ANNALES
POLONICI MATHEMATICI
104.3 (2012)

Existence of three solutions for a class of $(p_1,\dots,p_n)$-biharmonic systems with Navier boundary conditions

by Shapour Heidarkhani (Kermanshah and Tehran), Yu Tian (Beijing) and Chun-Lei Tang (Chongqing)

Abstract. We establish the existence of at least three weak solutions for the $(p_1,\dots,p_n)$-biharmonic system

$$\begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i)=\lambda F_{u_i}(x,u_1,\dots,u_n) & \text{in } \Omega,\\ u_i=\Delta u_i=0 & \text{on } \partial\Omega, \end{cases}$$

for $1\le i\le n$. The proof is based on a recent three critical points theorem.

1. Introduction. In this work, we study the existence of at least three weak solutions for the nonlinear elliptic system of $(p_1,\dots,p_n)$-biharmonic type under Navier boundary conditions

$$\begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i)=\lambda F_{u_i}(x,u_1,\dots,u_n) & \text{in } \Omega,\\ u_i=\Delta u_i=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

for $1\le i\le n$, where $\Omega\subset\mathbb{R}^N$ $(N\ge 2)$ is a non-empty bounded open set with smooth boundary $\partial\Omega$, $p_i\ge 1$ for $1\le i\le n$, $\lambda>0$, $F:\overline{\Omega}\times\mathbb{R}^n\to\mathbb{R}$ is a function such that $F(\cdot,t_1,\dots,t_n)$ is continuous in $\overline{\Omega}$ for all $(t_1,\dots,t_n)\in\mathbb{R}^n$, $F(x,\cdot,\dots,\cdot)$ is C^1 in $\mathbb{R}^n$ for every $x\in\overline{\Omega}$ and $F(x,0,\dots,0)=0$ for all $x\in\Omega$; finally F_t denotes the partial derivative of F with respect to t.

Here and in the next section, X will denote the Cartesian product of n Sobolev spaces $W^{2,p_i}(\Omega)\cap W_0^{1,p_i}(\Omega)$ for $i=1,\dots,n$, i.e., $X=(W^{2,p_1}(\Omega)\cap W_0^{1,p_1}(\Omega))\times\dots\times(W^{2,p_n}(\Omega)\cap W_0^{1,p_n}(\Omega))$ endowed with the norm

$$\|(u_1,\dots,u_n)\|=\sum_{i=1}^n\|u_i\|_{p_i}$$

2010 *Mathematics Subject Classification*: Primary 35G60; Secondary 35B38.
Key words and phrases: three solutions, critical point, $(p_1,\dots,p_n)$-biharmonic, multiplicity results, Navier boundary value problem.

DOI: 10.4064/ap104-3-4

where

$$\|u_i\|_{p_i} = \Big(\int\limits_{\Omega} |\Delta u_i(x)|^{p_i}\, dx \Big)^{1/p_i}$$

for $1 \le i \le n$.

We say that $u = (u_1, \ldots, u_n)$ is a *weak solution* to the system (1.1) if $u = (u_1, \ldots, u_n) \in X$ and

$$\int\limits_{\Omega} \sum_{i=1}^{n} |\Delta u_i(x)|^{p_i-2} \Delta u_i(x) \Delta v_i(x)\, dx - \lambda \int\limits_{\Omega} \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \ldots, u_n(x)) v_i(x)\, dx = 0$$

for every $(v_1, \ldots, v_n) \in X$.

There seems to be increasing interest in studying fourth-order boundary value problems, especially because the static form change of beam or the support of a rigid body can be described by a fourth-order equation, and a model to study travelling waves in suspension bridges involves a fourth-order nonlinear equation (for instance, see [15]), so this subject is important to physics. More general nonlinear fourth-order elliptic boundary value problems have been studied [1, 4, 5, 7, 8, 10–13, 16-20, 22, 26, 28] in recent years. In [12], using variational methods and under a suitable set of assumptions involving two parameters α and β (for instance, $\alpha^2 - 4\beta > 0$) the authors obtained two nontrivial solutions to the problem

$$\begin{cases} u^{\mathrm{iv}} + \alpha u'' + \beta u = f(x, u), & x \in (0, 1), \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0 \end{cases} \tag{1.2}$$

where α, β are real constants and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. In [13], applying Morse theory, the existence of three solutions to problem (1.2) with $\alpha = \beta = 0$ was proved. In [17], by also using the fixed-point index in cones and under the assumption $\alpha^2 - 4\beta = 0$, multiple solutions to the problem

$$\begin{cases} u^{\mathrm{iv}} + \alpha u'' + \beta u = \lambda f(x, u), & x \in (0, 1), \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0 \end{cases} \tag{1.3}$$

were obtained, where α, β are real constants, λ is a positive parameter and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, while in [7] the authors established multiple solutions for problem (1.3) by using a three critical points theorem (see Theorem 2.1) established in [3]. In [1] based on Ricceri's three critical points theorem [23] the existence of at least three (weak) solutions of the fourth-order boundary value problem

$$\begin{cases} u^{\mathrm{iv}} + \alpha u'' + \beta u = \lambda f(x,u) + \mu g(x,u), & x \in (0,1) \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{cases}$$

was considered where α, β are real constants, $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are L^2-Carathéodory functions and $\lambda, \mu > 0$. In [18], the authors studied the following superlinear p-biharmonic elliptic problem with Navier boundary condition:

$$\text{(1.4)} \qquad \begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$, $n \geq 2p+1$, $p > 1$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. By means of Morse theory, the authors proved the existence of a nontrivial solution to problem (1.4) having a linking structure around the origin. Moreover, in the case of both resonance near zero and nonresonance at $+\infty$, the existence of two nontrivial solutions was shown. Very recently, Li and Tang [19], employing Ricceri's three critical points theorem [23] investigated the problem

$$\text{(1.5)} \qquad \begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a non-empty bounded open set with C^1 boundary, $p > \max\{1, N/2\}$, $\lambda, \mu > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function, establishing the existence of an open interval $\Lambda \subseteq [0, \infty[$ and a number $q > 0$ with the following property: for every $\lambda \in \Lambda$ and every Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying

$$\sup_{|t| \leq \zeta} |g(\cdot, t)| \in L^1(\Omega)$$

for all $\zeta > 0$, there is a $\delta > 0$ such that, for each $\mu \in [0, \delta]$ equation (1.5) admits at least three weak solutions in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ whose norms are less than q; Li and Tang in [20] generalized these results to the system

