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Note on the Jacobian condition and the non-proper value set

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Abstract. We show that the non-proper value set of a polynomial map $(P,Q):\mathbb{C}^2\to\mathbb{C}^2$ satisfying the Jacobian condition $\det D(P,Q)\equiv \text{const}\neq 0$, if non-empty, must be a plane curve with one point at infinity.

1. Let $f=(P,Q):\mathbb{C}^2_{(x,y)}\to\mathbb{C}^2_{(u,v)}$ be a dominant polynomial map, $P,Q\in\mathbb{C}[x,y]$, and define $J(P,Q):=P_xQ_y-P_yQ_x$. Recall that the so-called *non-proper value set* A_f of f consists of all points $a\in\mathbb{C}^2$ such that the inverse $f^{-1}(K)$ is not compact for any compact neighborhood $K\subset\mathbb{C}^2$ of a. This set A_f, if non-empty, must be a plane curve such that each of its irreducible components can be parameterized by a non-constant polynomial map from $\mathbb{C}$ into $\mathbb{C}^2$ (see [J]). The mysterious Jacobian conjecture (see [BCW] and [E]), posed first by Keller in 1939 and still open, asserts that a polynomial map $f=(P,Q)$ of $\mathbb{C}^2$ with $J(P,Q)\equiv\text{const}\neq 0$ must have a polynomial inverse. This conjecture can be reduced to proving that the non-proper value set A_f is empty. Anyway one may think that in a counterexample to the Jacobian conjecture, if one exists, the non-proper value set must have a very special form. In [C] it was observed that in such a counterexample the irreducible components of A_f can be parameterized by polynomial maps $\xi\mapsto(p(\xi),q(\xi))$ with $\deg p/\deg q=\deg P/\deg Q$. In this paper we notice that the non-proper value set of a nonsingular polynomial map from $\mathbb{C}^2$ into itself, if non-empty, must be a curve with one point at infinity.

THEOREM 1. *Suppose* $f=(P,Q)$ *is a polynomial map of* $\mathbb{C}^2$ *with* $J(P,Q)\equiv\text{const}\neq 0$, $\deg P=\deg_y P=Kd$ *and* $\deg Q=\deg_y Q=Ke$, $\gcd(d,e)=1$,

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$$(1) \qquad \begin{aligned} P(x,y) &= Ay^{Kd} + \cdots + a_1(x)y + a_0(x), \qquad A \neq 0,\\ Q(x,y) &= By^{Ke} + \cdots + b_1(x)y + b_0(x), \qquad B \neq 0. \end{aligned}$$

If the non-proper value set A_f is not empty, then every irreducible component of A_f can be parameterized by polynomial maps of the form

$$(2) \qquad \xi \mapsto (A\xi^{md} + \textit{lower order terms in } \xi,\ B\xi^{me} + \textit{lower order terms in } \xi), \qquad m \in \mathbb{N}.$$

By definition A_f is the set of all values $a \in \mathbb{C}^2$ such that the number of solutions counted with multiplicities of the equation $f(x,y) = a$ is different from those for generic values in $\mathbb{C}^2$. Then, considering the components $P(x,y)$ and $Q(x,y)$ as elements of $\mathbb{C}[x][y]$, we can define the resultant

$$(3) \qquad \mathrm{Res}_y(P-u, Q-v) = R_0(u,v)x^N + \cdots + R_N(u,v),$$

where $R_i \in \mathbb{C}[u,v]$, $R_0 \neq 0$. From the basic properties of the resultant function we know that N is the geometric degree of f and $A_f = \{(u,v) \in \mathbb{C}^2 : R_0(u,v) = 0\}$. Note that a curve given by a polynomial parameter of the form (1) can be defined by a polynomial of the form $(A^e u^e - B^d v^d)^m + \sum_{0 \le id+je < mde} c_{ij} u^i v^j$ and its branch at infinity has a Newton–Puiseux series of the form $u = cv^{d/e} +$ lower order terms in v, where c is a dth root of B^d/A^e. Thus, Theorem 1 leads to

Corollary 1. *Let f be as in Theorem* 1. *Then*

$$(4) \qquad R_0(u,v) = C(A^e u^e - B^d v^d)^M + \sum_{0 \le id+je < Mde} c_{ij} u^i v^j$$

with $0 \neq C \in \mathbb{C}$ and $M \geq 0$.

