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Remarks on pluripolar hulls

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Abstract. The aim of the paper is to establish some results on pluripolar hulls and to define pluripolar hulls of certain graphs.

1. Introduction. Let Ω be a domain in $\mathbb{C}^n$. An upper semicontinuous function u on Ω is called *plurisubharmonic* if the restriction of u to the intersection of Ω with every complex line is subharmonic (we allow the function identically $-\infty$ to be plurisubharmonic). The cone of plurisubharmonic functions (resp. negative plurisubharmonic functions) is denoted by $\mathcal{PSH}(\Omega)$ (resp. $\mathcal{PSH}^-(\Omega)$). A subset E of $\mathbb{C}^n$ is called *pluripolar* if for every $a \in A$ we can find a neighbourhood U_a of a and $u \in \mathcal{PSH}(U_a)$ such that $u \equiv -\infty$ on $E \cap U_a$ and $u \not\equiv -\infty$. A basic theorem of Josefson (see [Kl, Theorem 4.7.4]) asserts that if E is pluripolar in Ω then there exists a plurisubharmonic function u on $\mathbb{C}^n$ such that $u \equiv -\infty$ on E but $u \not\equiv -\infty$. If E is pluripolar and contained in some domain Ω of $\mathbb{C}^n$ then we say that E is *complete pluripolar* in Ω if there exists $u \in \mathcal{PSH}(\Omega)$ such that $u^{-1}(-\infty) = E$.

It is easy to see that every complete pluripolar set $E \subset \Omega$ is a G_δ set. In the case $\Omega \subset \mathbb{C}$ by Deny's theorem (see e.g. [Lan]) every polar G_δ subset of Ω is complete polar. However, in higher dimensions the situation is much more complicated: the set $E = \{(z,0) : |z| < 1\}$ is closed and complete pluripolar in the bidisk $\Omega = \{(z,w) : |z| < 1,\ |w| < 1\}$, but is not complete pluripolar in any domain larger than $\overline{\Omega}$.

In order to see more concretely how a pluripolar subset E of Ω "propagates", following Levenberg and Poletsky we introduce two types of pluripolar hulls of E relative to Ω:

$$E^*_\Omega = \bigcap \{z \in \Omega : u(z) = -\infty,\ u \in \mathcal{PSH}(\Omega),\ u|_E \equiv -\infty\},$$
$$E^-_\Omega = \bigcap \{z \in \Omega : u(z) = -\infty,\ u \in \mathcal{PSH}^-(\Omega),\ u|_E \equiv -\infty\}.$$

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Obviously $E = E^*_\Omega$ if E is complete pluripolar. On the other hand, $E = E^*_\Omega$ does not imply E is complete pluripolar. Indeed, every countable and non-G_δ set E in $\mathbb{C}$ satisfies $E^*_{\mathbb{C}} = E$ but of course E is not complete polar (in $\mathbb{C}$). Nevertheless, this implication is true if we assume in addition that Ω is pseudoconvex and E is both F_σ and G_δ (see Proposition 2.1 in [Ze]). One of our aims is to discuss variants of the above mentioned result of Zeriahi where the emphasis is on regularity of plurisubharmonic functions whose singular locus coincides with E. In particular, we show in Theorem 3.2 that if E is closed in Ω and $E^*_{\Omega'} \cap \overline{\Omega} = E$, where Ω' is some domain larger than $\overline{\Omega}$, then there exists a plurisubharmonic function u on Ω, continuous on $\overline{\Omega}$ and strictly plurisubharmonic on $\Omega \setminus E$ such that $u^{-1}(-\infty) = E$.

In general, it is quite difficult to determine E^*_Ω and E^-_Ω; even simple looking sets like $F_1 = \{(z, z^\alpha) : z \neq 0\}$, $\alpha > 0$, $\alpha \notin \mathbb{Q}$, or $F_2 = \{(z, e^{-1/z}) : z \neq 0\}$ require considerable efforts (see [LP], [Wi1]) to establish their pluripolar hulls. Combining the description of $(F_2)^*_{\mathbb{C}^2}$ with Zeriahi's theorem, one can even show that F_2 is complete pluripolar in $\mathbb{C}^2$ (for details see [Wi1]). On the other hand, F_1 is not so, being a non-G_δ set (see [Wi3] for details).

