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# Existence results for a class of quasilinear integrodifferential equations of Volterra–Hammerstein type with nonlinear boundary conditions

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**Abstract.** The existence of a solution for a class of quasilinear integrodifferential equations of Volterra–Hammerstein type with nonlinear boundary conditions is established. Such equations occur in the study of the $p$-Laplace equation, generalized reaction-diffusion theory, non-Newtonian fluid theory, and in the study of turbulent flows of a gas in a porous medium. The results are obtained by using upper and lower solutions, and extend some previously known results.

In this paper we study existence results for the integrodifferential equation

$$(\Phi_m(u'))' = f(t, u, T_1u, T_2u, u'), \qquad t \in I = [0,1], \tag{1}$$

subject to one of the following boundary conditions:

$$g(u(0), u(1), u'(0), u'(1)) = 0, \qquad h(u(0)) = u(1), \tag{2}$$

or

$$p(u(0), u'(0)) = 0 = q(u(0), u'(0), u(1), u'(1)) \tag{3}$$

or

$$r(u(1), u'(1)) = 0 = w(u(0), u'(0), u(1), u'(1)), \tag{4}$$

where

$$T_1u(t) = \psi_1(t) + \int_0^t K_1(t,s)u(s)\,ds, \qquad T_1u(t) = \psi_2(t) + \int_0^t K_2(t,s)u(s)\,ds,$$

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$K_i \in C([0,1] \times [0,1], \mathbb{R}^+)$, $\psi_i \in C([0,1], \mathbb{R})$, $i = 1, 2$, $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function, and $\Phi_m(s) = |s|^{m-2}s$ for $m > 1$. Equations of the above form are mathematical models occurring in the study of the $m$-Laplace equation, in generalized reaction-diffusion theory ([6]), non-Newtonian fluid theory, and in the study of turbulent flows of a gas in a porous medium ([4]). In the non-Newtonian fluid theory, the quantity $m$ is a characteristic of the medium. Media with $m > 2$ are called dilatant fluids and those with $m < 2$ are called pseudoplastics. If $m = 2$, they are Newtonian fluids.

The equation

$$(\Phi_m(u'))' = f(t, u, u'), \qquad t \in I = [0,1], \tag{5}$$

with various boundary conditions has been studied by many authors (see [1–4, 6, 8–15] and references therein). On the contrary, it seems that little is known about problems (1)-(2), (1)-(3), and (1)-(4). Our results were motivated by the papers [1, 2, 5, 7] which studied periodic and Neumann nonlinear boundary conditions for equation (5). When $p = 2$, some related results have been obtained in [5, 7]. Our results extend those of [1, 2, 5, 7].

Definition 1. A function $\alpha \in C^1[0,1]$ with $\Phi_m(\alpha') \in C^1[0,1]$ is called a *lower solution* of (5) on $I = [0,1]$ if

$$(\Phi_m(\alpha'))' \geq f(t, \alpha, \alpha') \quad \text{for } t \in I.$$

Likewise, $\beta \in C^1[0,1]$ with $\Phi_m(\beta') \in C^1[0,1]$ is an *upper solution* of (5) on $I$ if

$$(\Phi_m(\beta'))' \leq f(t, \beta, \beta') \quad \text{for } t \in I.$$

In what follows we shall assume that

$$\alpha(t) \leq \beta(t), \qquad t \in I.$$

For $\alpha, \beta \in C(I), \alpha \leq \beta$, we define the set

$$E = \{u \in C^1(I) \mid \alpha(t) \leq u(t) \leq \beta(t), \forall t \in I\}.$$

In the following theorems we will use the following hypotheses:

($H_1$) $f$ is a continuous function in $\Omega = \{(t, y, z) \mid 0 \leq t \leq 1,\ (y, z) \in \mathbb{R}^2\}$.