Corollary 2. *Let f be as in Theorem* 1. *If $A_f \neq \emptyset$, then A_f is a curve with one point at infinity and the irreducible branches at infinity of A_f have Newton–Puiseux series of the form*

$$u = cv^{d/e} + \textit{lower order terms in } v$$

with coefficients c being dth roots of B^d/A^e.

As seen later, the representation in (1) of P and Q is only used to visualize the coefficient B^d/A^e. In fact, when $A_f \neq \emptyset$ the numbers d, e, B^d/A^e and the polynomial $R_0(u,v)$ are invariant under right actions of automorphisms of $\mathbb{C}^2$, since the set A_f does not depend on the coordinate (x,y). Furthermore, the coefficient B^d/A^e is uniquely determined from the relation

$$P_+^e(x,y) = (B^d/A^e)Q_+^d(x,y),$$

which is a consequence of the Jacobian condition when $\deg P > 1$ and $\deg Q > 1$. Here, P_+ and Q_+ are the leading homogeneous components of P and Q, respectively.

Theorem 1 will be proved in Sections 2–5 in an elementary way by using Newton–Puiseux expansions and the Newton theorem. It would be interesting to determine the form of $R_0(u,v)$ by examining directly the resultant function $\mathrm{Res}_y(P-u, Q-v)$.

2. Dicritical series of f. From now on, $f = (P,Q) : \mathbb{C}^2_{(x,y)} \to \mathbb{C}^2_{(u,v)}$ is a given polynomial map with $J(P,Q) \equiv \mathrm{const} \neq 0$, $\deg P = Kd > 0$ and $\deg Q = Ke > 0$, $\gcd(d,e) = 1$. The Jacobian condition will be used really in Lemma 3 and the proof of Theorem 1. Since A_f does not depend on the coordinate (x,y), to examine it we can assume that $\deg_y P = \deg P$, $\deg_y Q = \deg Q$ and

$$\begin{aligned} P(x,y) &= Ay^{Kd} + \cdots + a_1(x)y + a_0(x), \quad A \neq 0, \\ Q(x,y) &= By^{Ke} + \cdots + b_1(x)y + b_0(x), \quad B \neq 0. \end{aligned} \tag{5}$$

With this representation the Newton–Puiseux roots at infinity $y(x)$ of each of the equations $P(x,y) = 0$ and $Q(x,y) = 0$ are fractional power series of the form

$$y(x) = \sum_{k=0}^{\infty} c_k x^{1-k/m}, \quad m \in \mathbb{N},\ \gcd\{k : c_k \neq 0\} = 1,$$

for which the map $\tau \mapsto (\tau^m, y(\tau^m))$ is meromorphic and injective for τ large enough. In view of the Newton theorem we can represent

$$P(x,y) = A \prod_{i=1}^{\deg P} (y - u_i(x)), \quad Q(x,y) = B \prod_{j=1}^{\deg Q} (y - v_j(x)), \tag{6}$$

where $u_i(x)$ and $v_j(x)$ are the Newton–Puiseux roots at infinity of the equations $P = 0$ and $Q = 0$, respectively. We refer the readers to [A] and [BK] for the Newton theorem and the Newton–Puiseux roots.

We begin with the description of the non-proper value set A_f of f via Newton–Puiseux expansions. We will work with finite fractional power series $\varphi(x,\xi)$ of the form

$$\varphi(x,\xi) = \sum_{k=1}^{K-1} a_k x^{1-k/m} + \xi x^{1-K/m}, \quad m \in \mathbb{N},\ \gcd\{k : a_k \neq 0\} = 1, \tag{7}$$

where ξ is a parameter. For convenience, we set $\mathrm{mult}(\varphi) := m$. Such a series φ is called a *dicritical series* of f if

$$f(x, \varphi(x,\xi)) = f_\varphi(\xi) + \text{lower order terms in } x, \quad \deg f_\varphi > 0.$$

The following description of A_f was given in [C].