In the interesting paper [LP], a number of useful techniques to study pluripolar hulls have been established by Levenberg and Poletsky. The next goal of the present work is to apply the methods of Levenberg and Poletsky to describe E^*_Ω in case E is the graph of a holomorphic function over some pseudoconvex domain D minus a complex hypersurface and $\Omega = D \times \mathbb{C}$. It should be remarked that in one dimension a complete answer has been obtained in the work of Wiegerinck ([Wi2]). Here we encounter technical difficulties as a complex hypersurface in $\mathbb{C}^n, n \geq 2$, contains no isolated points. Therefore a complete description of the pluripolar hull in our case is still missing. We are able to obtain a partial answer in Proposition 4.1. Using this result and the above mentioned theorem of Wiegerinck it is not hard to show that $(F_2')^*_{\mathbb{C}^3} = F_2' \cup \{(0,0) \times \mathbb{C}\}$, where $F_2' = \{(z, w, e^{z/w}) : w \neq 0\}$. In particular, F_2' is not complete pluripolar.

The next section deals with a new kind of pluripolar hulls taken in the subclass $\mathcal{L}(\mathbb{C}^n)$ of plurisubharmonic functions with logarithmic growth on $\mathbb{C}^n$. The main result of this section is that the new pluripolar hull $\widetilde{E}_{\mathbb{C}^n}$ coincides with the former, $E^*_{\mathbb{C}^n}$. This result is inspired by a well known theorem of Siciak stating that every pluripolar set in $\mathbb{C}^n$ is the singular locus of a function in $\mathcal{L}(\mathbb{C}^n)$ (see [Sic]).

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2. Preliminaries. The result below is particularly useful when we want to "localize" E^*_Ω.

THEOREM 2.1 ([LP]). *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and $\{\Omega_\nu\}_{\nu=1}^\infty$ an increasing sequence of relatively compact domains with $\bigcup_{\nu=1}^\infty \Omega_\nu = \Omega$. Let $E \subset \Omega$ be a pluripolar set. Then*

$$E^*_\Omega = \bigcup_{\nu=1}^{\infty} (E \cap \Omega_\nu)^-_{\Omega_\nu}.$$

An important tool in the study of pluripolar hulls is the concept of the pluriharmonic measure (see [LP] for initial use of it and [Wi1], [Wi2], [E], [EW], etc. for further developments). Let $\Omega \subset \mathbb{C}^n$ be a domain and $E \subset \Omega$. The *pluriharmonic measure* at $z \in \Omega$ of E relative to Ω is the number

$$\omega(z, E, \Omega) = -\sup\{u(z) : u \in \mathcal{PSH}(\Omega),\ u|_E \le -1,\ u|_\Omega \le 0\}.$$

The following result provides a connection between the pluriharmonic measure and the pluripolar hull E^-_Ω.

LEMMA 2.2 ([LP]). *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$, and E be a pluripolar subset of Ω. Then*

$$E^-_\Omega = \{z \in \Omega : \omega(z, E, \Omega) > 0\}.$$

We also need

LEMMA 2.3 ([Wi1]). *Let Ω be a domain in $\mathbb{C}^n$, $E \subset \Omega$ and let $A \subset \Omega \setminus E$ be closed and pluripolar. Then for all $z \in \Omega \setminus A$ we have*

$$\omega(z, E, \Omega) = \omega(z, E, \Omega \setminus A).$$

The following result (Proposition 2.1 in [Ze]) characterizes complete pluripolar sets in terms of their pluripolar hulls.

THEOREM 2.4 ([Ze]). *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and $E \subset \Omega$ an F_σ set. Then E is a complete pluripolar set in Ω if and only if E is a G_δ set and $E^*_\Omega = E$.*

Now we turn to the pluripolar hull of a pluripolar set taken in the class $\mathcal{L}$ of plurisubharmonic functions on $\mathbb{C}^n$ with logarithmic growth. For a pluripolar subset E of $\mathbb{C}^n$ we set

$$\widetilde{E}_{\mathbb{C}^n} = \{z : u(z) = -\infty,\ u|_E \equiv -\infty,\ u \in \mathcal{L}(\mathbb{C}^n)\}.$$

It is proved in [Sic] that for every pluripolar subset E of $\mathbb{C}^n$ we can find $u \in \mathcal{L}(\mathbb{C}^n)$ such that $u \not\equiv -\infty$ and $E \subset u^{-1}(-\infty)$. The following theorem of Bedford and Taylor (Theorem 7.2 in [BT]) improves this result.