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda F_u(x,u,v) + \mu G_u(x,u,v) & \text{in } \Omega, \\ \Delta(|\Delta v|^{q-2}\Delta v) = \lambda F_v(x,u,v) + \mu G_v(x,u,v) & \text{in } \Omega, \\ u = \Delta u = v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a non-empty bounded open set with smooth boundary, $p > \max\{1, N/2\}$, $q > \max\{1, N/2\}$, $\lambda, \mu > 0$, $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that $F(\cdot, t_1, t_2)$ is continuous in Ω for all $(t_1, t_2) \in \mathbb{R}^2$, $F(x, \cdot, \cdot)$ is C^1 in $\mathbb{R}^2$ for every $x \in \Omega$ and $F(x, 0, 0) = 0$ for all $x \in \Omega$, and

$G : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a measurable function with respect to x in Ω for every $(t_1, t_2) \in \mathbb{R}^2$, and is a C^1-function of $(t_1, t_2) \in \mathbb{R}^2$ for every x in Ω.

We point out that our results extend in several directions the previous work of [1], [16], [20] and [19] by relaxing some hypotheses and sharpening the conclusion. The applicability of our results is illustrated by an example.

2. Main results. Our analysis is based on the following three critical points theorem (see also [24], [23], [6] and [21] for related results), which transfers the existence of three solutions of the system (1.1) into the existence of critical points of the Euler functional.

THEOREM 2.1 (see [9, Theorem 3.6]). *Let X be a reflexive real Banach space, let $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exist $r \in \mathbb{R}$ and $w \in X$ with $0 < r < \Phi(w)$ such that*

(i) $\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) < r\dfrac{\Psi(w)}{\Phi(w)}$,

(ii) *for each λ in*

$$\Lambda_r := \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}\right[$$

the functional $\Phi - \lambda\Psi$ is coercive.

Then for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

We need the following proposition in the proof of Theorem 2.3.

PROPOSITION 2.2. *Let X be as in the introduction and $T : X \to X^*$ be the operator defined by*

$$T(u_1, \ldots, u_n)(h_1, \ldots, h_n) = \int_\Omega \sum_{i=1}^{n} |\Delta u_i(x)|^{p_i - 2} \Delta u_i(x) \Delta h_i(x)\, dx$$

for every $(u_1, \ldots, u_n), (h_1, \ldots, h_n) \in X$. Then T admits a continuous inverse on X^*.

Proof. Taking into account (2.2) of [25] for $p_i \geq 2$ there exists a positive constant c_{p_i} such that

$$\langle |x|^{p_i-2}x - |y|^{p_i-2}y, x - y\rangle \geq c_{p_i}|x - y|^{p_i}$$

for every $x, y \in \mathbb{R}^N$ where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^N$.

Thus, it is easy to see that

$$(T(u_1,\dots,u_n)-T(v_1,\dots,v_n))(u_1-v_1,\dots,u_n-v_n) \geq \min\{c_{p_1},\dots,c_{p_n}\}\sum_{i=1}^{n}\|u_i-v_i\|_{p_i}^{p_i}$$

for every $(u_1,\dots,u_n),(v_1,\dots,v_n)\in X$, which means that T is strongly monotone. Therefore, since T is coercive and hemicontinuous in X (for more details, see [20]), by applying Theorem 26.A of [27], we conclude that T admits a continuous inverse on X^*. ∎

Let us recall that for $1\le i\le n$, $W_0^{1,p_i}(\Omega)$ is compactly embedded in

$$\begin{aligned} &L^{q_i}(\Omega) \text{ for all } q_i\in[p_i,p_iN/(N-p_i)[\text{ if } p_i<N,\\ &L^{q_i}(\Omega) \text{ for all } q_i>1 \text{ if } p_i=N, \\ &C^0(\overline{\Omega}) \text{ if } p_i>N,\end{aligned}\tag{2.1}$$

and that for $1\le i\le n$, $W^{2,p_i}(\Omega)$ is compactly embedded in

$$\begin{aligned} &L^{p_i^*}(\Omega) \text{ for all } p_i^*\in[p_i,p_iN/(N-2p_i)[\text{ if } p_i<N/2,\\ &L^{r_i}(\Omega) \text{ for all } r_i>p_i \text{ if } 2p_i=N, \\ &C^0(\overline{\Omega}) \text{ if } p_i>\max\{1,N/2\}.\end{aligned}\tag{2.2}$$

So, if $p_i>\max\{1,N/2\}$ for $1\le i\le n$ ($N=1$ is included in this case) the embedding $W^{2,p_i}(\Omega)\cap W_0^{1,p_i}(\Omega)\hookrightarrow C^0(\overline{\Omega})$ is compact, and if $p_i\le N/2$ for $1\le i\le n$, the embedding $W^{2,p_i}(\Omega)\cap W_0^{1,p_i}(\Omega)\hookrightarrow L^{q_i}\Omega)$ is compact for all $q_i\in[p_i,p_iN/(N-2p_i)[$.

Put

$$k=\max\left\{\sup_{u_i\in W^{2,p_i}(\Omega)\cap W_0^{1,p_i}(\Omega)\setminus\{0\}}\frac{\max_{x\in\overline{\Omega}}|u_i(x)|^{p_i}}{\|u_i\|_{p_i}^{p_i}} : 1\le i\le n\right\}.\tag{2.3}$$

In the case $p_i>\max\{1,N/2\}$ for $1\le i\le n$, since the embedding $W^{2,p_i}(\Omega)\cap W_0^{1,p_i}(\Omega)\hookrightarrow C^0(\overline{\Omega})$ for $1\le i\le n$ is compact, one has $k<\infty$.

For all $\gamma>0$ we define

$$K(\gamma)=\left\{t=(t_1,\dots,t_n)\in\mathbb{R}^n : \sum_{i=1}^{n}\frac{|t_i|^{p_i}}{p_i}\le\gamma\right\}.\tag{2.4}$$

Now, we state our main result.