LEMMA 1 ([C, Theorem 4.4]).

$$A_f = \bigcup_{\varphi \text{ is a dicritical series of } f} f_\varphi(\mathbb{C}).$$

To see this, note that by definition the non-proper value set A_f consists of all values $a \in \mathbb{C}^2$ such that there exists a sequence $\mathbb{C}^2 \ni p_i \to \infty$ with $f(p_i) \to a$. If φ is a dicritical series of f of the form (7), we can define the map $\Phi(t,\xi) := (t^{-m}, \varphi(t^{-m}, \xi))$. Then Φ sends $\mathbb{C}^* \times \mathbb{C}$ to $\mathbb{C}^2$ and the line $\{0\} \times \mathbb{C}$ to the line at infinity of $\mathbb{CP}^2$. The polynomial map $F_\varphi(t,\xi) := f \circ \Phi(t,\xi)$ sends the line $\{0\} \times \mathbb{C}$ to $A_f \subset \mathbb{C}^2$. Therefore, $f_\varphi(\mathbb{C})$ is an irreducible component of A_f, since $\deg f_\varphi > 0$. Conversely, if ℓ is an irreducible component of A_f, one can choose a smooth point (u_0, v_0) of A_f, $(u_0, v_0) \in \ell$, and an irreducible branch at infinity γ of the curve $P = u_0$ (or $Q = v_0$) such that the image $f(\gamma)$ is a branch curve intersecting ℓ transversally at (u_0, v_0). Let $u(x)$ be the Newton–Puiseux expansion of γ at infinity. Then we can construct a unique dicritical series $\varphi(x,\xi)$ such that $u(x) = \varphi(x, \xi_0 + \text{lower order terms in } x)$. For this dicritical series φ we have $f_\varphi(\mathbb{C}) = \ell$.

3. Associated sequence of a dicritical series. Let φ be a given dicritical series of f. Let us represent it as

$$\varphi(x,\xi) = \sum_{k=0}^{K-1} c_k x^{1-n_k/m_k} + \xi x^{1-n_K/m_K}, \tag{8}$$

where $0 \le n_0/m_0 < n_1/m_1 < \cdots < n_{K-1}/m_{K-1} < n_K/m_K = n_\varphi/m_\varphi$ and $c_i \in \mathbb{C}$ may be zero, so that the sequence $\{\varphi_i\}_{i=0,1,\ldots,K}$ of series defined by

$$\varphi_i(x,\xi) := \sum_{k=0}^{i-1} c_k x^{1-n_k/m_k} + \xi x^{1-n_i/m_i}, \quad i = 0,1,\ldots,K-1, \tag{9}$$

and $\varphi_K := \varphi$ has the following properties:

(S1) $\operatorname{mult}(\varphi_i) = m_i$.

(S2) For every $i < K$ at least one of the polynomials p_{φ_i} and q_{φ_i} has a root different from zero.

(S3) For every $\psi(x,\xi) = \varphi_i(x, c_i) + \xi x^{1-\alpha}$, $n_i/m_i < \alpha < n_{i+1}/m_{i+1}$, each of the polynomials p_ψ and q_ψ is either constant or a monomial in ξ.

The representation (8) of φ is thus the longest representation such that for each i there is a Newton–Puiseux root $y(x)$ of $P = 0$ or $Q = 0$ such that $y(x) = \varphi_i(x, c + \text{lower order terms in } x)$ and $c \neq 0$ if $c_i = 0$. This representation and the associated sequence $\{\varphi_i\}_{i=0,1,\ldots,K}$ are well defined and unique. Further, $\varphi_0(x,\xi) = \xi x$.

We will use the sequence $\{\varphi_i\}$ to determine the form of the polynomials $f_\varphi(\xi)$. For simplicity of notation, below we use lower indices "i" instead of the lower indices "φ_i".

For each φ_i, $i = 0, \ldots, K$, let us write

$$(10)\qquad \begin{aligned} P(x, \varphi_i(x, \xi)) &= p_i(\xi) x^{a_i/m_i} + \text{lower order terms in } x,\\ Q(x, \varphi_i(x, \xi)) &= q_i(\xi) x^{b_i/m_i} + \text{lower order terms in } x, \end{aligned}$$

where $p_i, q_i \in \mathbb{C}[\xi] \setminus \{0\}$, $a_i, b_i \in \mathbb{Z}$ and $m_i := \operatorname{mult}(\varphi_i)$.