THEOREM 2.5. *Let E be a complete pluripolar subset of $\mathbb{C}^n$. Then we can find $u \in \mathcal{L}(\mathbb{C}^n)$ such that $u \not\equiv -\infty$ and $u^{-1}(-\infty) = E$.*

Notation. If Ω is an open subset of $\mathbb{C}^n$ then by $\mathcal{H}(\Omega)$ we mean the set of holomorphic functions on Ω. We also denote by $\mathcal{PSH}^c(\Omega)$ the cone of functions u such that e^u is continuous on $\overline{\Omega}$ and $u \in \mathcal{PSH}(\Omega)$.

3. Variations on Zeriahi's theorem. The next result is a minor improvement of Theorem 2.4.

PROPOSITION 3.1. *Let Ω, Ω' be domains in $\mathbb{C}^n$ such that $\Omega \subset \Omega'$ and Ω' is pseudoconvex. Assume that E is a pluripolar subset of Ω which is F_σ and G_δ. If*

$$E = E^*_{\Omega'} \cap \Omega \tag{1}$$

then there exists a function $u \in \mathcal{PSH}(\Omega')$ such that

(a) $u^{-1}(-\infty) \cap \Omega = E$.
(b) *u is continuous on $\Omega \setminus E$.*

We could say that this result follows from *the proof* of Zeriahi's theorem. However, for the sake of completeness we indicate the details.

Proof. Since E is F_σ and G_δ we can express E and $\Omega \setminus E$ as increasing unions of compact subsets:

$$E = \bigcup_{j\ge 1} K_j, \qquad \Omega \setminus E = \bigcup_j L_j.$$

We also write $\Omega' = \bigcup_j \Omega'_j$, where Ω'_j are relatively compact subsets of Ω' and satisfy $K_j \cup L_j \subset \Omega'_j$ for all $j \ge 1$. Fix $j \ge 1$. Let a be a point in L_j. Then from (1) we have $a \notin E^*_{\Omega'}$. Hence, there exists $u_a^{(j)} \in \mathcal{PSH}(\Omega')$ such that $u_a^{(j)}|_E \equiv -\infty$ and $u_a^{(j)}(a) > -\infty$. By composing with a suitable increasing convex function we may assume that

$$u_a^{(j)}|_E \equiv -\infty, \qquad u_a^{(j)}(a) > -2/3, \qquad u_a^{(j)}|_{\Omega'_j} \le -1/2.$$

Since Ω' is pseudoconvex, using a result of Fornæss and Narasimhan in [FN] we get a sequence $\{u_k^{(j)}\}$ of real-valued, continuous plurisubharmonic functions on Ω' that decrease pointwise to $u_a^{(j)}$ on Ω'. Applying Dini's theorem we find k_a (sufficiently large) such that

$$u_{k_a}^{(j)}|_{K_j} \le -2^j, \qquad u_{k_a}^{(j)}(a) > -1, \qquad u_{k_a}^{(j)}|_{\overline{\Omega'_j}} \le 0.$$

As $u_{k_a}^{(j)}$ is continuous there exists a neighbourhood U_a of a such that $u_{k_a}^{(j)} > -1$ on U_a. Now a standard argument using the compactness of L_j implies that there exists a continuous plurisubharmonic function v_j on Ω' such that

(i) $v_j|_{K_j} \le -2^j$.
(ii) $v_j|_{L_j} > -1$.
(iii) $v_j|_{\overline{\Omega'_j}} \le 0$.

Then in view of (iii) the series

$$u(z) := \sum_{j\geq 1} 2^{-j} v_j(z)$$

defines a plurisubharmonic function on Ω'. It follows from (i) that $u \equiv -\infty$ on E. Furthermore from (ii) we have $u > -\infty$ on $\Omega' \backslash E$. Thus (a) is proved. Finally, (b) follows since the series converges uniformly on L_j for each j in view of (ii).

