to avoid proving results twice, once on a continuous time scale which leads to a differential inequality and once again on a discrete time scale which leads to a difference inequality. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [13]), i.e., when $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t : t\in\mathbb{N}_0\}$ where $q>1$. A cover story article in New Scientist [21] discusses several possible applications of time scales. In this paper, we will assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}}:=[a,b]\cap\mathbb{T}$.

In 1960 Z. Opial [16] proved the inequality

$$\int_a^b |y(t)|\,|y'(t)|\,dt \le \frac{b-a}{4}\int_a^b |y'(t)|^2\,dt, \tag{1.1}$$

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where $y(a) = y(b) = 0$ with the best constant $(b-a)/4$. This inequality, known in the literature as Opial's inequality, is one of the most important and fundamental integral inequalities in the qualitative analysis of properties of solutions of differential equations. Since its discovery an enormous amount of related work has been done, and many papers with new proofs, generalizations, extensions and discrete analogues have appeared. For more details, we refer the reader to the book [3]. Since continuous and discrete inequalities of Opial's type are important in the analysis of qualitative properties of solutions of differential and difference equations, we believe that dynamic inequalities of Opial's type on time scales will play the same role in the analysis of qualitative properties of solutions of dynamic equations. For different types of dynamic inequalities on time scales, we refer the reader to [1, 17, 18, 19] and the references cited therein. In [5] the authors extended (1.1) to time scales and proved that if $y : [0,b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(0) = 0$, then

$$\int_0^b |y(t) + y^\sigma(t)|\, |y^\Delta(t)|\, \Delta t \le b \int_0^b |y^\Delta(t)|^2\, \Delta t. \tag{1.2}$$

They also proved that if r and q are positive rd-continuous functions on $[0,b]_{\mathbb{T}}$, $\int_0^b (\Delta t/r(t)) < \infty$, q is nonincreasing and $y : [0,b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(0) = 0$, then

$$\int_0^b q^\sigma(t)|(y(t) + y^\sigma(t))y^\Delta(t)|\, \Delta t \le \int_0^b \frac{\Delta t}{r(t)} \int_0^b r(t)q(t)|y^\Delta(t)|^2\, \Delta t. \tag{1.3}$$

A function $g : \mathbb{T} \to \mathbb{R}$ is said to be *right-dense continuous* (rd-continuous) provided g is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T})$. Recently Karpuz, Kaymakçalan and Öcalan [14] proved an inequality similar to (1.3),

$$\int_a^b q(t)|(y(t) + y^\sigma(t))y^\Delta(t)|\, \Delta t \le K_q(a,b) \int_a^b |y^\Delta(t)|^2\, \Delta t, \tag{1.4}$$

where $q \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$ and $y : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = 0$, and

$$K_q(a,b) = \Big(2 \int_a^b q^2(u)(\sigma(u) - a)\Delta u\Big)^{1/2}. \tag{1.5}$$

They applied (1.4) to the second order dynamic equation

$$y^{\Delta\Delta}(t) + q(t)y^\sigma(t) = 0, \tag{1.6}$$

and proved that if $y(t)$ is a solution of (1.6) with $y(a) = y^{\Delta_\sigma}(b) = 0$, then

$$(1.7)\quad \Big(2\int_a^{\sigma(b)} Q^2(u)(\sigma(u)-a)\,\Delta u\Big)^{1/2} \geq 1, \quad \text{where} \quad Q(u) = \int_u^{\sigma(b)} q(t)\,\Delta t.$$

One can easily see that the inequalities of Opial's type established in [5] and [14] cannot be applied to the general equation

$$(1.8)\qquad (r(t)y^\Delta(t))^\Delta + q(t)y^\sigma(t) = 0, \qquad t \in [\alpha,\beta]_{\mathbb{T}},$$

on an arbitrary time scale $\mathbb{T}$, where r is a positive rd-continuous function, q is an rd-continuous function and

$$(1.9)\qquad \int_\alpha^\beta (1/r(t))\,\Delta t < \infty \quad \text{and} \quad \int_\alpha^\beta |q(t)|\,\Delta t < \infty.$$

Our aim in this paper is to prove some new dynamic inequalities of Opial's type on time scales and apply these inequalities to prove several results related to the following problems:

(i) obtain lower bounds for the spacing $\beta - \alpha$ where y is a solution of (1.8) satisfying $y(\alpha) = y^\Delta(\beta) = 0$ or $y^\Delta(\alpha) = y(\beta) = 0$,

(ii) obtain lower bounds for the spacing of generalized zeros of a solution of (1.8), and

(iii) obtain a lower bound for the smallest eigenvalue of the Sturm–Liouville eigenvalue problem

$$-y^{\Delta\Delta}(t) + q(t)y^\sigma(t) = \lambda y^\sigma(t), \qquad y(\alpha) = y(\beta) = 0.$$

By a *solution* of (1.8) on an interval $\mathbb{I}$, we mean a nontrivial real-valued function $y \in C_{\mathrm{rd}}(\mathbb{I})$ which has the property that $r(t)y^\Delta(t) \in C^1_{\mathrm{rd}}(\mathbb{I})$ and satisfies equation (1.8) on $\mathbb{I}$. We say that a solution y of (1.8) has a *generalized zero at* t if $y(t) = 0$, and has a *generalized zero in* $(t, \sigma(t))$ in case $y(t)y^\sigma(t) < 0$ and $\mu(t) > 0$. Equation (1.8) is *disconjugate* on $[t_0, b]_{\mathbb{T}}$ if there is no nontrivial solution of (1.8) with two (or more) generalized zeros in $[t_0, b]_{\mathbb{T}}$. Equation (1.8) is said to be *nonoscillatory* on $[t_0,\infty)_{\mathbb{T}}$ if there exists $c \in [t_0,\infty)_{\mathbb{T}}$ such that the equation is disconjugate on $[c, d]_{\mathbb{T}}$ for every $d > c$. In the opposite case, (1.8) is said to be *oscillatory* on $[t_0,\infty)_{\mathbb{T}}$. Oscillation of solutions of (1.8) may equivalently be defined as follows: A nontrivial solution $y(t)$ of (1.8) is called oscillatory if it has infinitely many (isolated) generalized zeros in $[t_0,\infty)_{\mathbb{T}}$; otherwise it is called nonoscillatory. So a solution $y(t)$ of (1.8) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. This means that the property of oscillation or nonoscillation concerns the behavior in the neighborhood of the infinite points. We say that (1.8) is *right disfocal* (resp. *left disfocal*) on $[\alpha,\beta]_{\mathbb{T}}$ if the solutions of (1.8) such that $y^\Delta(\alpha) = 0$ (resp. $y^\Delta(\beta) = 0$) have no generalized zeros in $[\alpha,\beta]_{\mathbb{T}}$.

