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Extension and normality of meromorphic mappings into complex projective varieties

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Abstract. The purpose of this article is twofold. The first is to show a criterion for the normality of holomorphic mappings into Abelian varieties; an extension theorem for such mappings is also given. The second is to study the convergence of meromorphic mappings into complex projective varieties. We introduce the concept of d-convergence and give a criterion of d-normality of families of meromorphic mappings.

1. Introduction. Let Ω be a domain in $\mathbb{C}^m$ and let X be a complex manifold.

We say that a sequence of holomorphic mappings $\{f_n\}_{n=1}^{\infty}$ from Ω into X *converges* to a holomorphic mapping f on Ω if for any compact set $K \subset \Omega$ there is a compact set $P \subset X$ such that $f(K), f_n(K) \subset P$ for all n and $\lim_{n\to\infty} \sup_{x\in K} d(f_n(x), f(x)) = 0$. Here X is equipped with some Hermitian metric. Note that this notion of convergence does not depend on the choice of the Hermitian metric.

A family $\mathcal{F}$ of holomorphic mappings from Ω into X is said to be *normal* on Ω if any sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ has a subsequence convergent on Ω.

It is known that if D is an analytic subset of a compact complex manifold X such that $X \setminus D$ is hyperbolically imbedded into X then the family $\mathcal{F}$ of all holomorphic mappings from Ω into X omitting D is a normal family. The following question arises naturally.

Question. *Are there any criteria for the normality of a family of holomorphic mappings $\mathcal{F}$ from Ω into X in the case where either each $f \in \mathcal{F}$ may intersect D, or $X \setminus D$ is not hyperbolically imbedded in X?*

Concerning this question, H. Fujimoto [Fu2] introduced the concept of a meromorphically normal family of meromorphic mappings with values in a

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complex projective space and gave the following result.

Theorem A. *Let $\mathcal{F}$ be a family of meromorphic mappings from a domain $\Omega \subset \mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ and let $\{H_j\}_{j=1}^{2n+1}$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position such that for each $f \in \mathcal{F}$, $f(\Omega) \not\subset H_j$ $(j = 1, \ldots, 2n+1)$ and for any fixed compact subset K of Ω, the $2(m-1)$-dimensional Lebesgue areas of $f^{-1}(H_j) \cap K$ $(j = 1, \ldots, 2n+1)$ counted with multiplicities for all f in $\mathcal{F}$ are bounded from above. Then $\mathcal{F}$ is a meromorphically normal family on Ω.*

Let f be a holomorphic mapping from a domain $\Omega \subset \mathbb{C}^m$ into a complex space X and let D be a divisor on X. We say that the mapping f *intersects D with multiplicity at least k* $(k \in \mathbb{Z}^+)$ on Ω if the image $f(\Omega)$ intersects D with multiplicity at least k at every point of their intersection.

In 1999, Tu [Tu] proved the following.

Theorem B. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain $\Omega \subset \mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ and let $\{H_j\}_{j=1}^{q}$ be q $(\geq 2n+1)$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position such that each f in $\mathcal{F}$ intersects H_j on Ω with multiplicity at least m_j $(j = 1, \ldots, q)$, where $m_1, \ldots, m_q$ are positive integers or $+\infty$, with $\sum_{j=1}^{q} 1/m_j < (q-n-1)/n$. Then $\mathcal{F}$ is a normal family on Ω.*

Since that time, the above results of H. Fujimoto and Z. H. Tu have been generalized to the case of moving hyperplanes or moving hypersurfaces by many authors, such as D. D. Thai, P. N. Mai and P. N. T. Trang [MTT], S. D. Quang and T. V. Tan [QT], Z. H. Tu and P. Li [Tu], [TL] and others. But so far, there are very few results on this problem in the case where X is not the complex projective space but a general projective variety.

Our aim in the present paper is twofold. Our first purpose is to study the normality of meromorphic mappings into Abelian varieties (cf. §3). We will prove the following.

Theorem 1.1. *Let $\mathcal{F}$ be a family of holomorphic mappings from a domain $\Omega \subset \mathbb{C}^m$ into an Abelian variety X and let D be a reduced ample divisor such that D intersects transversely any translate of any closed proper subgroup of X. Assume that each f in $\mathcal{F}$ intersects D with multiplicity at least 2. Then $\mathcal{F}$ is a normal family on Ω.*

By using Theorem 1.1 we shall prove an extension theorem for holomorpic mappings into an Abelian variety:

Theorem 1.2. *Let f be a holomorphic mapping from $\Omega \setminus S$ into an Abelian variety X, where S is an analytic subset of codimension 1 of $\Omega \subset \mathbb{C}^m$. Let D be a reduced ample divisor on X as in Theorem 1.1. Assume that f intersects D with multiplicity at least 2. Then f extends to a holomorphic mapping $\tilde{f}$ from Ω into X.*

In §4, we are going to discuss the normality of families of meromorphic mappings with values in an arbitrary complex projective variety. We introduce the notion of d-convergence of meromorphic mappings, which is a generalization of the notion of m-convergence of H. Fujimoto. We also give a criterion (cf. Theorem 4.13) of d-normality of families of meromorphic mappings. This is the second main purpose of the present paper.