It is reasonable to ask if the function u above can be chosen to be continuous on Ω if E is closed in Ω. We have the following

THEOREM 3.2. *Let Ω be a bounded domain in $\mathbb{C}^n$ and E be a closed pluripolar subset of Ω. Assume that*

$$E^*_{\Omega'} \cap \overline{\Omega} = E, \tag{2}$$

where Ω' is some domain in $\mathbb{C}^n$ that contains $\overline{\Omega}$. Then there exists a function $u \in \mathcal{PSH}^c(\Omega)$ such that

(a) $u^{-1}(-\infty) \cap \overline{\Omega} = E$.
(b) *u is smooth and strictly plurisubharmonic on $\Omega \setminus E$.*

Proof. We divide the proof into two steps.

STEP 1. Let Ω'' be a domain satisfying $\Omega \subset\subset \Omega'' \subset\subset \Omega'$. We will show that there exists $v \in \mathcal{PSH}(\Omega'')$ such that $v < 0$ on Ω'' and $v^{-1}(-\infty) \cap \overline{\Omega} = E$. Since the proof below is very similar to that of Proposition 3.1, we only sketch it (see also Lemma 4.2 in [EW] for a similar situation). Express E and $\overline{\Omega} \setminus E$ as increasing unions of compact sets:

$$E = \bigcup_{j\geq 1} K_j, \qquad \overline{\Omega} \setminus E = \bigcup_{j\geq 1} L_j.$$

Fix $j \geq 1$ and let $a \in L_j$. From (2) we get $u_a \in \mathcal{PSH}(\Omega')$ such that $u_a|_E \equiv -\infty$, $u_a(a) = -2/3$, and $u_a|_{\Omega''} \leq -1/2$. By taking convolution with standard smooth kernels we get a sequence of real-valued continuous plurisubharmonic functions $\{u_j\}$ on Ω'' that decrease pointwise to u_a on Ω''. Now the rest of the proof goes exactly as that of Proposition 3.1, and hence we omit the details.

STEP 2. Let Ω'' and v be as in Step 1. We will construct a function u satisfying (a) and (b). To this end, we use methods given in Lemma 1.2 of [Sib]. Let h be an increasing convex function on $[0,1]$ such that

$$h(x) = 0 \quad \forall x \in [0,1/2], \qquad h(x) < 1 \quad \forall x \neq 1, \qquad h(1) = 1.$$

Set $\widetilde{v}_k = h(e^{v/k})$. Clearly $\widetilde{v}_k \in \mathcal{PSH}(\Omega'')$, $\widetilde{v}_k$ vanishes on neighbourhoods of E in Ω'', $0 \leq \widetilde{v}_k \leq 1$ and $\lim_{k\to\infty} \widetilde{v}_k = 1$ on $\overline{\Omega} \setminus E$. By taking convolution with standard smooth kernels and shrinking Ω'' we can assume in addition that $\widetilde{v}_k$ is continuous on Ω''. Set

$$w_k = \max\{\widetilde{v}_1, \dots, \widetilde{v}_k\}.$$

Then $\{w_k\}$ is an increasing sequence of continuous plurisubharmonic functions on Ω'' that vanish on neighbourhoods of E in Ω''. Thus by Dini's theorem it converges uniformly on compact subsets of $\Omega \setminus E$. We claim that there exists a sequence $\{F_k\}$ of compact subsets and a sequence $\{n_k\}$ such that

(i) $\mathrm{Int}(F_{k-1}) \subset F_k, \bigcup F_k = \overline{\Omega} \setminus E$.
(ii) $\{z \in \Omega : \sum_{j=1}^{k-1} w_{n_j} \geq 1/2\} \subset F_k$.
(iii) $1 - 1/2^k \leq w_{n_k}$ on F_k.

First we choose n_1 so large that $F_1 := \{z : w_{n_1} \geq 1/2\} \neq \emptyset$. Assume that $F_1, \dots, F_{k-1}$ and $n_1, \dots, n_{k-1}$ have been chosen. Let F_k be any compact set in $\Omega \setminus E$ that contains $\{z \in \overline{\Omega} : \sum_{j=1}^{k-1} w_{n_j} \geq 1/2\} \cup \mathrm{Int}(F_{k-1})$. This is possible since the latter set is compact and disjoint from E. As the sequence $\{w_k\}$ converges to 1 uniformly on F_k we can choose n_k so large that

$$w_{n_k} \geq 1 - 1/2^k \quad \text{ on } F_k.$$

It is clear that the compact sets F_k can be taken so that $\bigcup F_k = \overline{\Omega} \setminus E$. Thus the claim is valid.