($H_2$) $f(t, y, z)$ satisfies the Nagumo condition in $E$, i.e. there exists a function $\Psi : [0, \infty) \to [0, \infty)$ with $1/\Psi$ integrable on every bounded interval $(a, b) \subset [0, \infty)$, such that

$$|f(t, y, z)| \leq \Psi(|z|) \quad \text{for } (t, y) \in E,\ z \in \mathbb{R},$$

where $\Psi$ satisfies

$$\int_0^\infty \Phi_m^{-1}(u)/\Psi(\Phi_m^{-1}(u))\, du = \infty.$$

From [11, 15], we have the following theorem:

THEOREM 1. *Let $\alpha, \beta \in C^1[0,1]$ be lower and upper solutions of* (5), *respectively, with $\alpha \le \beta$ in $I$. Assume that hypotheses* ($H_1$)–($H_2$) *are satisfied. Then for any $\alpha(0) \le A \le \beta(0)$, $\alpha(1) \le B \le \beta(1)$ there exists a solution $u$ of the boundary value problem*

$$(\Phi_m(u'))' = f(t, u, u'), \qquad u(0) = A, \qquad u(1) = B,$$

*satisfying $\alpha(t) \le u(t) \le \beta(t)$ on* $[0,1]$.

Before stating the main result on existence of solutions for problems (1)-(2), (1)-(3), and (1)-(4), we give the following

DEFINITION 2. A function $\alpha \in C^1[0,1]$ with $\Phi_m(\alpha') \in C^1[0,1]$ is called a *lower solution* of (1) on $[0,1]$ if

$$(\Phi_m(\alpha'))' \ge f(t, \alpha, T_1\alpha, T_2\alpha, \alpha') \quad \text{for } t \in I.$$

Likewise, $\beta \in C^1[0,1]$ with $\Phi_m(\beta') \in C^1[0,1]$ is an *upper solution* of (1) on $[0,1]$ if

$$(\Phi_m(\beta'))' \le f(t, \beta, T_1\beta, T_2\beta, \beta') \quad \text{for } t \in I.$$

Moreover, we define the following sets:

$$F = \{(y,z,u,v)| \mid \alpha(0) \le y \le \beta(0),\ \alpha(1) \le z \le \beta(1),\ u, v \in \mathbb{R}\};$$

$$G = \{g = g(y,z,u,v) \in C(F) \mid g \text{ is nondecreasing in } u, \text{ nonincreasing in } v, \text{ and } g(\alpha(0), \alpha(1), \alpha'(0), \alpha'(1)) \ge 0 \ge g(\beta(0), \beta(1), \beta'(0), \beta'(1))\};$$

$$H = \{h \mid h : [\alpha(0), \beta(0)] \to [\alpha(1), \beta(1)] \text{ is a homeomorphism, and } h(\alpha(0)) = \alpha(1),\ h(\beta(0)) = \beta(1)\};$$

$$P = \{p = p(s,t) \mid p : [\alpha(0), \beta(0)] \times \mathbb{R} \to \mathbb{R} \text{ is continuous and nonincreasing in } t \text{ and } p(\alpha(0), \alpha'(0)) \le 0 \le p(\beta(0), \beta'(0))\};$$

$$\Gamma = \{(s,t,u,v) \mid \alpha(0) \le s \le \beta(0),\ \alpha(1) \le u \le \beta(1),\ t, v \in \mathbb{R}\};$$

$$Z(p) = \{(s,t) \mid p(s,t) = 0,\ \alpha(0) \le s \le \beta(0),\ t \in \mathbb{R}\};$$

$$Q = \{q = q(s,t,u,v) \in C(\Gamma) \mid q \text{ is nondecreasing in } v, \text{ and } q(s,t,\alpha(1),\alpha'(1)) \le 0 \le q(s,t,\beta(1),\beta'(1)) \text{ for } (s,t) \in Z(p)\};$$

$$Z(r) = \{(u,v) \mid r(u,v) = 0, \alpha(1) \le u \le \beta(1),\ v \in \mathbb{R}\};$$

$$R = \{r = r(u,v) \mid r : [\alpha(1), \beta(1)] \times \mathbb{R} \to \mathbb{R} \text{ is continuous, nondecreasing in } v, \text{ and } r(\alpha(1), \alpha'(1)) \le 0 \le r(\beta(1), \beta'(1))\};$$

$$W = \{w = w(s,t,u,v) \in C(\Gamma) \mid w \text{ is nonincreasing in } t, \text{ and } w(\alpha(0), \alpha'(0), u, v) \le 0 \le w(\alpha(1), \alpha'(1), u, v) \text{ for } (u,v) \in Z(r)\}.$$

In what follows we impose the following conditions on (1):

(H$_3$) $f(t,u,v,w,z)$ is nonincreasing in $v$ and in $w$.