2. Preliminaries. Let F be a nonzero holomorphic function on a domain Ω in $\mathbb{C}^m$. For a sequence $\alpha = (\alpha_1, \ldots, \alpha_m)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\mathcal{D}^\alpha F = \partial^{|\alpha|} F / \partial z_1^{\alpha_1} \cdots \partial z_m^{\alpha_m}$. We define a map $\nu_F : \Omega \to \mathbb{Z}$ by

$$\nu_F(z) := \max\{l : \mathcal{D}^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < l\} \quad (z \in \Omega).$$

By a *divisor* on a domain Ω in $\mathbb{C}^m$ we mean a map $\nu : \Omega \to \mathbb{Z}$ such that, for each $a \in \Omega$, there are nonzero holomorphic functions F and G on a connected neighborhood $U \subset \Omega$ of a such that $\nu(z) = \nu_F(z) - \nu_G(z)$ for each $z \in U$ outside an analytic set of dimension $\leq m-2$. Two divisors are regarded as the same if they are identical outside of an analytic set of dimension $\leq m-2$. For a divisor ν on Ω we set $|\nu| := \overline{\{z : \nu(z) \neq 0\}}$, which is a purely $(m-1)$-dimensional analytic subset of Ω or an empty set.

Take a nonzero meromorphic function φ on a domain Ω in $\mathbb{C}^m$. For each $a \in \Omega$, we choose nonzero holomorphic functions F and G on a neighborhood $U \subset \Omega$ such that $\varphi = F/G$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq m-2$, and we define the divisors ν_φ, ν_φ^∞ by $\nu_\varphi := \nu_F$, $\nu_\varphi^\infty := \nu_G$; these are independent of the choices of F and G and so globally well-defined on Ω.

For a divisor ν on $\mathbb{C}$ and for positive integers k or $k = \infty$, we define the counting functions

$$\nu^{(k)}(z) := \min\{k, \nu(z)\}, \qquad n(t) := \sum_{|z| \leq t} \nu(z), \qquad n^{(k)}(t) := \sum_{|z| \leq t} \nu^{(k)}(z).$$

Set

$$N(r, \nu) := \int_1^r \frac{n(t)}{t}\, dt, \qquad N^{(k)}(r, \nu) := \int_1^r \frac{n^{(k)}(t)}{t}\, dt \qquad (1 < r < \infty).$$

Let $f : \mathbb{C} \to M$ be a holomorphic curve in a compact complex manifold M and let D be a divisor on M. Denote by L the line bundle defined by D. Take a Hermitian metric H on L, and denote by $\|\cdot\|$ the norm on the fibers L_x defined by H. The curvature form ω of the Hermitian line bundle L is defined as follows: for each $a \in M$, we choose a local holomorphic section $\sigma \in H^0(U, L)$, where U is an open neighborhood of a, and define the $(1,1)$ form ω on U by

$$\omega := \frac{i}{2\pi} \partial\bar{\partial} \log \|\sigma\|,$$

which is independent of the choice of σ and so globally well-defined on M.

We assume that $\omega \geq 0$ and $f(\mathbb{C}) \not\subset \operatorname{supp} D$.

The *characteristic function* of f with respect to D is defined by

$$T_f(r; D) := \int_1^r \frac{dt}{t} \int_{|z|<t} f^*\omega.$$

We choose $\sigma \in \Gamma(M, L)$ with $\operatorname{div}(\sigma) = D$ which satisfies $\|\sigma(x)\| < 1$ $(x \in M)$. We define the *proximity function* of f with respect to D by

$$m_f(r; D) := \frac{1}{2\pi} \int_{|z|=r} \log \frac{1}{\|\sigma \circ f\|} \, d\theta.$$

The First Main Theorem [NO] for holomorphic curves in the compact complex manifold M states that

$$T_f(r; D) = N(r, f^*D) + m_f(r; D) + O(1).$$

Assume that $\dim H^0(M, L) \geq 2$ and take a linearly independent system $\sigma_0, \ldots, \sigma_n$ of $H^0(M, L)$. Then we get a meromorphic mapping

$$\Phi_L : x \in M \mapsto [\sigma_0(x), \ldots, \sigma_n(x)] \in \mathbb{P}^n(\mathbb{C}).$$

If Φ_L gives rise to a holomorphic embedding, L is said to be *very ample*. If there is a number $k \in \mathbb{N}$ such that L^k is very ample line bundle, then L is said to be *ample*. The divisor D is said to be *ample* if the line bundle L is ample. One remarks that if D is ample then L is positive in the sense that we can find a Hermitian metric on L with positive curvature.