We will frequently use the following notions and results due to Hilger [10]. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The *forward jump operator* and the *backward jump operator* are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, where $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$, is said to be *left-dense* if $\rho(t) = t$ and $t > \inf \mathbb{T}$, it is *right-dense* if $\sigma(t) = t$, it is *left-scattered* if $\rho(t) < t$ and *right-scattered* if $\sigma(t) > t$. The *graininess function* μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ stands for $f(\sigma(t))$. Fix $t \in \mathbb{T}$ and let $y : \mathbb{T} \to \mathbb{R}$. Define $y^\Delta(t)$ to be the number (if it exists) with the property that given any $\epsilon > 0$ there is a neighborhood U of t with

$$|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

In this case, we say that $y^\Delta(t)$ is the (*delta*) *derivative* of y at t and that y is (*delta*) *differentiable* at t. Assume that $g : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$.

- If g is differentiable at t, then g is continuous at t.
- If g is continuous at t and t is right-scattered, then g is differentiable at t with
$$g^\Delta(t) = \frac{g(\sigma(t)) - g(t)}{\mu(t)}.$$
- If g is differentiable and t is right-dense, then
$$g^\Delta(t) = \lim_{s \to t} \frac{g(t) - g(s)}{t - s}.$$
- If g is differentiable at t, then $g(\sigma(t)) = g(t) + \mu(t) g^\Delta(t)$.

We will refer to the (delta) integral which we can define as follows: If $G^\Delta(t) = g(t)$, then the *Cauchy* (*delta*) *integral* of g is defined by $\int_a^t g(s)\,\Delta s := G(t) - G(a)$. It can be shown (see [6]) that if $g \in C_{\mathrm{rd}}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\,\Delta s$ exists for all $t_0 \in \mathbb{T}$, and satisfies $G^\Delta(t) = g(t)$, $t \in \mathbb{T}$. The integration on discrete time scales is defined by $\int_a^b f(t)\,\Delta t = \sum_{t \in [a,b)} \mu(t) f(t)$. For more details of the analysis on time scales, we refer the reader to the two books by Bohner and Peterson [6, 7] which summarize and organize much of the time scale calculus.

The rest of the paper is organized as follows: In Section 2, we will prove some new inequalities of Opial's type by making use of the Cauchy–Schwarz inequality ([6, Theorem 5.15]) on time scales and a simple consequence of Keller's chain rule. In Section 3, we will apply these inequalities to prove several results related to problems (i)–(iii) above. In particular, we obtain some results different from those in [14]. Also when $\mathbb{T} = \mathbb{R}$ some of our results reduce to those of Harris and Kong [9] and Brown and Hinton [8]; when $\mathbb{T} = \mathbb{N}$ our results are new for second order difference equations. Of particular interest in this paper is the case when q is oscillatory.

2. Opial's type inequalities. In this section, we prove some new inequalities of Opial's type on time scales. This will be done by making use of the Cauchy–Schwarz inequality ([6, Theorem 5.15])

$$\int_a^b |f(t)g(t)|\,\Delta t \le \Big[\int_a^b |f(t)|^2\,\Delta t\Big]^{1/2}\Big[\int_a^b |g(t)|^2\,\Delta t\Big]^{1/2}, \tag{2.1}$$

where a, $b \in \mathbb{T}$ and $f, g \in C_{\rm rd}(\mathbb{I}, \mathbb{R})$ and the formula

$$(y^\beta(t))^\Delta = \beta\int_0^1 [hy^\sigma(t) + (1-h)y(t)]^{\beta-1}\,dh\,y^\Delta(t) \quad \text{for } \beta > 0, \tag{2.2}$$

which is a simple consequence of Keller's chain rule [6, Theorem 1.90].

Theorem 1. *Let $\mathbb{T}$ be a time scale with $a, X \in \mathbb{T}$. Assume that $s \in C_{\rm rd}([a, X]_{\mathbb{T}}, \mathbb{R})$ and r is a positive rd-continuous function on $(a, X)_{\mathbb{T}}$ such that $\int_a^X r^{-1}(t)\,\Delta t < \infty$. If $y : [a, X]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = 0$ and y^Δ does not change sign in $(a, X)_{\mathbb{T}}$, then*

$$\int_a^X s(x)|y(x) + y^\sigma(x)|\,|y^\Delta(x)|\,\Delta x \le K_1(a, X)\int_a^X r(x)|y^\Delta(x)|^2\,\Delta x, \tag{2.3}$$

where

$$K_1(a, X) = \sqrt{2}\bigg(\int_a^X \frac{s^2(x)}{r(x)}\Big(\int_a^x \frac{\Delta t}{r(t)}\Big)\,\Delta x\bigg)^{1/2} + \sup_{a\le x\le X}\mu(x)\frac{|s(x)|}{r(x)}. \tag{2.4}$$

Proof. Since $y^\Delta(t)$ does not change sign in $(a, X)_{\mathbb{T}}$, we have

$$|y(x)| = \int_a^x |y^\Delta(t)|\,\Delta t \quad \text{for } x \in [a, X]_{\mathbb{T}}.$$

This implies that

$$|y(x)| = \int_a^x \frac{1}{\sqrt{r(t)}}\sqrt{r(t)}\,|y^\Delta(t)|\,\Delta t.$$

It follows by the Cauchy–Schwarz inequality (2.1) with

$$f(t) = \frac{1}{\sqrt{r(t)}}, \qquad g(t) = \sqrt{r(t)}|y^\Delta(t)|$$

that

$$\int_a^x |y^\Delta(t)|\,\Delta t \le \Big(\int_a^x \frac{1}{r(t)}\,\Delta t\Big)^{1/2}\Big(\int_a^x r(t)|y^\Delta(t)|^2\,\Delta t\Big)^{1/2}.$$

Then, for $a \le x \le X$ (noting $y(a) = 0$), we get

$$|y(x)| \le \Big(\int_a^x \frac{1}{r(t)}\,\Delta t\Big)^{1/2}\Big(\int_a^x r(t)|y^\Delta(t)|^2\,\Delta t\Big)^{1/2}. \tag{2.5}$$