Let $\{u_i(x) : i = 1, \ldots, \deg P\}$ and $\{v_j(x) : j = 1, \ldots, \deg Q\}$ be the collections of the Newton–Puiseux roots of $P = 0$ and $Q = 0$, respectively. As shown in Section 2, by the Newton theorem the polynomials $P(x, y)$ and $Q(x, y)$ can be factorized as

$$(11)\qquad P(x, y) = A \prod_{i=1}^{\deg P} (y - u_i(x)), \qquad Q(x, y) = B \prod_{j=1}^{\deg Q} (y - v_j(x)).$$

For each $i = 0, \ldots, K$, define

$$\begin{aligned} S_i &:= \{k : 1 \le k \le \deg P,\\ &\qquad\qquad u_k(x) = \varphi_i(x, a_{ik} + \text{lower order terms in } x), a_{ik} \in \mathbb{C}\},\\ T_i &:= \{k : 1 \le k \le \deg Q,\ v_k(x) = \varphi_i(x, b_{ik} + \text{lower terms in } x), b_{ik} \in \mathbb{C}\},\\ S_i^0 &:= \{k \in S_i : a_{ik} = c_i\}, \qquad T_i^0 := \{k \in T_i : b_{ik} = c_i\}. \end{aligned}$$

Write

$$\begin{aligned} p_i(\xi) &= A_i \bar{p}_i(\xi)(\xi - c_i)^{\# S_i^0}, \quad \bar{p}_i(\xi) := \prod_{k \in S_i \setminus S_i^0} (\xi - a_{ik}),\\ q_i(\xi) &= B_i \bar{q}_i(\xi)(\xi - c_i)^{\# T_i^0}, \quad \bar{q}_i(\xi) := \prod_{k \in T_i \setminus T_i^0} (\xi - b_{ik}). \end{aligned}$$

LEMMA 2. (i) $n_0 = 0$, $m_0 = 1$ *and*

$$\begin{aligned} A_0 &= A, \qquad \deg p_0 = a_0 = Kd,\\ B_0 &= B, \qquad \deg q_0 = b_0 = Ke. \end{aligned}$$

(ii) *For* $i = 1, \ldots, K$,

$$\begin{aligned} A_i &= A_{i-1} \bar{p}_{i-1}(c_{i-1}), \qquad \deg p_i = \# S_i = \# S_{i-1}^0,\\ \frac{a_i}{m_i} &= \frac{a_{i-1}}{m_{i-1}} + \# S_{i-1}^0 \left(\frac{n_{i-1}}{m_{i-1}} - \frac{n_i}{m_i} \right),\\ B_i &= B_{i-1} \bar{q}_{i-1}(c_{i-1}), \qquad \deg q_i = \# T_i = \# T_{i-1}^0,\\ \frac{b_i}{m_i} &= \frac{b_{i-1}}{m_{i-1}} + \# T_{i-1}^0 \left(\frac{n_{i-1}}{m_{i-1}} - \frac{n_i}{m_i} \right). \end{aligned}$$

Proof. Note that $\varphi_0(x, \xi) = \xi x$ and $\varphi_i(x, \xi) = \varphi_{i-1}(x, c_{i-1}) + \xi x^{1 - n_i/m_i}$ for $i > 0$. Then, substituting $y = \varphi_i(x, \xi)$, $i = 0, 1, \ldots, K$, into the Newton factorizations of $P(x, y)$ and $Q(x, y)$ in (11) one can easily verify the conclusions. ∎

4. The Jacobian condition. Let φ be a dicritical series of f and $\{\varphi_i\}$ be its associated series. Define

$$J_i(\xi) := a_i p_i(\xi)\dot{q}_i(\xi) - b_i\dot{p}_i(\xi)q_i(\xi).$$

The Jacobian condition will be considered in the following sense.

LEMMA 3. *Let $0 \le i < K$. If $a_i > 0$ and $b_i > 0$, then*

$$J_i(\xi) \equiv \begin{cases} -m_i J(P,Q) & \text{if } a_i + b_i = 2m_i - n_i, \\ 0 & \text{if } a_i + b_i > 2m_i - n_i. \end{cases}$$

Further, $J_i(\xi) \equiv 0$ if and only if $p_i(\xi)$ and $q_i(\xi)$ have a common root. In this case

$$p_i(\xi)^{b_i} = Cq_i(\xi)^{a_i}, \qquad C \in \mathbb{C}^*.$$

Proof. Since $a_i > 0$ and $b_i > 0$, differentiating $f(t^{-m_i}, \varphi_i(t^{-m_i}, \xi))$ with respect to t, we obtain

$$m_i J(P,Q)t^{n_i - 2m_i - 1} + \text{higher order terms in } t$$
$$= -J_i(\xi)t^{-a_i - b_i - 1} + \text{higher order terms in } t.$$