DEFINITION 2.1 (cf. [TL, Definition 4.4]). Let $\{\nu_i\}_{i=1}^{\infty}$ be a sequence of nonnegative divisors on a domain D in $\mathbb{C}^m$. It is said to *converge* to a nonnegative divisor ν on D if any $a \in D$ has a neighborhood U such that there exist nonzero holomorphic functions h and h_i on U with $\nu_i = \nu_{h_i}$ and $\nu = \nu_h$ on U such that $\{h_i\}_{i=1}^{\infty}$ converges to h uniformly on compact subsets of U.

The next lemma is a generalization of the classical lemma of Zalcman on normality criteria for holomorphic mappings.

LEMMA 2.2 (cf. [AK], [TTH]). *Let Ω be a domain in $\mathbb{C}^m$ and M be a compact complex Hermitian space. Let $\mathcal{F} \subset \operatorname{Hol}(\Omega, M)$. Then the family $\mathcal{F}$ is not normal if and only if there exist sequences $\{p_j\} \subset D$ with $p_j \to p_0 \in \Omega$, $\{f_j\} \subset \mathcal{F}$, $\{\rho_j\} \subset \mathbb{R}$ with $\rho_j > 0$ and $\rho_j \to 0$ and $\xi_j \in \mathbb{C}^m$ Euclidean unit vectors such that the sequence*

$$g_j(z) := f_j(p_j + \rho_j \xi_j z)$$

converges to a nonconstant holomorphic mapping $g : \mathbb{C} \to M$ uniformly on compact subsets of $\mathbb{C}$.

3. Holomorphic mappings into Abelian varieties

DEFINITION 3.1. Let V be a complex vector space of dimension n and let Λ be a lattice of rank $2n$. The quotient space $X = V/\Lambda$ is called an n-dimensional *complex torus*. X is an *Abelian variety* if it is a complex projective algebraic variety.

Let X be an Abelian variety and let D be a reduced ample divisor on X. In 2004, K. Yamanoi [Y] proved the following Second Main Theorem.

THEOREM 3.2. *Let X and D be as above and let f be an algebraically nondegenerate holomorphic curve $f : \mathbb{C} \to X$. Then for every $\epsilon > 0$,*

$$\| \ T_f(r; D) \leq N^{(1)}(r, f^*D) + \epsilon T_f(r; D).$$

Here, the curve f is said to be *algebraically nondegenerate* if $f(\mathbb{C})$ is not contained in any proper algebraic subset of X. Also the notation "$\| \ P$" means that the assertion P holds for all $r \in (1, +\infty)$ except a finite Lebesgue measure subset.

Applying the above result of K. Yamanoi, we deduce the following.

LEMMA 3.3. *Let X be an Abelian variety and let D be a reduced ample divisor on X such that D intersects transversely any translate of any closed algebraic proper subgroup of X. Then there is no nonconstant holomorphic curve $f : \mathbb{C} \to X$ which intersects D with multiplicity at least* 2.

Proof. Suppose that there exists such an f. Denote by Z the Zariski closure of $f(\mathbb{C})$. According to [NW2], Z is a translate of some closed algebraic subgroup of X. Then we can regard Z as an Abelian variety.

By assumption, $D' = D \cap Z$ is a reduced ample divisor on Z. Since $\overline{f(\mathbb{C})}^Z = Z$, we may regard f as an algebraically nondegenerate holomorphic curve $f : \mathbb{C} \to Z$ which intersects D' with multiplicity at least 2.

By Theorem 3.2, for $\epsilon < 1/2$ we have

$$\begin{aligned} \| \ T_f(r; D') &\leq N^{(1)}(r, f^*D') + \epsilon T_f(r; D') \leq \tfrac{1}{2} N(r, f^*D') + \epsilon T_f(r; D') \\ &\leq (1/2 + \epsilon) T_f(r; D'). \end{aligned}$$

Letting $r \to +\infty$, we get $\epsilon \geq 1/2$. This is a contradiction. ■

Proof of Theorem 1.1. Without loss of generality, we may assume that Ω is a polydisc in $\mathbb{C}^m$, $\Omega = \Delta^m$.

Suppose that $\mathcal{F}$ is not normal on Ω. Then, by Lemma 2.2, there exist a subsequence of $\mathcal{F}$ denoted by $\{f_j\}_{j=1}^\infty$ and $p_0 \in \Omega, \{p_j\}_{j=1}^\infty \subset \Omega$ with $p_j \to p_0, \{\rho_j\} \subset (0, +\infty)$ with $\rho_j \to 0$ and $\xi_j \in \mathbb{C}^m$ Euclidean unit vectors such that the sequence of holomorphic mappings

$$g_j := f_j(p_j + \rho_j \xi_j \cdot) : \Delta_{r_j} \to X \quad (r_j \uparrow \infty)$$

converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic mapping $g : \mathbb{C} \to X$.