(H$_4$) $f \in C([0,1]\times\mathbb{R}^4,\mathbb{R})$ and there exists a continuous function $h : [0,\infty)\to[0,\infty)$ such that

$$|f(t,u,v,w,z)| \le h(|z|) \quad \text{for } (t,u,v,w,z)\in\Omega,$$

where $\Omega = \{(t,u,v,w,z)\in I\times\mathbb{R}^3 : |u|\le r_1,\ |v|\le r_2,\ |w|\le r_3,\ z\in\mathbb{R}\}$ for some $r_1,r_2,r_3>0$, and also that

$$\int_0^\infty \Phi_m^{-1}(u)/h(\Phi_m^{-1}(u))\,du = \infty.$$

Now, we can prove our main results.

THEOREM 2. *Let $\alpha,\beta\in C^1[0,1]$ be lower and upper solutions of* (1), *respectively, with $\alpha\le\beta$ on $I=[0,1]$. Assume that hypotheses* (H$_3$)–(H$_4$) *are satisfied. Then for any $\alpha(0)\le A\le\beta(0)$, $\alpha(1)\le B\le\beta(1)$ there exists a solution $u$ of the boundary value problem*

$$(6)\qquad (\Phi_m(u'))' = f(t,u,T_1u,T_2u,u'),\quad u(0)=A,\quad u(1)=B,$$

*satisfying $\alpha(t)\le u(t)\le\beta(t)$ on $[0,1]$.*

*Proof.* Let $u_0(t)=\beta(t)$. Then

$$\begin{aligned} f(t,\alpha(t),[T_1u_0](t),[T_2u_0](t),\alpha'(t)) &\le f(t,\alpha,[T_1\alpha](t),[T_2\alpha](t),\alpha'(t))\\ &\le (\Phi_m(\alpha'))',\\ f(t,\beta(t),[T_1u_0](t),[T_2u_0](t),\beta'(t)) &\ge (\Phi_m(\beta'))',\quad t\in I=[0,1]. \end{aligned}$$

By Theorem 1, there exists a solution $u_1$ of the boundary value problem

$$(\Phi_m(u'))' = f(t,u,T_1u_0,T_2u_0,u'),\quad u(0)=A,\quad u(1)=B,$$

satisfying $\alpha(t)\le u_1(t)\le\beta(t)=u_0(t)$ on $[0,1]$.

We now consider the problem

$$(7)\qquad (\Phi_m(u'))' = f(t,u,T_1u_1,T_2u_1,u'),\quad u(0)=A,\quad u(1)=B.$$

Clearly,

$$f(t,\alpha,T_1u_1,T_2u_1,\alpha') \le (\Phi_m(\alpha'))',$$

and

$$f(t,u_1,T_1u_1,T_2u_1,u_1') \ge f(t,u_1,T_1u_0,T_1u_0,u_1') = (\Phi_m(u_1'))'.$$

By Theorem 1, there exists a solution $u_2$ of (7) satisfying $\alpha(t)\le u_2(t) \le u_1(t)$ on $[0,1]$.

By induction, we can construct a nonincreasing sequence $\{u_n(t)\}$ such that

$$\alpha(t)\le u_n(t)\le u_{n-1}(t)\le\cdots\le u_0(t)=\beta(t).$$

From condition ($H_4$), there exists a positive constant $N > 0$ such that $|u_n(t)| \le N$, $t \in I$, $n = 1, 2, \ldots$. On the other hand, $\{(\Phi_m(u_n'))'\}$ is uniformly bounded on $I$ by equation (6). Therefore, $\{u_n\}, \{\Phi_m(u_n')\}$ are uniformly bounded and equicontinuous. Applying the Arzelà–Ascoli theorem to the sequence $\{u_n\}$, we find that there exists a subsequence $\{u_{n_k}\}$ satisfying $\lim_{k\to\infty} \Phi_m(u_{n_k}') = v$. Thus, we obtain $\lim_{k\to\infty} u_{n_k}' = \Phi_m^{-1}(v)$, and so

$$u_{n_k}(t) = A + \int_0^t u_{n_k}'(s)\,ds \to A + \int_0^t \Phi_m^{-1}(v)\,ds = \overline{u}(t) \qquad (k \to \infty).$$

So there exists $\overline{u} \in C^1(I)$ such that $\lim_{k\to\infty} u_{n_k}(t) = \overline{u}(t)$. By the dominated convergence theorem, we know that $\overline{u}$ is a solution of problem (6).