Now we form the series

$$\widetilde{w}(z) = |z|^2 + \sum_{k \geq 1} (w_{n_k} - 1).$$

It defines a plurisubharmonic function on Ω. Moreover from (iii) we deduce that it converges uniformly on F_k for every $k \geq 1$. Thus by (i), $\widetilde{w}$ is real-valued, continuous on $\overline{\Omega} \setminus E$. Next (ii) implies that $\widetilde{w} \equiv -\infty$ on E and $\widetilde{w}$ is continuous at every point of E. Clearly $\widetilde{w}$ is strictly plurisubharmonic on $\Omega \setminus E$. Let φ be a continuous function on $\overline{\Omega}$ such that $\varphi > 0$ on Ω and $\varphi \equiv 0$ on $\partial\Omega$. Now using Richberg's regularization lemma ([Ri]) we get a smooth strictly plurisubharmonic function u on $\Omega \setminus E$ such that $\widetilde{w} \leq u \leq \widetilde{w} + \varphi$. Since E is closed and pluripolar, u can be extended through E to a plurisubharmonic function on Ω (still denoted by u). Finally, it is easy to see that u can be extended to a continuous function on $\overline{\Omega}$ and satisfies $u^{-1}(-\infty) \cap \overline{\Omega} = E$.

4. Pluripolar hulls of certain graphs. The next result is an analogue of Proposition 5 in [Wi2]. Needless to say, we rely heavily on Wiegerinck's methods.

PROPOSITION 4.1. *Let D be a pseudoconvex domain in $\mathbb{C}^n$, and $f \in \mathcal{H}(D \setminus A)$, where $A = \{g = 0\}$ and g is a holomorphic function on D.*

Denote by E the graph of f in $(D \setminus A) \times \mathbb{C}$. Then

$$Z = E \cup (A \times \mathbb{C})$$

is complete pluripolar in $D \times \mathbb{C}$.

Proof. We split the proof into two steps.

STEP 1. We assume that D is bounded, g is holomorphic on a Stein neighbourhood $\widetilde{D}$ of $\overline{D}$ and $f \in \mathcal{H}(\overline{D} \setminus A)$. Let B be a disk around $0 \in \mathbb{C}$. Let E' be the graph of f in $(D \setminus A) \times B$ and

$$Z' = E' \cup (A \times B).$$

We will show that there exists $u \in \mathcal{PSH}^-(D \times B)$ such that u is $-\infty$ exactly on Z'. By Theorem 1 in [Ch] we can expand

$$f(z) = \sum_{j \geq 0} \frac{f_j(z)}{g^j(z)}, \qquad z \in D \setminus A,$$

where f_j are holomorphic functions on $\widetilde{D}$ and satisfy

$$\lim_{j \to \infty} \|f_j\|_{\overline{D}}^{1/j} = 0.$$

Fix $\delta > 0$ so small that

$$K = \{z \in D : d(z, \partial D) \geq \delta,\ |g(z)| \geq \delta\} \neq \emptyset.$$

Set

$$\varepsilon_j = \sup_{k \geq j} \|f_k\|_{\overline{D}}^{1/k}, \qquad h_N(z,w) = \frac{1}{N} \log\left(\left|\left(w - \sum_{j=0}^{N} \frac{f_j(z)}{g^j(z)}\right) g^N(z)\right|\right),$$

where the integer N will be chosen later. It is clear that $\varepsilon_j \downarrow 0$. Let $M = \sup_{\overline{D}} |g|$. We now make some estimates. On $(K \times B) \cap E$ we have

$$\begin{aligned}
(3) \qquad h_N(z, f(z)) &= \frac{1}{N} \log\left|\sum_{j \geq N+1} \frac{f_j(z)}{g^j(z)}\right| + \log|g(z)| \\
&\leq \frac{1}{N} \log \sum_{j \geq N+1} \left|\frac{f_j(z)}{g^j(z)}\right| + \log|g(z)| \\
&\leq \frac{1}{N} \log \sum_{j \geq N+1} \left|\frac{\varepsilon_N^j}{g^j(z)}\right| + \log|g(z)| \\
&\leq \frac{1}{N} \log\left|\left(\frac{\varepsilon_N}{g(z)}\right)^{N+1} \frac{1}{1 - \varepsilon_N/\delta}\right| + \log|g(z)| \\
&= \left(1 + \frac{1}{N}\right)(\log \varepsilon_N - \log|g(z)|) - \frac{1}{N} \log\left|1 - \frac{\varepsilon_N}{\delta}\right| \\
&\quad + \log|g(z)| \\
&\leq \log \varepsilon_N + C_1,
\end{aligned}$$