For any fixed $z_0 \in \operatorname{Supp} g^* D$, there exist a relative compact neighborhood U of z_0 in $\mathbb{C}$, an open neighborhood V of $g(z_0)$ in X, a holomorphic function φ on V and a positive number j_0 such that $U \subset \Delta_{r_j}$, $g_j(U) \subset V$ for all $j \geq j_0$ and φ is a defining function of D on V. By the convergence of $\{g_j\}_{j \geq j_0}$, we see that $\{\varphi \circ g_j\}_{j \geq j_0}$ converges uniformly on compact subsets of U to $\varphi \circ g$. It is clear that all zeros of $\varphi \circ g_j$ have multiplicities at least 2. So, by Hurwitz's theorem, all zero points of $\varphi \circ g$ have multiplicities at least 2 (or $+\infty$). Thus, by Lemma 3.3, g must be constant. This is a contradiction. ∎

In order to prove Theorem 1.2, we need the following preparations.

DEFINITION 3.4 (cf. [TL, Definition 3.1]). Let Ω be a hyperbolic domain and let M be a complete complex Hermitian manifold with metric ds^2_M. A holomorphic mapping $f(z)$ from Ω into M is said to be *normal* if there exists a positive constant C such that for all $z \in \Omega$ and all $\xi \in T_z(\Omega)$,

$$ds^2_M(f(z), df(z)(\xi)) \leq C K_\Omega(z, \xi),$$

where K_Ω denotes the infinitesimal Kobayashi metric on Ω.

LEMMA 3.5 (cf. [TL]). *Let f be a holomorphic mapping from a bounded domain Ω in $\mathbb{C}^m$ into a complete complex Hermitian manifold M such that for every sequence of holomorphic mappings φ_j from the unit disc Δ in $\mathbb{C}$ into Ω, the sequence $\{f \circ \varphi_j\}_{k=1}^{\infty}$ of maps from Δ into M is a normal family on Δ. Then f is a normal holomorphic mapping from Ω into M.*

Proof of Theorem 1.2. Denote by $\operatorname{Reg}(S)$ (resp. $\operatorname{Sing}(S)$) the regular (resp. singular) part of the set S. We will prove that f extends over $\operatorname{Reg}(S)$ to a holomorphic mapping from $\Omega \setminus \operatorname{Sing}(S)$ into X.

For $z_0 \in \operatorname{Reg}(S)$, we take a sufficiently small relatively compact subdomain U of Ω containing z_0 such that $U \cap \operatorname{Sing}(S) = \emptyset$. It suffices to prove that f extends over $S \cap U$ to a holomorphic mapping.

Firstly, we shall prove that f is normal on $U \setminus S$. Indeed, suppose it is not. Then there exists a sequence $\{\varphi_i : \Delta \to U \setminus S\}_{j=1}^{\infty}$ of holomorphic mappings such that $\{f \circ \varphi_j\}$ is not normal.

We now show that $f \circ \varphi_j$ intersects D with multiplicity at least 2 for each φ_j. Indeed, take $a \in f \circ \varphi_j(\Delta) \cap D$, $x_0 \in \Delta$ and $\xi_0 \in U \setminus S$ so that $\varphi_j(x_0) = \xi_0$ and $f(\xi^0) = a$. We choose a holomophic function h defining D on a neighborhood of a. Denote by $(\xi_1, \ldots, \xi_m)$ the standard complex coordinates on Ω. Since f intersects D with multiplicity at least 2, we have

$$\frac{\partial(h \circ f)}{\partial \xi_i}(\xi^0) = 0 \quad (1 \leq i \leq m).$$

It follows that

$$(h \circ f \circ \varphi_j)'(x_0) = \sum_{i=1}^{m} \frac{\partial (h \circ f)}{\partial \xi_i}(\xi^0) \cdot (\varphi_j^i)'(x_0) = 0,$$

where $\varphi_j = (\varphi_j^1, \ldots, \varphi_j^m)$. This shows that $f \circ \varphi_j$ intersects D with multiplicity at least 2.

Then, Theorem 1.1 implies that the sequence $\{f \circ \varphi_j\}$ is normal. This is a contradiction.

Hence f is normal on $U \setminus S$.

We see that $S \cap U$ is a smooth analytic subset of U of codimension 1. Then f extends to a holomorphic mapping from U into X by Theorem 2.3(1) of Joseph and Kwack [JK2].

Therefore, f extends over $\operatorname{Reg}(S)$ to a holomorphic mapping from $\Omega \setminus \operatorname{Sing}(S)$ into X (denoted again by f). Since $\operatorname{Sing}(S)$ is an analytic subset of codimension at least two, by the result of Adachi, Suzuki and Yoshida [ASY], the map f extends over $\operatorname{Sing}(S)$ to a holomorphic mapping from Ω into X.

The proof of Theorem 1.2 is complete. ■

4. Convergence of meromorphic mappings. In this section we recall the notions of convergence introduced by S. Ivashkovich [I] and H. Fujimoto [Fu2]. Then we introduce the notion of *d*-convergence and give some criteria of *d*-normality of meromorphic mappings.