THEOREM 2.3. *Assume that there exist a positive constant r and two elements $w=(w_1,\dots,w_n)$ and $\overline{w}=(\overline{w}_1,\dots,\overline{w}_n)$ in X such that*

(A1) $\sum_{i=1}^{n}\frac{\|w_i\|_{p_i}^{p_i}}{p_i}>r$;

(A2) *if* $p_i > \max\{1, N/2\}$ *for* $1 \le i \le n$, *then*

$$\int\limits_{\Omega} \sup_{t\in K(kr)} F(x,t)\,dx < \Big(r\prod_{i=1}^{n} p_i\Big)\frac{\int_\Omega F(x,w(x))\,dx}{\sum_{i=1}^{n}\prod_{j=1,\,j\neq i}^{n} p_j\|w_i\|_{p_i}^{p_i}} \tag{2.5}$$

where k *is given by* (2.3), *and*

$$\limsup_{|t_1|+\cdots+|t_n|\to\infty} \frac{F(x,t)}{\sum_{i=1}^{n}|t_i|^{p_i}/p_i} < \frac{\int_\Omega \sup_{t\in K(kr)} F(x,t)\,dx}{m(\Omega)kr} \tag{2.6}$$

uniformly with respect to $x \in \Omega$ *where* $m(\cdot)$ *is Lebesgue measure;*

(A3) *if* $p_i \le \max\{1, N/2\}$ *for* $1 \le i \le n$, *then there exist positive constants* b_1, θ *and* s *with* $p_i < \theta < p_iN/(N-2p_i)$ *and* $s < p_i$ *for* $1 \le i \le n$ *satisfying*

$$|F(x,t)| \le b_1\Big(1+\sum_{i=1}^{n}|t_i|^s\Big) \quad \forall t_i\in\mathbb{R}, \tag{2.7}$$

$$\limsup_{\sum_{i=1}^{n}|t_i|\to 0} \frac{|F(x,t)|}{\sum_{i=1}^{n}|t_i|^{\theta}} < \infty, \tag{2.8}$$

$$\int\limits_{\Omega} F(x,\overline{w}(x))\,dx > 0. \tag{2.9}$$

Then, for each λ *in*

$$\Lambda_r := \begin{cases} \left]\dfrac{\sum_{i=1}^{n}\|w_i\|_{p_i}^{p_i}/p_i}{\int_\Omega F(x,w(x))\,dx}, \dfrac{r}{\int_\Omega \sup_{t\in K(kr)} F(x,t)\,dx}\right[& \text{if } p_i > \max\{1,N/2\}, \\ \left]\dfrac{\sum_{i=1}^{n}\|\overline{w}_i\|_{p_i}^{p_i}/p_i}{\int_\Omega F(x,\overline{w}(x))\,dx}, \dfrac{r}{\int_\Omega \sup_{\sum_{i=1}^{n}\|u_i\|_{p_i}^{p_i}/p_i\le r} F(x,u(x))\,dx}\right[& \text{if } p_i \le \max\{1,N/2\} \end{cases}$$

the system (1.1) *admits at least three distinct weak solutions in* X.

Proof. In order to apply Theorem 2.1 to our problem, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ for each $u = (u_1,\ldots,u_n) \in X$ as follows:

$$\Phi(u) = \sum_{i=1}^{n}\frac{\|u_i\|_{p_i}^{p_i}}{p_i} \tag{2.10}$$

and

$$\Psi(u) = \int\limits_{\Omega} F(x,u_1(x),\ldots,u_n(x))\,dx. \tag{2.11}$$

It is well known that Φ and Ψ are well defined and continuously differentiable functionals whose derivatives at the point $u = (u_1,\ldots,u_n) \in X$ are the functionals $\Phi'(u),\Psi'(u)\in X^*$ given by

$$\Phi'(u)(v)=\int_\Omega\sum_{i=1}^n|\Delta u_i(x)|^{p_i-2}\Delta u_i(x)\Delta v_i(x)\,dx,$$

$$\Psi'(u)(v)=\int_\Omega\sum_{i=1}^n F_{u_i}(x,u_1(x),\dots,u_n(x))v_i(x)\,dx$$

for every $v=(v_1,\dots,v_n)\in X$; moreover, Ψ is sequentially weakly upper semicontinuous.

Furthermore, Proposition 2.2 implies that Φ' admits a continuous inverse on X^* and since Φ' is monotone, we infer that Φ is sequentially weakly lower semicontinuous (see [27, Proposition 25.20]).

We claim that $\Psi':X\to X^*$ is a compact operator. If $p_i>\max\{1,N/2\}$ for $1\le i\le n$, it is enough to show that Ψ' is strongly continuous on X. To see this, for fixed $(u_1,\dots,u_n)\in X$ let $(u_{1m},\dots,u_{nm})\to(u_1,\dots,u_n)$ weakly in X as $m\to\infty$. Then the convergence is uniform on $\overline{\Omega}$ (see [27]). Since $F(x,\cdot,\dots,\cdot)$ is C^1 in $\mathbb{R}^n$ for every $x\in\overline{\Omega}$, the derivatives of F are continuous in $\mathbb{R}^n$ for every $x\in\overline{\Omega}$, so for $1\le i\le n$, $F_{u_i}(x,u_{1m},\dots,u_{nm})\to F_{u_i}(x,u_1,\dots,u_n)$ strongly as $m\to\infty$, which yields $\Psi'(u_{1m},\dots,u_{nm})\to\Psi'(u_1,\dots,u_n)$ strongly as $m\to\infty$. Thus we proved that Ψ' is strongly continuous on X, which implies that Ψ' is compact operator by Proposition 26.2 of [27]. If $p_i\le\max\{1,N/2\}$ for $1\le i\le n$, taking into account that the embedding $W^{2,p_i}\cap W_0^{1,p_i}\hookrightarrow L^{q_i}, q_i\in[p_i,p_iN/(N-2p_i)[$ for $1\le i\le n$ is compact, from condition (A3), we see that Φ' is compact. Hence the claim is true.

From (A1) and (2.10) we get $0<r<\Phi(w)$, as required in Theorem 2.1.

In what follows, we discuss two cases.