where C_1 depends on δ but not on ε_N, and N is chosen such that $\varepsilon_N < \delta/2$. Next, let $r > 0$. For $z \in K$ and $|w - f(z)| > r$ we have

$$\begin{aligned} (4) \qquad h_N(z,w) &= \frac{1}{N} \log \left| (w - f(z)) + \sum_{j \geq N+1} \frac{f_j(z)}{g(z)^j} \right| + \log |g(z)| \\ &\geq \frac{1}{N} \log \left| |w - f(z)| - \sum_{j \geq N+1} \left| \frac{f_j(z)}{g(z)^j} \right| \right| + \log \delta \\ &\geq \frac{1}{N} \log \left(r - 2 \left(\frac{\varepsilon_N}{\delta} \right)^{N+1} \right) + \log \delta \geq -C_2, \end{aligned}$$

where $C_2 > 0$ if N is sufficiently large. The last estimate is that, for $C_3 > 0$ large enough,

$$h_N(z,w) < C_3 \quad \text{on } D \times B,$$

where C_3 depends on M and radius of B. Next we set

$$u_N = \max(h_N - C_3, \log \varepsilon_N).$$

Choose a sequence $\{N_i\}$ of positive integers and a sequence $\{d_i\}$ of positive numbers with the following properties: $\sum d_i < \infty$ but $\sum d_i \log \varepsilon_{N_i} = -\infty$. This is possible since $\varepsilon_N \downarrow 0$. We form the series

$$u(z,w) = \sum_{i \geq 1} d_i u_{N_i}(z,w).$$

Notice that on $D \times B, u$ is the limit of a decreasing sequence of plurisubharmonic functions. We use (3) to see that $u \equiv -\infty$ on $E \cap (K \times B)$ and hence on E. Next from (4) we obtain $u \neq -\infty$ if $w \neq f(z)$. Combining these facts, we conclude that u is plurisubharmonic on $D \times B$, real-valued, continuous away from $E \cup (A \times \mathbb{C})$ and satisfies

$$E \subset \{(z,w) : u(z,w) = -\infty\} \subset E \cup (A \times \mathbb{C}).$$

Thus the function $u(z,w) + \log|g| - \log M$ satisfies our conditions.

STEP 2. We show that Z is complete pluripolar in $D \times \mathbb{C}$. Let $\{D_j\}$ be an increasing sequence of relatively compact hyperconvex subdomains of D such that $D = \bigcup D_j$. Let K be a closed ball such that $K \subset D_1$ and $K \cap A = \emptyset$. Denote by E_K the graph of f over K. Let $\{B_j\}$ be an increasing sequence of open disks such that $f(K) \subset B_1$ and $\mathbb{C} = \bigcup B_j$. Then we have

$$(E_K)^-_{D_j \times B_j} = (E \cap (D_j \times B_j))^-_{D_j \times B_j}.$$

On the other hand, by Step 1 we can find a plurisubharmonic function u_j on $D_j \times B_j$ so that $u_j = -\infty$ precisely on $(D_j \times B_j) \cap Z$. This implies that

$$\omega((z,w), E_K, D_j \times B_j) = 0 \quad \text{for } (z,w) \in (D_j \times B_j) \setminus Z.$$

Thus using Lemma 2.2 we find

$$(E_K)^-_{D_j\times B_j} \subset (D_j \times B_j)\cap Z.$$

Application of this and Theorem 2.1 gives

$$E^*_{D\times\mathbb{C}} \subset (D\times\mathbb{C})\cap Z.$$

Let $a \in D\times\mathbb{C}\setminus Z$. Then we can find $v\in\mathcal{PSH}(D\times\mathbb{C})$ such that $v(a)\neq-\infty$ and $v\equiv-\infty$ on E. Hence the plurisubharmonic function $\widetilde{v}=v+\log|g|$ is identically $-\infty$ on Z, whereas $\widetilde{v}(a)\neq-\infty$. It follows that $Z^*_{D\times\mathbb{C}}=Z$. Finally, it is clear that Z is a G_δ as well as an F_σ, and we infer from Theorem 2.4 that Z is complete pluripolar in $D\times\mathbb{C}$.