Recall that a meromorphic mapping f from Ω to X is defined by a holomorphic map $f_{[F]} : \Omega \setminus F \to X$, where F is an analytic subset of codimension at least 2, such that the closure $\bar{\Gamma}_{f_{[F]}}$ of its graph is an analytic subset of $\Omega \times X$. From now on this subset is denoted by Γ_f (without bar) and called the *graph* of the map f.

Definition 4.1. We shall say that a family $\{f_n\}_{n=1}^{\infty}$ of meromorphic mappings from a domain $\Omega \subset \mathbb{C}^m$ into a complex space X *strongly converges* (*s-converges*) to a meromorphic mapping $f : \Omega \to X$ if for any compact subset $K \subset \Omega$,

$$\mathop{\mathcal{H}\text{-lim}}_{n \to +\infty} \Gamma_{f_n} \cap (K \times X) = \Gamma_f \cap (K \times X).$$

Here $\mathcal{H}$-lim denotes the limit in the Hausdorff metric, supposing that both Ω and X are equipped with some Hermitian metrics. Note that this notion of convergence does not depend on the choice of Hermitian metrics on Ω and X.

In [I] S. Ivashkovich proved the following theorem.

Theorem 4.2. *Let $\{f_n\}$ be a sequence of meromorphic mappings from Ω to X that strongly converges to a meromorphic mapping f. Then:*

(a) *If f is holomorphic then for any relatively compact open $U \subset \Omega$, all restrictions $f_{n|U}$ are holomorphic for n large enough, and $\{f_n\}$ converges to f on D in the usual sense.*
(b) *If f_n are holomorphic then f is also holomorphic and $\{f_n\}$ converges to f on U in the usual sense.*

Denote by $I(f)$ the *indeterminacy locus* of f, defined to the smallest subset of Ω such that the restriction $f_{|\Omega \setminus I(f)}$ of f on $\Omega \setminus I(f)$ is holomorphic. Then $I(f)$ is an analytic set of codimension at least 2.

In fact, Theorem 4.2 helps us to state the notion of weak convergence introduced by S. Ivashkovich [I] simply as follows.

DEFINITION 4.3. We shall say that a sequence $\{f_n\}_{n=1}^{\infty}$ of meromorphic mappings from a domain $\Omega \subset \mathbb{C}^m$ to a complex space X *weakly converges* (*w-converges*) on Ω to a meromorphic mapping $f : \Omega \to X$ if $\{f_n\}_{n=1}^{\infty}$ converges strongly to f on $\Omega \setminus I(f)$.

It is obvious that strong convergence implies the weak one, but not vice versa. To see this, we consider the following example given by S. Ivashkovich in [I].

EXAMPLE 4.4. Let $f_n(z_1, z_2) = (z_1 : z_2 : 1/n)$ be holomorphic mappings of $\mathbb{C}^2$ into $\mathbb{P}^2(\mathbb{C})$ for $n = 1, 2, \ldots$. Then the sequence f_n weakly converges to $f = (z_1 : z_2 : 0)$, but does not strongly converge because $I(f) = \{(0, 0)\} \neq \emptyset$.

In the case of meromorphic mappings with values in a complex projective space, we have the following.

PROPOSITION 4.5 (cf. [IN, Theorem 2]). *A sequence $\{f_n\}$ of meromorphic mappings from Ω into $\mathbb{P}^N(\mathbb{C})$ weakly converges to f iff each $z \in \Omega$ has a neighborhood U on which f has a reduced representation $f = (f^0 : \cdots : f^N)$ and each f_n has a reduced representation $f_n = (f_n^0 : \cdots : f_n^N)$ such that f_n^j converges to f_j as holomorphic functions for all $j = 0, \ldots, N$.*

Hence even in the case where X is $\mathbb{P}^N(\mathbb{C})$, we do not have any information on the indeterminacy locus of f. In fact, to solve this problem, let us recall the notion of meromorphic convergence introduced by H. Fujimoto [Fu2] as follows.

DEFINITION 4.6 (cf. [Fu2, Definition 3.1]). We say that a sequence $\{f_n\}$ of meromorphic mappings from Ω into $\mathbb{P}^N(\mathbb{C})$ *meromorphically converges* (*m-converges*) to f if each $z \in \Omega$ has a neighborhood U on which f has a representation $f = (f^0 : \cdots : f^N)$ and each f_n has a reduced representation $f_n = (f_n^0 : \cdots : f_n^N)$ such that f_n^j converges to f_j as holomorphic functions on U for all $j = 0, \ldots, N$.

A natural question arises: *What is the relation between the notions of weak convergence and meromorphic convergence in the case of mappings* *into the complex projective space?* To answer this question, we need the following.