Since $y^\sigma = y + \mu y^\Delta$, we have

$$y(x) + y^\sigma(x) = 2y(x) + \mu y^\Delta(x). \tag{2.6}$$

Setting

$$z(x) := \int_a^x r(t)|y^\Delta(t)|^2 \, \Delta t, \tag{2.7}$$

we see that $z(a) = 0$, and

$$z^\Delta(x) = r(x)|y^\Delta(x)|^2 > 0. \tag{2.8}$$

From this, we get

$$|y^\Delta(x)|^2 = \frac{z^\Delta(x)}{r(x)} \quad \text{and} \quad |y^\Delta(x)| = \left(\frac{z^\Delta(x)}{r(x)}\right)^{1/2}. \tag{2.9}$$

From (2.5)–(2.9), we have

$$\begin{aligned} s(x)&|y(x) + y^\sigma(x)|\,|y^\Delta(x)| \\ &\le 2|s(x)|\,|y(x)|\,|y^\Delta(x)| + \mu(x)s(x)|y^\Delta|^2 \\ &\le 2|s(x)|\left(\frac{1}{r(x)}\right)^{1/2}\left(\int_a^x \frac{1}{r(t)}\,\Delta t\right)^{1/2}(z(x))^{1/2}(z^\Delta(x))^{1/2} \\ &\quad + \mu(x)|s(x)|\frac{z^\Delta(x)}{r(x)}. \end{aligned}$$

This implies that

$$\begin{aligned} &\int_a^X s(x)|y(x) + y^\sigma(x)|\,|y^\Delta(x)|\,\Delta x \\ &\le 2\int_a^X |s(x)|\left(\frac{1}{r(x)}\right)^{1/2}\left(\int_a^x \frac{1}{r(t)}\,\Delta t\right)^{1/2}(z(x))^{1/2}(z^\Delta(x))^{1/2}\,\Delta x \\ &\quad + \int_a^X \mu(x)\frac{|s(x)|}{r(x)} z^\Delta(x)\,\Delta x \\ &\le 2\int_a^X |s(x)|\left(\frac{1}{r(x)}\right)^{1/2}\left(\int_a^x \frac{1}{r(t)}\,\Delta t\right)^{1/2}(z(x))^{1/2}(z^\Delta(x))^{1/2}\,\Delta x \\ &\quad + \max_{a\le x\le X}\left(\mu(x)\frac{|s(x)|}{r(x)}\right)\int_a^X z^\Delta(x)\,\Delta x. \end{aligned} \tag{2.10}$$

Supposing that the integrals in (2.10) exist and again applying the Cauchy–Schwarz inequality (2.1) to the first integral on the right hand side, we have

$$\int_a^X s(x)|y(x)+y^\sigma(x)|\,|y^\Delta(x)|\,\Delta x \le 2\Big(\int_a^X s^2(x)\frac{1}{r(x)}\Big(\int_a^x \frac{1}{r(t)}\,\Delta t\Big)\,\Delta x\Big)^{1/2}\Big(\int_a^X z(x)z^\Delta(x)\,\Delta x\Big)^{1/2} + \sup_{a\le x\le X}\Big(\mu(x)\frac{|s(x)|}{r(x)}\Big)\int_a^X z^\Delta(x)\,\Delta x. \tag{2.11}$$

From (2.8), and the chain rule (2.2), we obtain

$$2z(x)z^\Delta(x) \le (z^2(x))^\Delta. \tag{2.12}$$

Substituting (2.12) into (2.11) and using the fact that $z(a)=0$, we see that

$$\begin{aligned}\int_a^X & s(x)|y(x)+y^\sigma(x)|\,|y^\Delta(x)|\,\Delta x\\ &\le 2\Big(\int_a^X s^2(x)\frac{1}{r(x)}\Big(\int_a^x \frac{1}{r(t)}\,\Delta t\Big)^2\Delta x\Big)^{1/2}\Big(\frac{1}{2}\Big)^{1/2}\Big(\int_a^X (z^2(t))^\Delta\,\Delta t\Big)^{1/2}\\ &= \sqrt{2}\Big(\int_a^X s^2(x)\frac{1}{r(x)}\Big(\int_a^x \frac{1}{r(t)}\,\Delta t\Big)\,\Delta x\Big)^{1/2} z(X) + \sup_{a\le x\le X}\Big(\mu(x)\frac{|s(x)|}{r(x)}\Big)z(X).\end{aligned}$$

Using (2.7), we obtain (2.3). ∎

We omit the proof of the following theorem, since it is similar to the proof of Theorem 1, with $[a,X]_{\mathbb{T}}$ replaced by $[b,X]_{\mathbb{T}}$ and $|y(x)| = \int_x^b |y^\Delta(t)|\,\Delta t$.

Theorem 2. *Let $\mathbb{T}$ be a time scale with $X, b\in\mathbb{T}$. Assume that $s\in C_{\mathrm{rd}}([a,X]_{\mathbb{T}},\mathbb{R})$ and r is a positive rd-continuous function on $(a,X)_{\mathbb{T}}$ such that $\int_a^X r^{-1}(t)\,\Delta t<\infty$. If $y:[X,b]_{\mathbb{T}}\to\mathbb{R}$ is delta differentiable with $y(b)=0$ and y^Δ does not change sign in $(X,b)_{\mathbb{T}}$, then*

$$\int_X^b s(x)|y(x)+y^\sigma(x)|\,|y^\Delta(x)|\,\Delta x \le K_2(X,b)\int_X^b r(x)|y^\Delta(x)|^2\,\Delta x, \tag{2.13}$$

where

$$K_2(X,b) = \sqrt{2}\Big(\int_X^b \frac{s^2(x)}{r(x)}\Big(\int_x^b \frac{\Delta t}{r(t)}\Big)\,\Delta x\Big)^{1/2} + \sup_{X\le x\le b}\mu(x)\frac{|s(x)|}{r(x)}. \tag{2.14}$$

In the following, we assume that there exists $h\in(a,b)$ which is the unique solution of the equation

$$K(a,b) = K_1(a,h) = K_2(h,b) < \infty, \tag{2.15}$$

where $K_1(a,h)$ and $K_2(h,b)$ are defined in Theorems 1 and 2, and establish a new inequality of Opial's type when $y(a) = 0 = y(b)$.

THEOREM 3. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$. Assume that $s \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$ and r is a positive rd-continuous function on $[a,b]_{\mathbb{T}}$ such that $\int_a^b r^{-1}(t)\,\Delta t < \infty$. If $y : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = 0 = y(b)$ and y^Δ does not change sign in $(a,b)_{\mathbb{T}}$, then*

$$\int_a^b s(x)|y(x) + y^\sigma(x)|\,|y^\Delta(x)|\,\Delta x \le K(a,b)\int_a^b r(x)|y^\Delta(x)|^2\,\Delta x, \tag{2.16}$$

where $K(a,b)$ is as in (2.15).