Case 1. If $p_i>\max\{1,N/2\}$ for $1\le i\le n$, from (2.3) for each $(u_1,\dots,u_n)\in X$ we have

$$\sup_{x\in\Omega}|u_i(x)|^{p_i}\le k\|u_i\|_{p_i}^{p_i}\qquad\text{for } i=1,\dots,n,$$

so that

$$\sup_{x\in\Omega}\sum_{i=1}^n\frac{|u_i(x)|^{p_i}}{p_i}\le k\sum_{i=1}^n\frac{\|u_i\|_{p_i}^{p_i}}{p_i},\tag{2.12}$$

and hence using (2.10) and (2.12), we obtain

$$\begin{aligned}\Phi^{-1}(]-\infty,r])&=\bigg\{u\in X:\sum_{i=1}^n\frac{\|u_i\|_{p_i}^{p_i}}{p_i}\le r\bigg\}\\&\subseteq\bigg\{u\in X:\sum_{i=1}^n\frac{|u_i(x)|^{p_i}}{p_i}\le kr\text{ for all }x\in\Omega\bigg\}.\end{aligned}$$

Therefore, owing to assumption (2.5), we have

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r])}\int_\Omega F(x,u(x))\,dx\\ &\le \int_\Omega \sup_{t\in K(kr)} F(x,t)\,dx\\ &< r\frac{\int_\Omega F(x,w(x))\,dx}{\sum_{i=1}^n \|w_i\|_{p_i}^{p_i}/p_i} = r\frac{\Psi(w)}{\Phi(w)}.\end{aligned}$$

Furthermore, from (2.6) there exist two constants $\gamma,\tau\in\mathbb{R}$ with

$$\gamma < \frac{\int_\Omega \sup_{t\in K(kr)} F(x,t)\,dx}{r}$$

such that

$$km(\Omega)F(x,t) \le \gamma\sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} + \tau \quad \text{for all } x\in\overline{\Omega}$$

and all $(t_1,\ldots,t_n)\in\mathbb{R}^n$. Fix $u=(u_1,\ldots,u_n)\in X$. Then

$$F(x,u(x)) \le \frac{1}{km(\Omega)}\left(\gamma\sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} + \tau\right) \quad \text{for all } x\in\overline{\Omega}. \tag{2.13}$$

Now, in order to prove the coercivity of the functional $\Phi-\lambda\Psi$, we first assume that $\gamma>0$. So, for any fixed $\lambda\in\Lambda_r$, from (2.10)–(2.13) we have

$$\begin{aligned}\Phi(u)-\lambda\Psi(u) &= \sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i} - \lambda\int_\Omega F(x,u(x))\,dx\\ &\ge \sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i} - \frac{\lambda\gamma}{km(\Omega)}\left(\sum_{i=1}^n \frac{1}{p_i}\int_\Omega |u_i(x)|^{p_i}\,dx\right) - \frac{\lambda\tau}{k}\\ &\ge \sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i} - \frac{\lambda\gamma}{km(\Omega)}\left(km(\Omega)\sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i}\right) - \frac{\lambda\tau}{k}\\ &= \sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i} - \lambda\gamma\sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i} - \frac{\lambda\tau}{k}\\ &\ge \left(1-\gamma\frac{r}{\int_\Omega \sup_{t\in K(kr)} F(x,t)\,dx}\right)\sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i} - \frac{\lambda\tau}{k},\end{aligned}$$

and thus

$$\lim_{\|u\|\to\infty}(\Phi(u)-\lambda\Psi(u)) = \infty.$$

On the other hand, if $\gamma\le 0$, we clearly get $\lim_{\|u\|\to\infty}(\Phi(u)-\lambda\Psi(u))=\infty$. Both cases lead to the coercivity of the functional $\Phi-\lambda\Psi$.

Case 2. If $p_i \le \max\{1, N/2\}$ for $1 \le i \le n$, from (2.10) we get

$$\Phi^{-1}(]-\infty, r]) = \left\{u \in X : \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} \le r\right\}$$
$$\subseteq \{u \in X : \|u_i\|_{p_i} \le \sqrt[p_i]{p_i r} \text{ for } 1 \le i \le n\}.$$

From (2.8), for some positive constant b_2, there exists $\eta \in (0,1)$ satisfying

$$|F(x,t)| \le b_2 \sum_{i=1}^{n} |t_i|^{\theta}$$

for all $|t_i| \le \eta$ for $1 \le i \le n$. In view of assumption (A3) and (2.7), if we put

$$b_3 = \max\left\{b_2, \sup_{|t_i|>\eta} \frac{b_1(1+\sum_{i=1}^{n}|t_i|^s)}{\sum_{i=1}^{n}|t_i|^{\theta}}\right\},$$

then $|F(x,t)| \le b_3 \sum_{i=1}^{n} |t_i|^{\theta}$ for all $t_i \in \mathbb{R}$. Therefore, by (2.1) and (2.2), we have (for a suitable constant $b_4 > 0$)

$$\begin{aligned}
\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) &\le \sup_{\Phi(u)\le r} \int_{\Omega} |F(x,u(x))|\,dx \\
&\le b_3 \sup_{\Phi(u)\le r} \int_{\Omega} \sum_{i=1}^{n} |u_i(x)|^{\theta}\,dx \\
&\le b_4 \sup_{\sum_{i=1}^{n}\|u_i\|_{p_i}^{p_i}/p_i \le r} \sum_{i=1}^{n} \|u_i\|_{p_i}^{\theta} \\
&\le b_4 n(\sqrt[p_i]{p_i r})^{\theta} \qquad \text{for } 1 \le i \le n.
\end{aligned}$$

So

$$\lim_{r\to 0} \frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} = 0.$$

Since, from assumption (2.9), $\Psi(\overline{w}) > 0$, from the above we have

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) < r\frac{\Psi(\overline{w})}{\Phi(\overline{w})}.$$

Moreover, any fixed $\lambda \in \Lambda_r$, from assumption (2.7) one has

$$\Phi(u) - \lambda\Psi(u) \ge \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} - \lambda \int_{\Omega} b_1\Big(1 + \sum_{i=1}^{n} |u_i(x)|^s\Big)\,dx.$$

Noting that $s < q_i$ for all $q_i \in [p_i, p_i N/(N-2p_i)[$, we see that

$$\Phi(u) - \lambda\Psi(u) \ge \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} - \lambda \int_{\Omega} b_1\Big(1 + C\sum_{i=1}^{n} |u_i(x)|^{q_i}\Big)\,dx$$

for some $C > 0$. Then, using the embedding $W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega) \hookrightarrow L^{q_i}\Omega)$ for all $q_i \in [p_i, p_i N/(N-2p_i)[$, for each $\lambda \in \Lambda_r$ we have

$$\lim_{\|u\| \to \infty} (\Phi(u) - \lambda \Psi(u)) = \infty.$$

So, assumptions (i) and (ii) in Theorem 2.1 are satisfied. Hence, as the weak solutions of the system (1.1) are exactly the solutions of the equation $\Phi'(u) - \lambda \Psi'(u) = 0$, the system (1.1) admits at least three distinct weak solutions in X. ∎

Now we want to present a verifiable consequence of the main result where the test function w is specified.