For a domain G in $\mathbb{C}^m$, denote by $\mathcal{D}^+(G)$ the space of all nonnegative divisors on G. Let S be an analytic subset of codimension at least 2 in G. It is known that any divisor $\nu \in \mathcal{D}^+(G \setminus S)$ can be uniquely extended to $\hat{\nu} \in \mathcal{D}^+(G)$. Moreover, we have

LEMMA 4.7 (cf. [Fu2, p. 26]). *If $\{\nu_p : p = 1, 2, \ldots\}$ in $\mathcal{D}^+(G\setminus S)$ converges to ν on $G \setminus S$, then $\{\hat{\nu}_p\}$ converges to $\hat{\nu}$ in $\mathcal{D}^+(G)$, where $\{\hat{\nu}_p\}$ and $\hat{\nu}$ are the extensions of ν_p and ν in G respectively.*

LEMMA 4.8 (cf. [Fu2, Proposition 3.5]). *Let $\{f_i\}$ be a sequence from meromorphic mappings of a domain Ω in $\mathbb{C}^m$ into $\mathbb{P}^N(\mathbb{C})$ and let S be a thin analytic subset of Ω. Suppose that $\{f_i\}$ meromorphically converges on $\Omega \setminus S$ to a meromorphic mapping f from $\Omega \setminus S$ into $\mathbb{P}^N(\mathbb{C})$. If there exists a hyperplane H in $\mathbb{P}^N(\mathbb{C})$ such that $f(\Omega \setminus S) \not\subset H$ and $\{f_i^* H\}$ is a convergent sequence of divisors on Ω, then $\{f_i\}$ is meromorphically convergent on Ω.*

We now show that weak convergence implies the meromorphic one.

PROPOSITION 4.9. *Let $\{f_n\}$ be a sequence of meromorphic mappings from a domain $\Omega \subset \mathbb{C}^m$ into $\mathbb{P}^N(\mathbb{C})$. Assume that $\{f_n\}$ weakly converges to a meromorphic mapping $f : \Omega \to \mathbb{P}^N(\mathbb{C})$. Then $\{f_n\}$ meromorphically converges to f on Ω.*

Proof. Take a hyperplane H in $\mathbb{P}^N(\mathbb{C})$ such that $f(\Omega \setminus S) \not\subset H$. We may assume that $H = \{\omega_0 = 0\}$, where $(\omega_0 : \cdots : \omega_N)$ are homogeneous coordinates of $\mathbb{P}^N(\mathbb{C})$. By Proposition 4.5, the sequence $\{f_n^* H|_{\Omega\setminus I(f)}\}$ of divisors converges to $f^* H|_{\Omega\setminus S}$. Then by Lemma 4.7, $\{f_n^* H\}$ converges to $f^* H$ on Ω. Hence $\{f_n\}$ meromorphically converges to f on Ω by Lemma 4.8. ∎

However, the next example shows that meromorphic convergence does not imply the weak one.

EXAMPLE 4.10. Let $f_n = (z : z + 1/n)$ and $f = (z : z)$ be holomorphic mappings of $\mathbb{C}$ into $\mathbb{P}^1(\mathbb{C})$. Then $f_n^0 = z$ (resp. $f_n^1 = z + 1/n$) converges to $f^0 = z$ (resp. $f^1 = z$) as holomorphic functions, but $\{f_n\}$ does not converge to f, since $f_n(0) = (0 : 1) \neq f(0) = (1 : 1)$ for every $n \geq 1$.

Thus we have the following relatios between convergence notions: strong convergence $\Rightarrow$ weak convergence $\Rightarrow$ meromorphic convergence (see also [IN, Theorem 2]). Of course Examples 4.4 and 4.10 show that all $\Leftarrow$ directions are untrue.

Let X be a complex projective variety and let D be an ample divisor on X. We now generalize the notion of meromorphic convergence of H. Fujimoto to the case of meromorphic mappings from Ω into X as follows.

Definition 4.11. We say that a sequence $\{f_n\}_{n=1}^{\infty}$ of meromorphic mappings from a domain $\Omega \subset \mathbb{C}^m$ to X *d-converges* on Ω to a meromorphic mapping $f : \Omega \to X$ with respect to D if $\{f_n\}_{n=1}^{\infty}$ converges strongly to f on $\Omega \setminus \mathrm{Supp}(f^*D)$ and the divisors f_n^*D converge to f^*D.

Denote by L the line bundle defined by D on X. Take a holomorphic section $\sigma \in \Gamma(X, L)$ such that $\mathrm{div}(\sigma) = D$. Since D is ample, there exists a positive number l such that L^l is very ample. Take a basis $(\sigma_0, \ldots, \sigma_N)$ of the $\mathbb{C}$-vector space $\Gamma(X, L^l)$ such that $\sigma_0 = \sigma^l$. Hence there is a holomorphic embedding $\Phi : X \to \mathbb{P}^N(\mathbb{C})$ of X into $\mathbb{P}^N(\mathbb{C})$ defined by $\Phi(x) = (\sigma_0 : \cdots : \sigma_N)$ in homogeneous coordinates $(\omega_0 : \cdots : \omega_N)$ of $\mathbb{P}^N(\mathbb{C})$. Put $H = \{\omega_0 = 0\}$. Then H is a hyperplane of $\mathbb{P}^N(\mathbb{C})$ satisfying $\Phi^*H = lD$.