Comparing the two sides we get the first conclusion. The remaining ones are left to the reader as an elementary exercise. ∎

5. Proof of Theorem 1. (i) Assume that $A_f \neq \emptyset$. Then A_f is a plane curve in $\mathbb{C}^2$. Let ℓ be an irreducible component of A_f. By Lemma 1 there is a dicritical series φ of f such that ℓ can be parameterized by the polynomial map $f_\varphi(\xi) = (p_\varphi(\xi), q_\varphi(\xi))$, i.e. $\ell = f_\varphi(\mathbb{C})$. We will show that

$$(12) \qquad f_\varphi(\xi) = (AC_\varphi^d \xi^{D_\varphi d} + \cdots, BC_\varphi^e \xi^{D_\varphi e} + \cdots), \qquad C_\varphi \neq 0,\ D_\varphi \in \mathbb{N}.$$

Then by changing variable $\xi \mapsto C_\varphi^{-1}\xi$ we get the desired parameterization $\xi \mapsto (A\xi^{D_\varphi d} + \cdots, B\xi^{D_\varphi e} + \cdots)$ of ℓ.

(ii) Consider the associated sequence $\{\varphi_i\}_{i=1}^K$ of φ. Since $A_f \neq \emptyset$, we have

$$\deg P > 1, \qquad \deg Q > 1.$$

Otherwise, f is bijective and $A_f = \emptyset$. Since φ is a dicritical series of f, without loss of generality we can assume that

$$\deg p_K > 0, \qquad a_K = 0, \qquad b_K \le 0.$$

Then from the construction of the sequence φ_i it follows that

$$(13) \qquad \begin{cases} p_i(c_i) = 0 \text{ and } a_i > 0, & i = 0, 1, \ldots, K-1, \\ q_i(c_i) = 0 & \text{if } b_i > 0. \end{cases}$$

This allows us to use the Jacobian condition in the sense of Lemma 3. Then, by induction using Lemma 2, Lemma 3 and (13) we can obtain without difficulty the following.

ASSERTION. *For $i = 0, 1, \ldots, K-1$ we have*

(a) $$a_i > 0, \quad b_i > 0,$$

(b) $$\frac{a_i}{b_i} = \frac{\#S_i}{\#T_i} = \frac{d}{e},$$

(c) $$\frac{\#S_i^0}{\#T_i^0} = \frac{d}{e}, \quad \bar{p}_i(\xi)^e = \bar{q}_i(\xi)^d.$$

(iii) Now, we prove (12). By Lemma 2(iii) and (b-c) we have

$$\begin{aligned}\frac{b_K}{m_K} &= \frac{b_{K-1}}{m_{K-1}} + \#T^0_{K-1}\bigg(\frac{n_{K-1}}{m_{K-1}} - \frac{n_K}{m_K}\bigg)\\ &= \frac{e}{d}\bigg[\frac{a_{K-1}}{m_{K-1}} + \#S^0_{K-1}\bigg(\frac{n_{K-1}}{m_{K-1}}\frac{n_K}{m_K}\bigg)\bigg]\\ &= \frac{e}{d}\,\frac{a_K}{m_K} = 0,\end{aligned}$$

as $a_K = 0$. Hence, $f_\varphi(\xi) = (p_K(\xi), q_K(\xi))$ by definition and (a). Using Lemma 2(ii)–(iii) to compute the coefficients A_K and B_K we get

$$A_K = A\Big(\prod_{k\le K-1} \bar{p}_k(c_k)\Big), \quad B_K = B\Big(\prod_{k\le K-1} \bar{q}_k(c_k)\Big).$$

Let C_φ be a dth root of $\prod_{k\le K-1} \bar{p}_k(c_k)$ and $D_\varphi := \gcd(\#S^0_{K-1}, \#T^0_{K-1})$. Then, by Lemma 2(ii) and (b-c) we have $A_K = AC_\varphi^d$, $B_K = BC_\varphi^e$, $\deg p_K = \#S^0_{K-1} = D_\varphi d$ and $\deg q_K = \#T^0_{K-1} = D_\varphi e$. Thus,

$$f_\varphi(\xi) = (AC_\varphi^d \xi^{D_\varphi d} + \cdots, BC_\varphi^e \xi^{D_\varphi e} + \cdots). \ \blacksquare$$

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