Fix $x^0 \in \Omega$ and pick r_1, r_2 with $0 < r_1 < r_2$ such that

$$B(x^0, r_1) \subset B(x^0, r_2) \subseteq \Omega$$

where $B(x^0, r_i)$ denotes the (open) ball with center at x^0 and radius r_i for $i = 1, \dots, n$. Put

$$\sigma_i = \sigma_i(N, p_i, r_1, r_2) := \frac{12(N+2)^2(r_1+r_2)}{(r_2-r_1)^3} \left(\frac{k\pi^{N/2}(r_2^N - r_1^N)}{\Gamma(1+N/2)} \right)^{1/p_i} \tag{2.14}$$

for $1 \le i \le n$, and

$$\theta_i = \theta_i(N, p_i, r_1, r_2) \tag{2.15}$$
$$:= \begin{cases} \dfrac{3N}{(r_2-r_1)(r_1+r_2)} \left(\dfrac{k\pi^{N/2}((r_1+r_2)^N - (2r_1)^N)}{2^N \Gamma(1+N/2)} \right)^{1/p_i} & \text{if } N < \dfrac{4r_1}{r_2-r_1}, \\ \dfrac{12r_1}{(r_2-r_1)^2(r_1+r_2)} \left(\dfrac{k\pi^{N/2}((r_1+r_2)^N - (2r_1)^N)}{2^N \Gamma(1+N/2)} \right)^{1/p_i} & \text{if } N \ge \dfrac{4r_1}{r_2-r_1}. \end{cases}$$

Corollary 2.4. *Assume that there exists $\overline{w} = (\overline{w}_1, \dots, \overline{w}_n) \in X$ such that assumption* (A3) *in Theorem* 2.3 *holds. Furthermore, suppose that there exist two positive constants c and d with*

$$\sum_{i=1}^n \frac{(d\theta_i)^{p_i}}{p_i} > \frac{c}{\prod_{i=1}^n p_i}$$

such that:

(B1) $F(x,t) \ge 0$ *for each* $(x,t) \in (\overline{\Omega} \setminus B(x^0, r_1)) \times [0,d]^n$;

(B2) *if* $p_i > \max\{1, N/2\}$ *for* $1 \le i \le n$, *then*

$$\sum_{i=1}^n \frac{(d\sigma_i)^{p_i}}{p_i} \int_\Omega \sup_{t \in K(c/\prod_{i=1}^n p_i)} F(x,t)\,dx < \frac{c}{\prod_{i=1}^n p_i} \int_{B(x^0,r_1)} F(x,d,\dots,d)\,dx \tag{2.16}$$

where σ_i and θ_i are given by (2.14) and (2.15), respectively, and

$$\limsup_{|t_1|+\dots+|t_n|\to\infty} \frac{F(x,t)}{\sum_{i=1}^n |t_i|^{p_i}/p_i} < \frac{\prod_{i=1}^n p_i}{m(\Omega)c} \int_\Omega \sup_{t\in K(c/\prod_{i=1}^n p_i)} F(x,t)\,dx \tag{2.17}$$

for all $x\in\overline{\Omega}$.

Then, for $r := c/k\prod_{i=1}^n p_i$ and each λ in

$$\Lambda' := \begin{cases} \left]\dfrac{\sum_{i=1}^n \frac{(d\sigma_i)^{p_i}}{kp_i}}{\int_{B(x^0,r_1)} F(x,d,\dots,d)\,dx}, \dfrac{r}{\int_\Omega \sup_{t\in K(kr)} F(x,t)\,dx}\right[& \text{if } p_i > \max\{1,N/2\},\\ \left]\dfrac{\sum_{i=1}^n \frac{\|\overline{w}_i\|_{p_i}^{p_i}}{p_i}}{\int_\Omega F(x,\overline{w}(x))\,dx}, \dfrac{r}{\int_\Omega \sup_{\sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i}\le r} F(x,u(x))\,dx}\right[& \text{if } p_i \le \max\{1,N/2\}, \end{cases}$$

the system (1.1) admits at least three distinct weak solutions in X.

Proof. Set $w(x) = (w_1(x),\dots,w_n(x))$ where for $1\le i\le n$,

$$w_i(x) = \begin{cases} 0 & \text{if } x\in\overline{\Omega}\setminus B(x^0,r_2),\\ \dfrac{d(3(l^4-r_2^4)-4(r_1+r_2)(l^3-r_2^3)+6r_1r_2(l^2-r_2^2))}{(r_2-r_1)^3(r_1+r_2)} & \text{if } x\in B(x^0,r_2)\setminus B(x^0,r_1),\\ d & \text{if } x\in B(x^0,r_1), \end{cases}$$

with $l=\operatorname{dist}(x,x^0)=\sqrt{\sum_{i=1}^N (x_i-x_i^0)^2}$. We have

$$\frac{\partial w_i(x)}{\partial x_i} = \begin{cases} 0 & \text{if } x\in(\overline{\Omega}\setminus B(x^0,r_2))\cup S(x^0,r_1),\\ \dfrac{12d(l^2(x_i-x_i^0)-(r_1+r_2)l(x_i-x_i^0)+r_1r_2(x_i-x_i^0))}{(r_2-r_1)^3(r_1+r_2)} & \text{if } x\in B(x^0,r_2)\setminus B(x^0,r_1), \end{cases}$$

$$\frac{\partial^2 w_i(x)}{\partial^2 x_i} = \begin{cases} 0 & \text{if } x\in(\overline{\Omega}\setminus B(x^0,r_2))\cup B(x^0,r_1),\\ \dfrac{12d(r_1r_2+(2l-r_1-r_2)(x_i-x_i^0)^2/l-(r_2+r_1-l)l)}{(r_2-r_1)^3(r_1+r_2)} & \text{if } x\in B(x^0,r_2)\setminus B(x^0,r_1), \end{cases}$$

and

$$\sum_{i=1}^{N}\frac{\partial^2 w_i(x)}{\partial^2 x_i} = \begin{cases} 0 & \text{if } x\in(\overline{\Omega}\setminus B(x^0,r_2))\cup B(x^0,r_1),\\ \dfrac{12d((N+2)l^2-(N+1)(r_1+r_2)l+Nr_1r_2))}{(r_2-r_1)^3(r_1+r_2)} & \text{if } x\in B(x^0,r_2)\setminus B(x^0,r_1). \end{cases}$$