We now apply this result to study pluripolar hulls of certain graphs which are analogues of [Wi1] (see Theorem 7 in [Wi1]).

Proposition 4.2. *Let D be a pseudoconvex domain in $\mathbb{C}^n$ $(n\geq 2)$, p and q be holomorphic functions on D such that $q\not\equiv 0, A=\{z\in D: q(z)=0\}\neq\emptyset$ and $A'=\{z\in D: p(z)=q(z)=0\}$ is of (complex) codimension 2 in D. Let φ be a holomorphic function on $\mathbb{C}$ which is not a polynomial. Denote by E the graph of $f=\varphi(p/q)$ over $D\setminus A$. Then*

$$E^*_{D\times\mathbb{C}} = E\cup(A'\times\mathbb{C}). \tag{5}$$

In particular, E is not complete pluripolar in $D\times\mathbb{C}$ if $A'\neq\emptyset$.

Proof. First we show that $A'\times\mathbb{C}\subset E^*_{D\times\mathbb{C}}$. We may assume that $A'\neq\emptyset$. Fix $z_0\in A'$ and $c\in\mathbb{C}\setminus\{0\}$. Then $z_0\in X_c=\{z\in D: p(z)-cq(z)=0\}$. Let $u\in\mathcal{PSH}(D\times\mathbb{C})$ be such that $u\equiv-\infty$ on E. Let $v(z)=u(z,\varphi(c))$. Clearly $v\equiv-\infty$ on $X_c\setminus A'$. As A' is of codimension 2 in D we deduce that $v\equiv-\infty$ on X_c. In particular $v\equiv-\infty$ on A'. It follows that $u\equiv-\infty$ on $A'\times\mathbb{C}$. Thus the right hand side of (5) is contained in the left hand side. It remains to prove the reverse inclusion. For this, take an arbitrary point $(z_0,w_0)\in A\times\mathbb{C}$ such that $p(z_0)\neq 0$; we claim that $(z_0,w_0)\notin E^*_{D\times\mathbb{C}}$. Indeed, set

$$F=\{(\xi,\varphi(1/\xi)):\xi\in\mathbb{C}\setminus\{0\}\},\qquad B=\{z\in D: p(z)=0\}.$$

As φ is not a polynomial, the point $\xi=0$ is an essential singularity for $\varphi(1/\xi)$. It follows from Theorem 2 in [Wi2] that F is complete pluripolar in $\mathbb{C}^2$. By Theorem 2.5 we get a function $u\in\mathcal{L}(\mathbb{C}^2)$ such that $u=-\infty$ precisely on F. Define

$$\widetilde{u}(z,w)=u\biggl(\frac{q(z)}{p(z)},w\biggr)+\log|p(z)|.$$

Obviously $\widetilde{u}\in\mathcal{PSH}((D\setminus B)\times\mathbb{C})$. Since $u\in\mathcal{L}(\mathbb{C}^2)$ the function $\widetilde{u}$ is locally bounded from above near every point of $B\times\mathbb{C}$. So it extends through $B\times\mathbb{C}$ to a plurisubharmonic function (still denoted by $\widetilde{u}$) on $D\times\mathbb{C}$. As $\widetilde{u}\equiv-\infty$

on $E \setminus (B \times \mathbb{C})$, an open subset of E, we infer $\widetilde{u} \equiv -\infty$ on E. Now the claim follows since $\widetilde{u}(z_0, w_0) = u(0, w_0) + \log|p(z_0)| > -\infty$. On the other hand, using Proposition 4.1 one gets

$$E^*_{D\times\mathbb{C}} \subset E \cup (A \times \mathbb{C}).$$

Putting all this together we are done.

5. Hulls in the class $\mathcal{L}(\mathbb{C}^n)$. We start with the following

PROPOSITION 5.1. *Let E be a pluripolar subset of $\mathbb{C}^n$. Then $E^*_{\mathbb{C}^n} = \widetilde{E}_{\mathbb{C}^n}$.*

For simplicity of notation, throughout this section we write $E^* = E^*_{\mathbb{C}^n}$ and $\widetilde{E} = \widetilde{E}_{\mathbb{C}^n}$.