THEOREM 3. *Let $\alpha, \beta \in C^1[0,1]$ be lower and upper solutions of* (1), *respectively*, *with $\alpha \le \beta$ on $I = [0,1]$. Assume that hypotheses* ($H_3$)–($H_4$) *are satisfied*, *and $g \in G$, $h \in H$. Then the boundary value problem* (1)-(2) *has a solution $u = u(t)$ with $\alpha(t) \le u(t) \le \beta(t)$ on $[0,1]$.*

*Proof.* For each $\alpha(0) \le c \le \beta(0)$, there exists (by Theorem 2) a solution $u_c$ of the BVP

$$(\Phi_m(u'))' = f(t, u, T_1u, T_2u, u'), \qquad u(0) = c, \qquad u(1) = h(c),$$

satisfying $\alpha(t) \le u_c(t) \le \beta(t)$ on $[0,1]$. If $c = \alpha(0)$, then $u_c'(0) \ge \alpha'(0)$ and $u_c'(1) \le \alpha'(1)$. Hence,

$$\begin{aligned} g(u_c(0), u_c(1), u_c'(0), u_c'(1)) &= g(\alpha(0), \alpha(1), u_c'(0), u_c'(1)) \\ &\ge g(\alpha(0), \alpha(1), \alpha'(0), \alpha'(1)) \ge 0 \end{aligned} \tag{8}$$

by the monotonicity of $g$ in the last two variables. Similarly, if $c = \beta(0)$, we have $u_c'(0) \le \beta'(0), u_c'(1) \ge \beta'(1)$, and therefore,

$$g(u_c(0), u_c(1), u_c'(0), u_c'(1)) \le g(\beta(0), \beta(1), \beta'(0), \beta'(1)) \le 0, \quad c = \beta(0). \tag{9}$$

Define

$$\begin{aligned} M &= \{c \in [\alpha(0), \beta(0)] : g(u_c(0), u_c(1), u_c'(0), u_c'(1)) < 0\}, \\ N &= \{c \in [\alpha(0), \beta(0)] : g(u_c(0), u_c(1), u_c'(0), u_c'(1)) > 0\}. \end{aligned}$$

If the theorem is not true, then $M \cup N = [\alpha(0), \beta(0)]$ and both $M, N$ are nonempty by (8)-(9). We claim that $M$ is closed. To see this, let $c_n \in M$ with $\lim_{n\to\infty} c_n = c_0$. Then with $u_n = u_{c_n}$, it follows that $g(u_n(0), u_n(1), u_n'(0), u_n'(1)) < 0$ and there exists a subsequence of $u_n$ which converges, uniformly on $[0,1]$, to a solution $u_0$ of (1) satisfying $u_0(0) = c_0, u_0(1) = h(c_0)$ and $g(u_0(0), u_0(1), u_0'(0), u_0'(1)) \le 0$. By assumption, equality cannot occur, so that $g(u_0(0), u_0(1), u_0'(0), u_0'(1)) < 0$, and thus $c_0 \in M$. Therefore, $M$ is closed, so $N$ is open. Likewise, we may show $N$ is closed. This is a contradiction which proves the theorem.