Proof. Since

$$\int_a^b s(x)|y(x) + y^\sigma(x)|\,|y^\Delta(x)|\,\Delta x = \int_a^X s(x)|y(x) + y^\sigma(x)|\,|y^\Delta(x)|\,\Delta x + \int_X^b s(x)|y(x) + y^\sigma(x)|\,|y^\Delta(x)|\,\Delta x,$$

the rest of the proof is a combination of Theorems 1 and 2; we omit the details. ∎

Setting $r = s$ in Theorems 1 and 2 give us the following results.

COROLLARY 4. *Let $\mathbb{T}$ be a time scale with $a, X \in \mathbb{T}$, and let r be a positive rd-continuous function on $[a,X]_{\mathbb{T}}$ such that $\int_a^X r^{-1}(t)\,\Delta t < \infty$. If $y : [a,X]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = 0$ and y^Δ does not change sign in $(a,X)_{\mathbb{T}}$, then*

$$\int_a^X r(x)|y(x) + y^\sigma(x)|\,|y^\Delta(x)|\,\Delta x \le K_1^*(a,X)\int_a^X r(x)|y^\Delta(x)|^2\,\Delta x, \tag{2.17}$$

where

$$K_1^*(a,X) = \sqrt{2}\bigg(\int_a^X r(x)\bigg(\int_a^x \frac{\Delta t}{r(t)}\bigg)\,\Delta x\bigg)^{1/2} + \sup_{a\le x\le X}\mu(x). \tag{2.18}$$

COROLLARY 5. *Let $\mathbb{T}$ be a time scale with $X, b \in \mathbb{T}$, and let r be a positive rd-continuous function on $(X,b)_{\mathbb{T}}$ such that $\int_X^b r^{-1}(t)\,\Delta t < \infty$. If $y : [X,b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(b) = 0$ and y^Δ does not change sign in $(X,b)_{\mathbb{T}}$, then*

$$\int_X^b r(x)|y(x) + y^\sigma(x)|\,|y^\Delta(x)|\,\Delta x \le K_2^*(X,b)\int_X^b r(x)|y^\Delta(x)|^2\,\Delta x, \tag{2.19}$$

where

$$K_2^*(X,b) = \sqrt{2}\Big(\int_X^b r(x)\Big(\int_x^b \frac{\Delta t}{r(t)}\Big)\,\Delta x\Big)^{1/2} + \sup_{X\le x\le b}\mu(x). \tag{2.20}$$

Now assume that there exists $h \in (a,b)$ which is the unique solution of the equation

$$K^*(a,b) = K_1^*(a,h) = K_2^*(h,b) < \infty,$$

where $K_1^*(a,h)$ and $K_2^*(h,b)$ are defined in Corollaries 4 and 5. Theorem 3 gives the following result when $r = s$.

Corollary 6. *Let $\mathbb{T}$ be a time scale with $a,b \in \mathbb{T}$ and let r be a positive rd-continuous function on $(a,b)_{\mathbb{T}}$ such that $\int_a^b r^{-1}(t)\,\Delta t < \infty$. If $y : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = 0 = y(b)$ and y^Δ does not change sign in $(a,b)_{\mathbb{T}}$, then*

$$\int_a^b r(x)|y(x) + y^\sigma(x)|\,|y^\Delta(x)|\,\Delta x \le K^*(a,b)\int_a^b r(x)|y^\Delta(x)|^2\,\Delta x. \tag{2.21}$$

On a time scale $\mathbb{T}$, we note from the chain rule (2.2) that

$$\begin{aligned}((t-a)^2)^\Delta &= 2\int_0^1 [h(\sigma(t)-a) + (1-h)(t-a)]\,dh\\ &\ge 2\int_0^1 [h(t-a) + (1-h)(t-a)]\,dh = 2(t-a).\end{aligned}$$

This implies that

$$\int_a^X (x-a)\,\Delta x \le \int_a^X \frac{1}{2}((x-a)^2)^\Delta\,\Delta x = \frac{(X-a)^2}{2}. \tag{2.22}$$

From this and (2.19) (by putting $r(t) = 1$), we get

$$\begin{aligned}K_1^*(a,X) &= \sqrt{2}\Big(\int_a^X (x-a)\,\Delta x\Big)^{1/2}\\ &\le \sqrt{2}\Big(\frac{(X-a)^2}{2}\Big)^{1/2} + \max_{a\le x\le X}\mu(x)\\ &= \max_{a\le x\le X}\mu(x) + (X-a).\end{aligned} \tag{2.23}$$

So by setting $r = 1$ in (2.17) and using (2.23), we have the following result.

Corollary 7. *Let $\mathbb{T}$ be a time scale with $a, X \in \mathbb{T}$. If $y : [a,X]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = 0$ and y^Δ does not change sign in $(a,X)_{\mathbb{T}}$,* *then*

$$\int_a^X |y(x)+y^\sigma(x)|\,|y^\Delta(x)|\,\Delta x \le \Big((X-a)+\sup_{a\le x\le X}\mu(x)\Big)\int_a^X |y^\Delta(x)|^2\,\Delta x. \tag{2.24}$$

Remark 1. One can put $r(t) = 1$ in Corollary 5 to obtain a result similar to Corollary 7. The details are left to the interested reader.

In Corollary 6, we note that if $r(t) = 1$, then the unique solution of (2.15) is $h = (a + b)/2$. This gives the following result.

Corollary 8. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$. If $y : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = 0 = y(b)$ and y^Δ does not change sign in $(a, b)_{\mathbb{T}}$, then*

$$\int_a^b |y(x) + y^\sigma(x)|\,|y^\Delta(x)|\,\Delta x \le \left(\frac{b-a}{2} + \sup_{a\le x\le b}\mu(x)\right)\int_a^b |y^\Delta(x)|^2\,\Delta x. \tag{2.25}$$

Remark 2. In Corollary 8 if $\mathbb{T} = \mathbb{R}$, then $\mu(x) = 0$, $\sigma(x) = x$, $y(x) = y^\sigma(x)$ and the inequality (2.25) reduces to the original Opial inequality (1.1).

3. Applications. In this section, we will apply the Opial inequalities proved in Section 2 to obtain some results related to problems (i)–(iii) above for equation (1.8).