It is easy to see that $w=(w_1,\dots,w_n)\in X$ and, in particular,

$$\|w_i\|_{p_i}^{p_i} = \frac{(12d)^{p_i}2\pi^{N/2}}{(r_2-r_1)^{3p_i}(r_1+r_2)^{p_i}\Gamma(N/2)} \times \int_{r_1}^{r_2} |(N+2)\xi^2-(N+1)(r_1+r_2)\xi+Nr_1r_2|^{p_i}\xi^{N-1}\,d\xi \tag{2.18}$$

for $1\le i\le n$. Hence, from (2.14), (2.15) and (2.18) we get

$$\frac{(d\theta_i)^{p_i}}{k} < \|w_i\|_{p_i}^{p_i} < \frac{(d\sigma_i)^{p_i}}{k} \tag{2.19}$$

for $1\le i\le n$. However, taking into account that $\sum_{i=1}^{n}\frac{(d\theta_i)^{p_i}}{p_i} > \frac{c}{\prod_{i=1}^{n}p_i}$, from (2.19) one has

$$\sum_{i=1}^{n}\frac{\|w_i\|_{p_i}^{p_i}}{p_i} > r,$$

which is assumption (A1).

Since $0\le w_i(x)\le d$ for each $x\in\Omega$ for $1\le i\le n$, condition (B1) ensures that

$$\int_{\overline{\Omega}\setminus B(x^0,r_2)} F(x,w(x))\,dx + \int_{B(x^0,r_2)\setminus B(x^0,r_1)} F(x,w(x))\,dx \ge 0. \tag{2.20}$$

Moreover, from (2.16) and (2.20), we have

$$\int_{\Omega}\sup_{t\in K(kr)}F(x,t)\,dx < \frac{c\int_{B(x^0,r_1)}F(x,d,\dots,d)\,dx}{\left(\sum_{i=1}^{n}\frac{(d\sigma_i)^{p_i}}{p_i}\right)\left(\prod_{i=1}^{n}p_i\right)} \le \frac{c}{k}\,\frac{\int_{\Omega}F(x,w(x))\,dx}{\sum_{i=1}^{n}\prod_{j=1,j\ne i}^{n}p_j\|w_i\|_{p_i}^{p_i}}$$
$$= \left(r\prod_{i=1}^{n}p_i\right)\frac{\int_{\Omega}F(x,w(x))dx}{\sum_{i=1}^{n}\prod_{j=1,\,j\ne i}^{n}p_j\|w_i\|_{p_i}^{p_i}},$$

so assumption (2.5) in (A2) is satisfied. Also, (2.17) yields (2.6). Hence, taking into account that $\Lambda'\subseteq\Lambda_r$, using Theorem 2.3, we obtain the desired conclusion. ■

REMARK 2.5. For any $u \in L^2(\Omega)$, we have $u = \sum_{k=1}^{\infty} a_k e_k$, where the a_k are coefficients and e_k is an eigenvector corresponding to the eigenvalue λ_k for $k = 1, \ldots, \infty$ of the operator $-\Delta$, the selfadjoint extension of the operator $-\sum_{k=1}^{N} \partial^2/\partial x_k^2$ with the domain $C_0^2(\Omega) \subset L^2(\Omega)$, where e_k for $k = 1, \ldots, \infty$ form an orthonormal base. Then we can get $-\Delta u = \sum_{k=1}^{\infty} a_k \lambda_k e_k$. Using the equality above, it follows that

$$\|\Delta u\|_{L^2}^2 = \sum_{k=1}^{\infty} a_k^2 \lambda_k^2 \geq \lambda_1^2 \sum_{k=1}^{\infty} a_k^2 = \lambda_1^2 \|u\|_{L^2}^2,$$

so

$$\|\Delta u\|_{L^2} \geq \lambda_1 \|u\|_{L^2}. \tag{2.21}$$

Taking into account that $\|u\|_\infty \leq \frac{1}{2}\|u\|_{L^2}^{1/2} \|\Delta u\|_{L^2}^{1/2}$ (see [14]), from (2.21) we obtain

$$\|u\|_\infty \leq \frac{1}{2} \lambda_1^{-1/2} \|\Delta u\|_{L^2}. \tag{2.22}$$

Moreover, using the Hölder inequality we have

$$\|\Delta u\|_{L^2} = \Big(\int_\Omega |\Delta u|^2 \, dx \Big)^{1/2} \leq (\|1\|_{L^p} \| \, |\Delta u|^2 \|_{L^q})^{1/2} \leq m(\Omega)^{1/2p} \|\Delta u\|_{L^{2q}}$$

where $1/p + 1/q = 1$, which in conjunction with (2.22) yields

$$\|u\|_\infty \leq \frac{1}{2} \lambda_1^{-1/2} m(\Omega)^{1/2p} \|\Delta u\|_{L^{2q}}.$$

We recall an estimate for λ_1, the principal eigenvalue of the operator Δ, on a planar convex domain: $\lambda_1 \geq \frac{\pi^2}{4}\big(\frac{L^2}{4A^2} + \frac{1}{d^2}\big)$ where A, L and d denote the area, boundary length and diameter of the domain, respectively (see [2]).

We present an example to illustrate Corollary 2.4 as follows:

EXAMPLE 2.6. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$, $p_1 = p_2 = 4$ and $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y, t_1, t_2) = \begin{cases} 0 & \text{for } t_i < 0,\ i = 1, 2, \\ (x^2 + y^2) t_2^{100} e^{-t_2} & \text{for } t_1 < 0,\ t_2 \geq 0, \\ (x^2 + y^2) t_1^{100} e^{-t_1} & \text{for } t_1 \geq 0,\ t_2 < 0, \\ (x^2 + y^2) \sum_{i=1}^{2} t_i^{100} e^{-t_i} & \text{for } t_i \geq 0,\ i = 1, 2, \end{cases}$$

for $(x, y, t_1, t_2) \in \Omega \times \mathbb{R}^2$. In fact, by choosing $r_1 = 1$ and $r_2 = 2$, taking into account that $k = \frac{9 \cdot 6^4}{289\pi^3}$, we have

$$\sigma_1 = \sigma_2 = \frac{3456}{\pi^{1/2}} \sqrt[4]{\frac{27}{289}} \quad \text{and} \quad \theta_1 = \theta_2 = \frac{12}{\pi^{1/2}} \sqrt[4]{\frac{45}{1156}}.$$