Assume that f_n d-converges to f on Ω with respect to D. We regard f_n and f as holomorphic mappings from $\Omega \setminus D$ into $\mathbb{P}^N(\mathbb{C})$. By assumption, $\{f_n^*H\}$ converges to f^*H. Hence, by Lemma 4.8, $\{f_n\}$ converges to f meromorphically on Ω.

In the case where X is the complex projective space, Ivashkovich and Neji [IN] proved the equivalence of meromorphic convergence and Γ-convergence. In the next proposition, we will show that they coincide with d-convergence.

Proposition 4.12. *Let $\{f_n\}$ be a sequence of meromorphic mapings from a domain $\Omega \subset \mathbb{C}^m$ into $\mathbb{P}^N(\mathbb{C})$ and let D be an arbitrary divisor on $\mathbb{P}^N(\mathbb{C})$. Then $\{f_n\}$ d-converges to a meromorphic mapping f on Ω with respect to D if and only if it converges to f meromorphically on Ω.*

Proof. The "if" part is clear. We now prove the "only if" part. Assume that $\{f_n\}$ d-converges to f on Ω with respect to D.

For fixed homogeneous coordinates $(\omega_0 : \cdots : \omega_N)$ of $\mathbb{P}^N(\mathbb{C})$, let Q be a homogeneous polynomial in $\omega_0, \ldots, \omega_N$ defining D.

We define meromorphic mappings $\{F_n\}_{n=1}^{\infty}$ of Ω into $\mathbb{P}^{N+1}(\mathbb{C})$ as follows: for any $z \in \Omega$, if f_n has a reduced representation $f_n = (f_{n0} : \cdots : f_{nN})$ on a neighborhood $U_z \subset \Omega$ then F_n has a reduced representation $F_n = (f_{n0}^d : \ldots : f_{nN}^d : Q(f_n))$ on U_z, where d is the degree of Q. Let H be the hyperplane in $\mathbb{P}^{N+1}(\mathbb{C})$ defined by $H = \{\omega_{N+1} = 0\}$, where $(\omega_0 : \cdots : \omega_N : \omega_{N+1})$ are homogeneous coordinates of $\mathbb{P}^{N+1}(\mathbb{C})$.

It is easy to see that $\{F_n\}$ converges to a meromorphic mapping F from $\Omega \setminus \mathrm{Supp}\, f^*D$ into $\mathbb{P}^{N+1}(\mathbb{C})$, and if f has a reduced representation $f = (f_0 : \cdots : f_n)$ on an open subset $U \subset \Omega$ then F has reduced representation $F = (f_0^d : \cdots : f_n^d : Q(f))$ on U. Since $\{f_n^*D\}$ converges to f^*D, we see that $\{F_n^*H\}$ converges to F^*H. By Lemma 4.8, $\{F_n\}$ converges to F meromorphically on Ω. This implies that $\{f_n\}$ converges to f meromorphically on Ω. ∎

As usual, we say a family $\mathcal{F}$ of meromorphic mappings from a domain Ω into X is *d-normal* if every sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ has a subsequence d-convergent on Ω.

Let $\{D_i\}_{i=1}^{l}$ be l divisors on X. Assume that $\dim X = n$. We say that $\{D_i\}_{i=1}^{l}$ is *in general position* if for any distinct indices $1 \le i_1, \ldots, i_k \le l$, the codimension of any irreducible component of $\bigcap_{j=1}^{k} \operatorname{Supp} D_{i_j}$ is k for $k \le n$, and $\bigcap_{j=1}^{k} \operatorname{Supp} D_{i_j} = \emptyset$ for $k > n$. Denote by $c_1(D_i)$ the Chern class of the line bundle defined by D_i.

We now give a criterion of d-normality of meromorphic mappings:

THEOREM 4.13. *Let $\mathcal{F}$ be a family of meromorphic mappings from a domain $\Omega \subset \mathbb{C}^m$ into a complex projective variety X of dimension n, and let $D_1, \ldots, D_l$ $(l > n \operatorname{rank}_{\mathbb{Z}}\{c_1(D_i)\})$ be l ample divisors on X in general position such that:*

- (i) *For any fixed compact subset K of Ω, the $2(n-1)$-dimensional Lebesgue areas of $f^{-1}(D_i) \cap K$ $(1 \le j \le n+1)$ counted with multiplicities for all f in $\mathcal{F}$ are bounded from above.*
- (ii) *For any fixed compact subset K of Ω, the $2(m-1)$-dimensional Lebesgue areas of $f^{-1}(Q_j) \cap K$ $(n+2 \le i \le l)$ regardless of multiplicities for all f in $\mathcal{F}$ are bounded from above.*

Then $\mathcal{F}$ is a d-normal family on Ω with respect to some D_i $(1 \le i \le n+1)$.

In order to prove Theorem 1.4, we need the following preparations.