Theorem 9. *Assume that r is a positive rd-continuous function on $(\alpha, \beta)_{\mathbb{T}}$ such that $\int_a^X r^{-1}(t)\,\Delta t < \infty$. Suppose y is a nontrivial solution of* (1.8)*. If $y(\alpha) = y^\Delta(\beta) = 0$, then*

$$\sqrt{2}\left(\int_\alpha^\beta \frac{Q^2(t)}{r(t)}\left(\int_\alpha^t \frac{\Delta u}{r(u)}\right)\Delta t\right)^{1/2} + \sup_{\alpha\le t\le\beta}\mu(t)\left|\frac{Q(t)}{r(t)}\right| \ge 1, \tag{3.1}$$

where $Q(t) = \int_t^\beta q(s)\,\Delta s$. If $y^\Delta(\alpha) = y(\beta) = 0$, then

$$\sqrt{2}\left(\int_\alpha^\beta \frac{Q^2(t)}{r(t)}\left(\int_t^\beta \frac{\Delta u}{r(u)}\right)\Delta t\right)^{1/2} + \sup_{\alpha\le t\le\beta}\mu(t)\left|\frac{Q(t)}{r(t)}\right| \ge 1, \tag{3.2}$$

where $Q(t) = \int_\alpha^t q(s)\,\Delta s$.

Proof. We prove (3.1). Multiplying (1.8) by y^σ and integrating by parts, we have

$$\begin{aligned}\int_\alpha^\beta y^\sigma(t)(r(t)y^\Delta(t))^\Delta\,\Delta t &= y(t)r(t)y^\Delta(t)\Big|_\alpha^\beta - \int_\alpha^\beta r(t)(y^\Delta(t))^2\,\Delta t\\ &= -\int_\alpha^\beta q(t)(y^\sigma(t))^2\,\Delta t.\end{aligned}$$

Using the assumptions $y(\alpha) = y^{\Delta}(\beta) = 0$ and $Q(t) = \int_t^{\beta} q(s)\,\Delta s$, we get

$$\int_{\alpha}^{\beta} r(t)(y^{\Delta}(t))^2\,\Delta t = \int_{\alpha}^{\beta} q(t)(y^{\sigma}(t))^2\,\Delta t = -\int_{\alpha}^{\beta} Q^{\Delta}(t)(y^{\sigma}(t))^{2}\,\Delta t.$$

Integrating the right hand side by parts and using the fact that $y(\alpha) = 0 = Q(\beta)$, we see that

$$\begin{aligned}\int_{\alpha}^{\beta} r(t)(y^{\Delta}(t))^2\,\Delta t &= \int_{\alpha}^{\beta} Q(t)(y(t)+y^{\sigma}(t))y^{\Delta}(t)\,\Delta t\\ &\le \int_{\alpha}^{\beta} |Q(t)|\,|y(t)+y^{\sigma}(t)|\,|y^{\Delta}(t)|\,\Delta t.\end{aligned}$$

Applying the inequality (2.3) with $s = Q$, we have

$$\begin{aligned}\int_{\alpha}^{\beta} r(t)(y^{\Delta}(t))^2\,\Delta t \le \Bigg[\sqrt{2}\bigg(\int_{\alpha}^{\beta} \frac{Q^2(t)}{r(t)}\bigg(\int_{\alpha}^{t}\frac{\Delta u}{r(u)}\bigg)\,\Delta t\bigg)^{1/2} + \sup_{\alpha\le t\le\beta}\mu(t)\bigg|\frac{Q(t)}{r(t)}\bigg|\Bigg]\\ \times \int_{\alpha}^{\beta} r(t)|y^{\Delta}(t)|^2\,\Delta t.\end{aligned}$$

This implies (3.1). The proof of (3.2) is similar by using Theorem 2 instead of Theorem 1. ∎

As a special case of Theorem 9, when $r(t) = 1$, we have the following results for equation (1.6), different from those results obtained by Karpuz, Kaymakçalan and Öcalan in [14].

Corollary 10. *Suppose y is a nontrivial solution of* (1.6). *If $y(\alpha) = y^{\Delta}(\beta) = 0$, then*

$$\sqrt{2}\Big(\int_{\alpha}^{\beta} Q^2(t)(t-\alpha)\,\Delta t\Big)^{1/2} + \sup_{\alpha\le t\le\beta}\mu(t)|Q(t)| \ge 1,$$

where $Q(t) = \int_t^{\beta} q(s)\Delta s$. If $y^{\Delta}(\alpha) = y(\beta) = 0$, then

$$\sqrt{2}\Big(\int_{\alpha}^{\beta} Q^2(t)(\beta-t)\,\Delta t\Big)^{1/2} + \sup_{\alpha\le t\le\beta}\mu(t)|Q(t)| \ge 1,$$

where $Q(t) = \int_{\alpha}^{t} q(s)\,\Delta s$.

Remark 3. Note that if $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ and equation (1.8) (when $r(t) = 1$) becomes

$$y''(t) + q(t)y(t) = 0. \tag{3.3}$$

In this case Corollary 10 reduces to the following result obtained by Brown and Hinton [8].

Corollary 11 ([8]). *If y is a solution of* (3.3) *such that* $y(\alpha) = y'(\beta) = 0$, *then*

$$2\int_{\alpha}^{\beta} Q^2(s)(s-\alpha)\,ds > 1, \tag{3.4}$$

where $Q(t) = \int_t^{\beta} q(s)\,ds$. *If instead* $y'(\alpha) = y(\beta) = 0$, *then*

$$2\int_{\alpha}^{\beta} Q^2(s)(\beta-s)\,ds > 1, \tag{3.5}$$

where $Q(t) = \int_{\alpha}^{t} q(s)\,ds$.

Remark 4. Note that if $\mathbb{T} = \mathbb{N}$, then $\mu(t) = 1$ and equation (1.8) (when $r(t) = 1$) becomes

$$\Delta^2 y(n) + q(n)y(n+1) = 0, \tag{3.6}$$

and Corollary 10 reduces to the following result.

Corollary 12. *If y is a solution of* (3.6) *such that* $y(\alpha) = \Delta y(\beta) = 0$, *then*

$$\sqrt{2}\Big(\sum_{n=\alpha+1}^{\beta-1} (Q(n))^2(n-\alpha)\Big)^{1/2} + \sup_{\alpha\le n\le\beta} |Q(n)| > 1,$$

where $Q(n) = \sum_{s=n}^{\beta-1} q(s)$. *If instead* $\Delta y(\alpha) = y(\beta) = 0$, *then*

$$\sqrt{2}\Big(\sum_{n=\alpha}^{\beta-1} (Q(n))^2(\beta-n)\Big)^{1/2} + \sup_{\alpha\le n\le\beta} |Q(n)| > 1,$$

where $Q(n) = \sum_{s=\alpha}^{n-1} q(s)$.