Clearly, by choosing $x^0=(0,0)$, $c=4$ and $d=100$ we observe that assumption (B1) is satisfied. For (B2),

$$\begin{aligned}\sum_{i=1}^{2}\frac{(d\sigma_i)^{p_i}}{p_i}\int_{\Omega}&\sup_{(t_1,t_2)\in K(c/\prod_{i=1}^2 p_i)} F(x,t_1,t_2)\,dx\,dy\\
&=\kappa\int_{\Omega}\sup_{(t_1,t_2)\in K(1/4)} F(x,y,t_1,t_2)\,dx\,dy\\
&\le\kappa\int_{\Omega}\sup_{(t_1,t_2)\in K(1/4)}(x^2+y^2)\sum_{i=1}^{2}t_i^{100}e^{-t_i}\,dx\,dy\\
&=\kappa\max_{(t_1,t_2)\in K(1/4)}\sum_{i=1}^{2}t_i^{100}e^{-t_i}\int_{x^2+y^2\le 9}(x^2+y^2)\,dx\,dy\\
&\le\kappa\Big(2\max_{|t|\le 1}t^{100}e^{-t}\Big)\int_{x^2+y^2\le 9}(x^2+y^2)\,dx\,dy\\
&\le 81\pi\kappa e\le\frac{\pi}{4}100^{100}e^{-100}=\frac{1}{2}100^{100}e^{-100}\int_{x^2+y^2\le 1}(x^2+y^2)\,dx\,dy\\
&=\frac{c}{\prod_{i=1}^2 p_i}\int_{S(x^0,r_1)}F(x,y,d,d)\,dx\,dy,\end{aligned}$$

where $\kappa=\frac{1}{2}(100\cdot\frac{3456}{\pi^{1/2}}\sqrt[4]{\frac{27}{289}})^4$. So, Corollary 2.4 is applicable to the system

$$\begin{cases}\Delta(|\Delta u_1|^2\Delta u_1)=\lambda(x^2+y^2)(u_1^+)^{99}e^{-u_1^+}(100-u_1^+) & \text{in } \Omega,\\ \Delta(|\Delta u_2|^2\Delta u_2)=\lambda(x^2+y^2)(u_2^+)^{99}e^{-u_2^+}(100-u_2^+) & \text{in } \Omega,\\ u_1=\Delta u_1=u_2=\Delta u_2=0 & \text{on } \partial\Omega,\end{cases}$$

where $u_i^+=\max\{u_i,0\}$, for every $\lambda\in\,\big]\frac{289\pi^2(100\cdot\frac{3456}{\pi^{1/2}}\sqrt[4]{\frac{27}{289}})^4}{18\cdot 6^4\cdot(100)^{100}e^{-100}},\frac{289\pi^2}{2^6\cdot 3^{10}e}\big[$.

Put

$$\tau_i=\tau_i(N,p_i,r_1,r_2):=\frac{12(N+2)^2(r_1+r_2)}{(r_2-r_1)^3}\bigg(\frac{k(r_2^N-r_1^N)}{r_1^N}\bigg)^{1/p_i}\quad\text{for } 1\le i\le n. \tag{2.23}$$

Here is a remarkable consequence of Corollary 2.4.

COROLLARY 2.7. *Let $F:\mathbb{R}^n\to\mathbb{R}$ be a C^1 function in $\mathbb{R}^n$ such that $F(0,\dots,0)=0$. Assume that there exist positive constants c and d with $\sum_{i=1}^n(d\theta_i)^{p_i}/p_i>c/\prod_{i=1}^n p_i$ and an element $\overline{w}=(\overline{w}_1,\dots,\overline{w}_n)\in X$ such that*

(C1) *$F(t)\ge 0$ for each $t\in[0,d]^n$;*

(C2) *if* $p_i > \max\{1, N/2\}$ *for* $1 \le i \le n$, *then*

$$m(\Omega)\sum_{i=1}^{n}\frac{(d\tau_i)^{p_i}}{p_i}\max_{t\in K(c/\prod_{i=1}^n p_i)}F(t) < \frac{c}{\prod_{i=1}^n p_i}F(d,\dots,d) \tag{2.24}$$

where τ_i *is given by* (2.23), *and*

$$\limsup_{|t_1|+\dots+|t_n|\to\infty}\frac{F(t)}{\sum_{i=1}^n |t_i|^{p_i}/p_i} < \frac{\prod_{i=1}^n p_i}{c}\max_{t\in K(c/\prod_{i=1}^n p_i)}F(t_1,\dots,t_n); \tag{2.25}$$

(C3) *if* $p_i \le \max\{1, N/2\}$ *for* $1 \le i \le n$, *then there exist positive constants* b_1, θ *and* s *with* $p_i < \theta < p_iN/(N-2p_i)$ *and* $s < p_i$ *for* $1 \le i \le n$ *satisfying*

$$|F(t)| \le b_1\Big(1+\sum_{i=1}^n |t_i|^s\Big) \quad \forall t_i \in \mathbb{R}, \tag{2.26}$$

$$\limsup_{\sum_{i=1}^n |t_i|\to 0}\frac{|F(t)|}{\sum_{i=1}^n |t_i|^\theta} < \infty, \tag{2.27}$$

$$\int_\Omega F(\overline{w}(x))\,dx > 0. \tag{2.28}$$

Then, for $r := c/k\prod_{i=1}^n p_i$ *and each* λ *in*

$$\Lambda'' := \begin{cases} \Big]\dfrac{\sum_{i=1}^n \frac{(d\tau_i)^{p_i}}{kp_i}}{F(d,\dots,d)}, \dfrac{r}{m(\Omega)\max_{t\in K_1(kr)}F(t)}\Big[& \text{if } p_i > \max\{1,N/2\},\\ \Big]\dfrac{\sum_{i=1}^n \frac{\|\overline{w}_i\|_{p_i}^{p_i}}{p_i}}{\int_\Omega F(\overline{w}(x))\,dx}, \dfrac{r}{\int_\Omega \sup_{\sum_{i=1}^n \|u_i\|_{p_i}^{p_i}/p_i\le r}F(u(x))\,dx}\Big[& \\ & \text{if } p_i \le \max\{1,N/2\}, \end{cases}$$

the system

$$\begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i) = \lambda F_{u_i}(u_1,\dots,u_n) & \text{in } \Omega,\\ u_i = \Delta u_i = 0 & \text{on } \partial\Omega \end{cases}$$

for $1 \le i \le n$, *admits at least three distinct weak solutions in* X.