THEOREM 4. *Let $\alpha, \beta \in C^1[0,1]$ be lower and upper solutions of* (1), *respectively, with $\alpha \le \beta$ on $I = [0,1]$. Assume that hypotheses* ($H_3$)–($H_4$) *are satisfied, and $p \in P$, $q \in Q$. Then the boundary value problem* (1)-(3) *has a solution $u = u(t)$ with $\alpha(t) \le u(t) \le \beta(t)$ on $[0,1]$.*

THEOREM 5. *Let $\alpha, \beta \in C^1[0,1]$ be lower and upper solutions of* (1), *respectively, with $\alpha \le \beta$ on $I = [0,1]$. Assume that hypotheses* ($H_3$)–($H_4$) *are satisfied, and $r \in R$, $w \in W$. Then the boundary value problem* (1)-(4) *has a solution $u = u(t)$ with $\alpha(t) \le u(t) \le \beta(t)$ on $[0,1]$.*

COROLLARY 1. *Let $\alpha, \beta \in C^1[0,1]$ be lower and upper solutions of* (1), *respectively, with $\alpha \le \beta$ on $I = [0,1]$. Assume that hypotheses* ($H_3$)–($H_4$) *are satisfied, and $m = (\beta(1) - \alpha(1))/(\beta(0) - \alpha(0))$. Furthermore, suppose there exists $c > 0$ such that $\beta'(1) - \alpha'(1) \ge c(\beta'(0) - \alpha'(0))$ and let $d$ satisfy $\beta'(1) - c\beta'(0) \ge d \ge \alpha'(1) - c\alpha'(0)$. Then equation* (1) *has a solution $u$ with $\alpha(t) \le u(t) \le \beta$ and $u(1) = mu(0) + \alpha(1) - m\alpha(0)$, $u'(1) = cu'(0) + d$.*

COROLLARY 2. *Let $\alpha, \beta \in C^1[0,1]$ be lower and upper solutions of* (1), *respectively, with $\alpha \le \beta$ on $I = [0,1]$. Suppose there exists a constant $L > 0$ such that for all $(t,u) \in E$ and $u_1, u_2 \in \mathbb{R}$,*

$$|f(t,u,T_1u,T_2u,u_1') - f(t,u,T_1u,T_2u,u_2')| \le L|u_1' - u_2'|.$$

*Let $A, B, a_1, a_2, b_1, b_2$ be real numbers such that $a_i, b_i \ge 0$ $(i = 1,2)$, $a_1 + a_2 > 0$, $b_1 + b_2 > 0$ and*

$$a_1\alpha(0) - a_2\alpha'(0) - A \le 0 \le a_1\beta(0) - a_2\beta'(0) - A,$$
$$b_1\alpha(1) + b_2\alpha'(1) - B \le 0 \le b_1\beta(1) + b_2\beta'(1) - B.$$

*Then equation* (1) *has a solution $u$ such that*

$$a_1u(0) - a_2u'(0) - A = 0 = b_1u(1) + b_2u(1) - B, \quad \alpha(t) \le u(t) \le \beta(t).$$

EXAMPLE. To illustrate Theorem 4 for the case when the boundary conditions are nonlinear, let $f$ satisfy conditions ($H_3$)–($H_4$) and assume

$$f\Big(t, -1, \psi_1(t) + \int_0^t K_1(t,s)\,ds, \psi_2(t) + \int_0^t K_2(t,s)\,ds, 0\Big) \le 0$$
$$\le f\Big(t, 1, \psi_1(t) + \int_0^t K_1(t,s)\,ds, \psi_2(t) + \int_0^t K_2(t,s)\,ds, 0\Big), \quad 0 \le t \le 1,$$

where $K_i \in C([0,1] \times [0,1], \mathbb{R}^+)$, $\psi_i \in C([0,1], \mathbb{R})$, $i = 1,2$, so that $\alpha = -1$, $\beta = 1$ are lower and upper solutions, respectively, of (1). Let $p = p(s,t)$,

$q = q(s,t,u,v)$ be defined by

$$p(s,t) = s^2 - t - 1, \qquad q(s,t,u,v) = s + t + cu + dv,$$

where $c \geq 5/4$, $d > 0$ are real constants. It is easy to check that $p \in P$, $q \in Q$, so that by Theorem 4, there exists a solution $u$ of the boundary value problem

$$(\Phi_m(u'))' = f(t,u,T_1u,T_2u,u'),$$
$$(u(0))^2 - u'(0) - 1 = 0 = u(0) + u'(0) + cu(1) + du'(1),$$

satisfying $-1 \leq u(x) \leq 1$ on $[0,1]$.

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