Theorem 13. *Assume that r is a positive rd-continuous function on* $(\alpha,\beta)_{\mathbb{T}}$ *such that* $\int_a^X r^{-1}(t)\,\Delta t < \infty$. *Suppose that y is a nontrivial solution of* (1.8). *If* $y(\alpha) = y^{\Delta}(\beta) = 0$, *then*

$$\sup_{\alpha\le t\le\beta}\left|\frac{Q(t)}{r(t)}\right|\left[\sqrt{2}\left(\int_{\alpha}^{\beta}\frac{1}{r(t)}\left(\int_{\alpha}^{t}\frac{\Delta u}{r(u)}\right)\Delta t\right)^{1/2} + \sup_{\alpha\le t\le\beta}\mu(t)\right] \ge 1, \tag{3.7}$$

where $Q(t) = \int_t^{\beta} q(s)\,\Delta s$. *If* $y^{\Delta}(\alpha) = y(\beta) = 0$, *then*

$$\sup_{\alpha\le t\le\beta}\left|\frac{Q(t)}{r(t)}\right|\left[\sqrt{2}\left(\int_{\alpha}^{\beta}\frac{1}{r(t)}\left(\int_{t}^{\beta}\frac{\Delta u}{r(u)}\right)\Delta t\right)^{1/2} + \sup_{\alpha\le t\le\beta}\mu(t)\right] \ge 1, \tag{3.8}$$

where $Q(t) = \int_{\alpha}^{t} q(s)\,\Delta s$.

Proof. We prove (3.7). Multiplying (1.8) by y^σ and integrating by parts as in the proof of Theorem 9 we get

$$\int_\alpha^\beta r(t)(y^\Delta(t))^2\,\Delta t = \int_\alpha^\beta q(t)(y^\sigma(t))^2\,\Delta t = -\int_\alpha^\beta Q^\Delta(t)(y^\sigma(t))^2\,\Delta t.$$

Integrating the right hand side by parts and using the fact that $y(\alpha) = 0 = Q(\beta)$, we see that

$$\begin{aligned}\int_\alpha^\beta r(t)(y^\Delta(t))^2\,\Delta t &\le \int_\alpha^\beta |Q(t)|\,|y(t)+y^\sigma(t)|\,|y^\Delta(t)|\,\Delta t\\ &\le \sup_{\alpha\le t\le\beta}\left|\frac{Q(t)}{r(t)}\right|\int_\alpha^\beta r(t)|y(t)+y^\sigma(t)|\,|y^\Delta(t)|\,\Delta t.\end{aligned}$$

Applying (2.17) with (2.18) and cancelling the term $\int_\alpha^\beta r(t)(y^\Delta(t))^2\,\Delta t$, we get (3.7). The proof of (3.8) is similar by using Corollary 5 instead of Corollary 4. ■

As a special case of Theorem 13, when $r(t) = 1$, we have the following result.

Corollary 14. *Suppose that y is a nontrivial solution of* (1.6). *If $y(\alpha) = y^\Delta(\beta) = 0$, then*

$$\sup_{\alpha\le t\le\beta}|Q(t)|\Big[(\beta-\alpha)+\sup_{\alpha\le t\le\beta}\mu(t)\Big]\ge 1,$$

where $Q(t) = \int_t^\beta q(s)\,\Delta s$. If $y^\Delta(\alpha) = y(\beta) = 0$, then

$$\sup_{\alpha\le t\le\beta}|Q(t)|\Big[(\beta-\alpha)+\sup_{\alpha\le t\le\beta}\mu(t)\Big]\ge 1,$$

where $Q(t) = \int_\alpha^t q(s)\,\Delta s$.

When $\mathbb{T} = \mathbb{R}$ we see that $\mu(t) = 0$, and as a special case of Corollary 14, we have the following result due to Harris and Kong [9] for the second order differential equation (3.3).

Corollary 15 ([9]). *Suppose that y is a nontrivial solution of* (3.3). *If $y(\alpha) = y'(\beta) = 0$, then*

$$(\beta-\alpha)\sup_{\alpha\le t\le\beta}\left|\int_t^\beta q(s)\,ds\right|\ge 1. \tag{3.9}$$

If $y'(\alpha) = y(\beta) = 0$, then

$$(\beta-\alpha)\sup_{\alpha\le t\le\beta}\left|\int_\alpha^t q(s)\,ds\right|\ge 1, \tag{3.10}$$

When $\mathbb{T} = \mathbb{N}$, we see that $\mu(t) = 1$ and as a special case of Corollary 14 we have the following result for the second order difference equation (3.6).

COROLLARY 16. *If y is a solution of* (3.6) *such that* $y(\alpha) = \Delta y(\beta) = 0$, *then*

$$\sup_{\alpha \le n \le \beta} |Q(n)|(\beta + 1 - \alpha) > 1,$$

where $Q(n) = \sum_{s=n}^{\beta-1} q(s)$. *If instead* $\Delta y(a) = y(b) = 0$, *then*

$$\sup_{\alpha \le n \le \beta} |Q(n)|(\beta + 1 - \alpha) > 1,$$

where $Q(n) = \sum_{s=\alpha}^{n-1} q(s)$.

REMARK 5. Note that application of (2.3) allows us to use an arbitrary anti-derivative Q in the above arguments.

REMARK 6. The above results yield sufficient conditions for disfocality of (1.8), i.e., conditions ensuring that there does not exist a nontrivial solution y satisfying either $y(\alpha) = y^{\Delta}(\beta) = 0$ or $y^{\Delta}(\alpha) = y(\beta) = 0$.

Our concern now is to determine a lower bound for the distance between consecutive generalized zeros of solutions of (1.8). Perhaps the best known existence result of this type for the dynamic equation (1.6) on a time scale $\mathbb{T}$ is due to Bohner et al. [4]. In particular they extended the classical Lyapunov inequality (see [15]) and proved that if $y(t)$ is a solution of (1.6) with $y(\alpha) = y(\beta) = 0$ $(\alpha < \beta)$ then

$$\int_{\alpha}^{\beta} q(t)\,\Delta t > \frac{4}{f(d)},$$

where $q(t)$ is a positive rd-continuous function defined on $\mathbb{T}$, $f(d) = (d - \alpha)(d - \beta)$, and d is the element of $\mathbb{T}$ closest to the midpoint of $[\alpha, \beta]$. As a particular case they derived that

$$\int_{\alpha}^{\beta} q(t)\,\Delta t > \frac{4}{\beta - \alpha}. \tag{3.11}$$

In the following, we assume that there exists a unique $h \in [\alpha, \beta]_{\mathbb{T}}$ such that

$$\int_{\alpha}^{h} \frac{\Delta t}{r(t)} = \int_{h}^{\beta} \frac{\Delta t}{r(t)} \quad \text{with} \quad \int_{\alpha}^{\beta} \frac{\Delta t}{r(t)} < \infty. \tag{3.12}$$

Note again that when $r(t) = 1$, we have $h - \alpha = \beta - h$. So the unique solution is $h = (\alpha + \beta)/2$.