Proof. Set $F(x,t) = F(t)$ for all $x \in \overline{\Omega}$ and $t = (t_1,\dots,t_n) \in \mathbb{R}^n$. Clearly, all assumptions of Corollary 2.4 are satisfied. In particular, since $m(B(x^0,r_1)) = r_1^N\frac{\pi^{N/2}}{\Gamma(1+N/2)}$, assumption (2.24) implies (2.16). So, we have the conclusion by using Corollary 2.4. ∎

Acknowledgements. The authors express their sincere gratitude to the referee for reading this paper very carefully and especially for valuable suggestions concerning improvements.

The research of Shapour Heidarkhani was supported in part by a grant from IPM (No. 90470020).

The research of C.-L. Tang was supported by National Natural Science Foundation of China (No. 11071198).

References

[1] G. A. Afrouzi, S. Heidarkhani and D. O'Regan, *Existence of three solutions for a doubly eigenvalue fourth-order boundary value problem*, Taiwanese J. Math. 15 (2011), 201–210.

[2] P. Antunes and P. Freitas, *New bounds for the principal Dirichlet eigenvalue of planar regions*, Experiment. Math. 15 (2006), 333–342.

[3] D. Averna and G. Bonanno, *A mountain pass theorem for a suitable class of functions*, Rocky Mountain J. Math. 39 (2009), 707–727.

[4] M. B. Ayed and M. Hammami, *On a fourth order elliptic equation with critical nonlinearity in dimension six*, Nonlinear Anal. 64 (2006), 924–957.

[5] Z. Bai and H. Wang, *On positive solutions of some nonlinear fourth-order beam equations*, J. Math. Anal. Appl. 270 (2002), 357–368.

[6] G. Bonanno, *Some remarks on a three critical points theorem*, Nonlinear Anal. 54 (2003), 651–665.

[7] G. Bonanno and B. Di Bella, *A boundary value problem for fourth-order elastic beam equations*, J. Math. Anal. Appl. 343 (2008), 1166–1176.

[8] —, —, *A fourth-order boundary value problem for a Sturm–Liouville type equation*, Appl. Math. Comput. 217 (2010), 3635–3640.

[9] G. Bonanno and S. A. Marano, *On the structure of the critical set of non-differentiable functions with a weak compactness condition*, Appl. Anal. 89 (2010), 1–10.

[10] A. Cabada, J. A. Cid and L. Sanchez, *Positivity and lower and upper solutions for fourth-order boundary value problems*, Nonlinear Anal. 67 (2007), 1599–1612.

[11] J. Chabrowski and J. M. do Ó, *On some fourth-order semilinear elliptic problems in* R^N, ibid. 49 (2002), 861–884.

[12] M. R. Grossinho, L. Sanchez and S. A. Tersian, *On the solvability of a boundary value problem for a fourth-order ordinary differential equation*, Appl. Math. Lett. 18 (2005), 439–444.

[13] G. Han and Z. Xu, *Multiple solutions of some nonlinear fourth-order beam equation*, Nonlinear Anal. 68 (2008), 3646–3656.

[14] A. A. Ilyin, *Best constants in Sobolev inequalities on the sphere and in Euclidean space*, J. London Math. Soc. (2) 59 (1999), 263–286.

[15] A. C. Lazer and P. J. McKenna, *Large amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis*, SIAM Rev. 32 (1990), 537–578.

[16] L. Li and S. Heidarkhani, *Existence of three solutions to a double eigenvalue problem for the p-biharmonic equation*, Ann. Polon. Math. 104 (2012), 71–80.

[17] X.-L. Liu and W.-T. Li, *Existence and multiplicity of solutions for fourth-order boundary value problems with parameters*, J. Math. Anal. Appl. 327 (2007), 362–375.

[18] S. Liu and M. Squassina, *On the existence of solutions to a fourth-order quasilinear resonant problem*, Abstr. Appl. Anal. 7 (2002), 125–133.

[19] C. Li and C.-L. Tang, *Three solutions for a Navier boundary value problem involving the p-biharmonic*, Nonlinear Anal. 72 (2010), 1339–1347.

[20] L. Li and C.-L. Tang, *Existence of three solutions for* (p, q)*-biharmonic systems*, ibid. 73 (2010), 796–805.

[21] S. A. Marano and D. Motreanu, *On a three critical points theorem for non-differentiable functions and applications to nonlinear boundary value problems*, ibid. 48 (2002), 37–52.
[22] A. M. Micheletti and A. Pistoia, *Multiplicity results for a fourth-order semilinear elliptic problem*, ibid. 31 (1998), 895–908.
[23] B. Ricceri, *A three critical points theorem revisited*, ibid. 70 (2009), 3084–3089.
[24] —, *On a three critical points theorem*, Arch. Math. (Basel) 75 (2000), 220–226.
[25] J. Simon, *Régularité de la solution d'une équation non linéaire dans R^N*, in: Journées d'Analyse Non Linéaire (Besançon, 1977), P. Bénilan and J. Robert (eds.), Lecture Notes in Math. 665, Springer, Berlin, 1978, 205–227.
[26] W. Wang and P. Zhao, *Nonuniformly nonlinear elliptic equations of p-biharmonic type*, J. Math. Anal. Appl. 348 (2008), 730–738.
[27] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, Vol. II, Springer, Berlin, 1985.
[28] J. Zhang and S. Li, *Multiple nontrivial solutions for some fourth-order semilinear elliptic problems*, Nonlinear Anal. 60 (2005), 221–230.

Shapour Heidarkhani
Department of Mathematics
Faculty of Sciences
Razi University
67149 Kermanshah, Iran
and
School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
P.O. Box 19395-5746, Tehran, Iran
E-mail: s.heidarkhani@razi.ac.ir

Yu Tian
School of Science
Beijing University of Posts and Telecommunications
Beijing 100876, P.R. China
E-mail: tianyu2992@163.com

Chun-Lei Tang
School of Mathematics and Statistics
Southwest University
Chongqing 400715, P.R. China
E-mail: tangcl@swu.edu.cn

Received 26.5.2011
and in final form 28.1.2012 (2461)