LEMMA 4.14 (cf. [St, Proposition 4.12]). *Let $\{N_i\}_{i=1}^{\infty}$ be a sequence of purely $(m-1)$-dimensional analytic subsets of a domain D in $\mathbb{C}^m$. If the $2(m-1)$-dimensional Lebesgue areas of $N_i \cap K$ regardless of multiplicities $(i = 1, 2, \ldots)$ are bounded from above for any fixed compact subset K of D, then $\{N_i\}$ is normal in the sense of convergence of closed subsets in D.*

LEMMA 4.15 (cf. [St, Theorem 2.24]). *A sequence $\{\nu_i\}_{i=1}^{\infty}$ of nonnegative divisors on a domain D in $\mathbb{C}^m$ is normal in the sense of convergence of divisors on D if and only if the $2(m-1)$-dimensional Lebesgue areas of $\nu_i \cap E$ $(i = 1, 2, \ldots)$ counted with multiplicities are bounded from above for any compact subset E of D.*

Proof of Theorem 4.13. Without loss of generality, we may assume that Ω is a polydisc in $\mathbb{C}^m$, $\Omega = \Delta^m$.

Let $\{f_k\}_{k=1}^{\infty} \subset \mathcal{F}$ be an arbitrary sequence. By Lemma 4.14, there exists a subsequence (again denoted by $\{f_k\}_{i=k}^{\infty}$) such that

$$\lim_{k\to\infty} \operatorname{Supp} f_k^* D_i = S_i \quad (i = 1, \ldots, l) \tag{4.1}$$

as a sequence of closed subsets of Ω, where each S_i $(i = 1, \dots, q)$ is either an empty set or a purely $(m-1)$-dimensional analytic subset of Ω.

Set $E := \bigcup_{i=1}^{l} S_i$. Then E is a thin analytic subset of Ω. For any fixed $z_0 \in \Omega \setminus E$, there exist a relatively compact Stein neighborhood U_{z_0} of z_0 in $\Omega \setminus E$ and a positive integer k_0 such that for all $k \geq k_0$,

$$\operatorname{Supp} f_k^* D_i \cap U_{z_0} = \emptyset. \tag{4.2}$$

It is easy to see that $I(f_k) \subset \operatorname{Supp} f_k^* D_i$ for all k; then $\{f_k|_{U_{z_0}}\}_{k=k_0}^{\infty} \subset \operatorname{Hol}(U_{z_0}, X)$. We now prove that $\{f_k|_{U_{z_0}}\}_{k=k_0}^{\infty}$ is a normal family on U_{z_0}. Indeed, suppose it is not; then by Lemma 2.2 there exist a subsequence (again denoted by $\{f_k|_{U_{z_0}}\}_{k=k_0}^{\infty}$) and $p_0 \in U_{z_0}, \{p_k\}_{k\geq k_0}^{\infty} \subset U_{z_0}$ with $p_k \to p_0, \{\rho_k\} \subset (0, +\infty)$ with $\rho_k \to 0$ and $\xi_k \in \mathbb{C}^m$ Euclidean unit vectors such that the sequence of holomorphic maps

$$g_k := f_k(p_k + \rho_k \xi_k \cdot) : \Delta_{r_k} \to X, \quad k \geq k_0 \ (r_k \to \infty)$$

converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic map $g : \mathbb{C} \to X$. Then by (4.2) and by Hurwitz's theorem, it is easy to see that for each $i \in \{1, \dots, l\}$ we have either

- $g(\mathbb{C}) \cap \operatorname{Supp} D_i = \emptyset$, or
- $g(\mathbb{C}) \subset \operatorname{Supp} D_i$.

Denote by Z the Zariski closure of $g(\mathbb{C})$ which is a subvariety of X. Then we can regard g as a nonconstant holomorphic curve $g : \mathbb{C} \to Z \setminus \bigcup_{D_i \not\supset Z} D_i$ with Zariski dense image. Since $\{D_i\}_{i=1}^{l}$ are in general position in X, by Theorem 1.2(ii) in [NW1], we find that $l \leq n \operatorname{rank}_{\mathbb{Z}}\{[c_1(D_i)]\}$. This is a contradiction, hence $\{f_k|_{U_{z_0}}\}_{k=k_0}^{\infty}$ is a normal family on U_{z_0}.

By the usual diagonal argument, we can find a subsequence (again denoted by $\{f_k\}_{k=1}^{\infty}$) which converges uniformly on compact subsets of $\Omega \setminus E$ to a holomorphic map f. Since $\{D_i\}_{i=1}^{n+1}$ are in general position, there exists a fixed index i_0 $(1 \leq i_0 \leq n+1)$ such that $f(\Omega \setminus E) \not\subset \operatorname{Supp} D_i$. By assumption (ii) and by Lemma 4.15, $f_k^* D_{i_0}$ converges in the sense of convergence of divisors on Ω to a divisor. This implies that $\{f_k\}_{k\geq k_0}$ d-converges to f on Ω. Thus $\mathcal{F}$ is a d-normal family on Ω. We have completed the proof of Theorem 4.13. ■

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