THEOREM 17. *Assume that* (3.12) *holds and* $Q^{\Delta}(t) = q(t)$. *Suppose that y is a nontrivial solution of* (1.8) *and* $y^{\Delta}(t)$ *does not change sign in* $(\alpha, \beta)_{\mathbb{T}}$.

If $y(\alpha) = y(\beta) = 0$, *then*

$$\sqrt{2}\Big(\int_\alpha^\beta \frac{Q^2(t)}{r(t)}\Big(\int_\alpha^h \frac{\Delta u}{r(u)}\Big)\,\Delta t\Big)^{1/2} + \sup_{\alpha\le t\le\beta} \mu(t)\Big|\frac{Q(t)}{r(t)}\Big| \ge 1. \tag{3.13}$$

Proof. As in the proof of Theorem 13, by multiplying (1.8) by $y^\sigma(t)$, integrating by parts and using $y(\alpha) = y(\beta) = 0$, we have

$$\int_\alpha^\beta r(t)|y^\Delta(t)|^2\,dt \le \int_\alpha^\beta |Q(t)|\,|y(t) + y^\sigma(t)|^\gamma |y^\Delta(t)|\,dt. \tag{3.14}$$

Applying (2.16), we get

$$\int_\alpha^\beta r(t)|y^\Delta(t)|^2\,dt \le K(\alpha,\beta)\int_\alpha^\beta r(t)|y^\Delta(t)|^2\,dt,$$

where $K(\alpha,\beta)$ is as in (2.15). Cancelling $\int_\alpha^\beta r(t)|y^\Delta(t)|^2\,\Delta t$, we get (3.13). ■

As a special case of Theorem 17 when $r(t) = 1$ (note that in this case $h = (\alpha+\beta)/2)$), we have the following result for equation (1.6).

THEOREM 18. *Assume that* $Q^\Delta(t) = q(t)$. *Suppose that* y *is a nontrivial solution of* (1.6) *and* $y^\Delta(t)$ *does not change sign in* $(\alpha,\beta)_{\mathbb{T}}$. *If* $y(\alpha) = y(\beta) = 0$, *then*

$$\sqrt{\beta-\alpha}\Big(\int_\alpha^\beta Q^2(t)\,\Delta t\Big)^{1/2} + \sup_{\alpha\le t\le\beta} \mu(t)|Q(t)| \ge 1.$$

As special cases of Theorem 18, when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$, we have the following results for the second order differential equation (3.3) and second order difference equation (3.6).

COROLLARY 19. *Assume that* $Q'(t) = q(t)$. *Suppose that* y *is a nontrivial solution of* (3.3) *and* $y'(t)$ *does not change sign in* (α,β). *If* $y(\alpha) = y(\beta) = 0$, *then*

$$\int_\alpha^\beta \Big(\int_\alpha^t q(u)\,du\Big)^2 dt \ge \frac{1}{\beta-\alpha}. \tag{3.15}$$

COROLLARY 20. *Assume that* $\Delta Q(n) = q(n)$. *Suppose that* y *is a nontrivial solution of* (3.6) *and* $\Delta y(n)$ *does not change sign in* (α,β). *If* $y(\alpha) = y(\beta) = 0$, *then*

$$\sqrt{\beta-\alpha}\Big(\sum_{n=\alpha}^{n-1} Q^2(n)\Big)^{1/2} + \sup_{\alpha\le n\le\beta} |Q(n)| \ge 1.$$

REMARK 7. We mentioned here that inequality (3.15) is different from (3.11) and improve the inequality

$$\int_\alpha^\beta\Big(\int_\alpha^t q(u)\,du\Big)^2\,dt-\frac{1}{\beta-\alpha}\Big(\int_\alpha^\beta\Big(\int_\alpha^t q(u)\,du\Big)\,dt\Big)^2\ge\frac{\pi^2}{8(\beta-\alpha)},$$

obtained by Brown and Hinton [8, Corollary 4.2] for equation (3.3).

As an application, we will show how Opial and Wirtinger inequalities may be used to find a lower bound for the smallest eigenvalue of a Sturm–Liouville eigenvalue problem on a time scale $\mathbb{T}$. For more details on Sturm–Liouville problems, we refer the reader to [2]. Consider the Sturm–Liouville eigenvalue problem

$$-y^{\Delta\Delta}(t)+q(t)y^\sigma(t)=\lambda y^\sigma(t),\qquad y(0)=y(\beta)=0,\tag{3.16}$$

where q is an rd-continuous function and λ is a constant and assume that λ_0 is the smallest eigenvalue of (3.16). Our main aim is to find a lower bound of λ_0. To find it, we will apply the Opial type inequality (2.16) and a Wirtinger inequality due to Hilscher [11],

$$\int_\alpha^\beta\frac{M(t)M^\sigma(t)}{|M^\Delta(t)|}(y^\Delta(t))^2\,\Delta t\ge\frac{1}{\psi^2}\int_\alpha^\beta|M^\Delta(t)|(y^\sigma(t))^2\,\Delta t,\tag{3.17}$$

for a positive function $M\in C^1_{\mathrm{rd}}(\mathbb{I})$ with either $M^\Delta(t)>0$ or $M^\Delta(t)<0$ on $\mathbb{I}$, $y\in C^1_{\mathrm{rd}}(\mathbb{I})$ with $y(\alpha)=0=y(\beta)$, for $\mathbb{I}=[\alpha,\beta]_{\mathbb{T}}\subset\mathbb{T}$ and

$$\psi=\Big(\sup_{t\in\mathbb{I}^k}\frac{M(t)}{M^\sigma(t)}\Big)^{1/2}+\Big(\sup_{t\in\mathbb{I}^k}\frac{\mu(t)|M^\Delta(t)|}{M^\sigma(t)}+\sup_{t\in\mathbb{I}^k}\frac{M(t)}{M^\sigma(t)}\Big)^{1/2}.$$

We denote

$$A(Q)=\sqrt{\beta-\alpha}\Big(\int_\alpha^\beta Q^2(t)\,\Delta t\Big)^{1/2}+\sup_{\alpha\le t\le\beta}\mu(t)|Q(t)|.$$

THEOREM 21. *Assume that λ_0 is the smallest eigenvalue of* (3.16) *and $q(t)=Q^\Delta(t)+\gamma$, where $\gamma<\lambda_0$. Then*

$$|\lambda_0-\gamma|\ge\frac{1-A(Q)}{(1+\sqrt{2})^2\sigma^2(\beta)}.\tag{3.18}$$

Proof. Let $y(t)$ be an eigenfunction of (3.16) corresponding to λ_0. Multiplying (3.16) by $y^\sigma(t)$ and proceeding as in the proof of Theorem 9 we get

$$-\int_0^\beta y^{\Delta\Delta}y^\sigma(t)\,\Delta t+\int_0^\beta q(t)(y^\sigma(t))^2\,\Delta t=\lambda_0\int_0^\beta(y^\sigma(t))^2\,\Delta t.$$

This implies, after integrating by parts and using the fact that $y(0) = y(\beta) = 0$, that

$$\begin{aligned}(\lambda_0 - \gamma)\int_0^\beta (y^\sigma(t))^2\,\Delta t &= \int_0^\beta (y^\Delta(t))^2\,\Delta t + \int_0^\beta Q^\Delta(t)(y^\sigma(t))^2\,\Delta t\\ &= \int_0^\beta (y^\Delta(t))^2\,\Delta t - \int_0^\beta Q(t)[y(t)+y^\sigma(t)]y^\Delta(t)\,\Delta t\\ &\geq \int_0^\beta (y^\Delta(t))^2\,\Delta t - \int_0^\beta |Q(t)|\,|y(t)+y^\sigma(t)|\,|y^\Delta(t)|\,\Delta t.\end{aligned}$$

Proceeding as in the proof of Theorem 9, by applying (2.16) with $r(t) = 1$ and $s = Q$ to the term

$$\int_0^\beta |Q(t)|\,|y(t)+y^\sigma(t)|\,|y^\Delta(t)|\,\Delta t,$$

we obtain

$$|\lambda_0 - \gamma|\int_0^\beta (y^\sigma(t))^2\,\Delta t \geq \int_0^\beta (y^\Delta(t))^2\,\Delta t - A(Q)\int_0^\beta |y^\Delta(t)|^2\,\Delta t.$$

Now, applying Wirtinger's inequality (3.17) with $M(t) = t$, we have

$$|\lambda_0 - \gamma|\psi_1^2\sigma^2(\beta)\int_0^\beta (y^\Delta(t))^2\,\Delta t \geq \int_0^\beta (y^\Delta(t))^2\,\Delta t - A(Q)\int_0^\beta (y^\Delta(t))^2\,\Delta t,$$

where

$$\psi_1 = \left(\sup_{t\in\mathbb{I}^k}\frac{t}{\sigma(t)}\right)^{1/2} + \left(\sup_{t\in\mathbb{I}^k}\frac{\mu(t)}{\sigma(t)} + \sup_{t\in\mathbb{I}^k}\frac{t}{\sigma(t)}\right)^{1/2} \leq 1+\sqrt{2}.$$

This implies that

$$|\lambda_0 - \gamma|(1+\sqrt{2})^2\sigma^2(\beta) \geq 1 - A(Q). \ \blacksquare$$

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References

[1] R. P. Agarwal, M. Bohner, D. O'Regan and S. H. Saker, *Some Wirtinger-type inequalities on time scales and their applications*, Pacific J. Math. 252 (2011), 1–18.

[2] R. P. Agarwal, M. Bohner and P. J. Y. Wong, *Sturm–Liouville eigenvalue problems on time scales*, Appl. Math. Comput. 99 (1999), 153–166.

[3] R. P. Agarwal and P. Y. H. Pang, *Opial Inequalities with Applications in Differential and Difference Equations*, Kluwer, Dordrecht, 1995.
[4] M. Bohner, S. Clark and J. Ridenhour, *Lyapunov inequalities for time scales*, J. Inequal. Appl. 7 (2002), 61–77.
[5] M. Bohner and B. Kaymakçalan, *Opial inequalities on time scales*, Ann. Polon. Math. 77 (2001), 11–20.
[6] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.
[7] —, —, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.
[8] R. C. Brown and D. B. Hinton, *Opial's inequality and oscillation of 2nd order equations*, Proc. Amer. Math. Soc. 125 (1997), 1123–1129.
[9] B. J. Harris and Q. Kong, *On the oscillation of differential equations with an oscillatory coefficient*, Trans. Amer. Math. Soc. 347 (1995), 1831–1839.
[10] S. Hilger, *Analysis on measure chains—a unified approach to continuous and discrete calculus*, Results Math. 18 (1990), 18–56.
[11] R. Hilscher, *A time scale version of a Wirtinger type inequality and applications*, J. Comput. Appl. Math. 141 (2002), 219–226.
[12] H. L. Hong, W. C. Lian and C. C. Yeh, *The oscillation of half-linear differential equations with oscillatory coefficients*, Math. Comput. Modelling 24 (1996), 77–86.
[13] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, 2001.
[14] B. Karpuz, B. Kaymakçalan and Ö. Öcalan, *A generalization of Opial's inequality and applications to second-order dynamic equations*, Differential Equations Dynam. Systems 18 (2010), 11–18.
[15] A. M. Lyapunov, *Problème Général de la Stabilité du Movement*, Ann. of Math. Stud. 17, Princeton Univ. Press, Princeton, NJ, 1949.
[16] Z. Opial, *Sur une inegalité*, Ann. Polon. Math. 8 (1960), 29–32.
[17] S. H. Saker, *Some nonlinear dynamic inequalities on time scales and applications*, J. Math. Inequal. 4 (2010), 561–579.
[18] —, *Some nonlinear dynamic inequalities on time scales*, Math. Inequal. Appl. 14 (2011), 633–645.
[19] —, *Lyapunov inequalities for half-linear dynamic equations on time scales and disconjugacy*, Dynam. Contin. Discrete Impuls. Systems Ser. B Appl. Algorithms 18 (2001), 149–161.
[20] H. M. Srivastava, K. L. Tseng, S. J. Tseng and J. C. Lo, *Some weighted Opial-type inequalities on time scales*, Taiwanese J. Math. 14 (2010), 107–122.
[21] V. Spedding, *Taming nature's numbers*, New Scientist, July 19, 2003, 28–31.
[22] F. H. Wong, W. C. Lin, S. L. Yu and C. C. Yeh, *Some generalization of Opial's inequalities on time scales*, Taiwanese J. Math. 12 (2008), 463–471.

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