Proof of Proposition 5.1. Obviously, $E^* \subset \widetilde{E}$. The reverse inclusion is an easy consequence of the Bedford–Taylor theorem (Theorem 2.5). Indeed, let $z_0 \notin E^*$. Then there is $u \in \mathcal{PSH}(\mathbb{C}^n)$ such that $u(z_0) > -\infty$ and $u|_E \equiv -\infty$. Let $E' = \{z : u(z) = -\infty\}$. Then E' is complete pluripolar in $\mathbb{C}^n$. So using Theorem 2.5, we find $v \in \mathcal{L}(\mathbb{C}^n)$ such that v is $-\infty$ exactly on E'. It follows that $z_0 \notin \widetilde{E}$. We are done.

PROPOSITION 5.2. *Let Ω be a domain in $\mathbb{C}^n$ and E be a pluripolar subset of Ω. Assume that $E^* \cap \Omega = E$ and E is F_σ and G_δ. Then there exists $u \in \mathcal{L}(\mathbb{C}^n)$ such that $u^{-1}(-\infty) \cap \Omega = E$.*

Proof. Applying Proposition 3.1 with $\Omega' = \mathbb{C}^n$ we obtain $v \in \mathcal{PSH}(\mathbb{C}^n)$ such that $v^{-1}(-\infty) \cap \Omega = E$. Now the set $v^{-1}(-\infty)$ is complete pluripolar in $\mathbb{C}^n$, so by Theorem 2.5 we get $u \in \mathcal{L}(\mathbb{C}^n)$ so that $u^{-1}(-\infty) = v^{-1}(-\infty)$. We are done.

Let $E \subset \mathbb{C}^n$. Then the *Siciak extremal function* associated to E is defined as follows:

$$V_E(z) = \sup\{u(z) : u \in \mathcal{L}(\mathbb{C}^n),\ u|_E \le 0\}.$$

Siciak has proved in [Sic] that if E is a pluripolar set in $\mathbb{C}^n$ then so is $A_{V_E} = \{z \in \mathbb{C}^n : V_E(z) < \infty\}$. The result below describes the pluripolar hull of A_{V_E} and $A_{V_{E^*}}$.

PROPOSITION 5.3. *Let $E \subset \mathbb{C}^n$ be a pluripolar set. Then*

$$(A_{V_E})^* = A_{V_{E^*}} = E^*.$$

Consequently, if $E = E^*$ then $E^* = A_{V_E} = \{z \in \mathbb{C}^n : V_E(z) < \infty\}$.

Proof. First we prove that $A_{V_{E^*}} = E^*$. Since $V_{E^*} \equiv 0$ on E^* we deduce that $E^* \subset A_{V_{E^*}}$. For the reverse inclusion, we fix $z \in A_{V_{E^*}}$ and let $u \in \mathcal{L}(\mathbb{C}^n)$ with $u \equiv -\infty$ on E. From the definition of E^* we have $u \equiv -\infty$ on E^*. Hence for every $m > 0$ we have $u + m \in \mathcal{L}(\mathbb{C}^n)$ and $u + m \equiv -\infty$ on E. Consequently, $u(z) + m \le V_{E^*}(z)$. This leads to $u(z) \le V_{E^*}(z) - m$.

Now let m go to ∞ to obtain $u(z) = -\infty$. Applying Proposition 5.1 we get $z \in \widetilde{E} = E^*$. The proof is complete.

6. Miscellanea. To finish this paper, we include the following elementary facts.

PROPOSITION 6.1. *Let E be a compact pluripolar subset of a pseudoconvex domain Ω in $\mathbb{C}^n$. Let $\widehat{E}_\Omega$ denote the holomorphic hull of E in Ω, i.e.,*

$$\widehat{E}_\Omega = \{z \in \Omega : |f(z)| \le \|f\|_E,\ \forall f \text{ holomorphic on } \Omega\}.$$

Then $\widehat{E}_\Omega \subset E^*_\Omega$.

Proof. Let $u \in \mathcal{PSH}(\Omega)$ be such that $u \equiv -\infty$ on E. Since Ω is pseudoconvex, it is well known that $\widehat{E}_\Omega$ coincides with the hull of E with respect to plurisubharmonic functions, and we infer that $u \equiv -\infty$ on $\widehat{E}_\Omega$. The desired conclusion follows.

PROPOSITION 6.2. *Let D, G be domains in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively and let $E \subset D, F \subset G$ be pluripolar sets. Then*

$$(E \times F)^*_{D\times G} = E^*_D \times F^*_G, \tag{6}$$
$$(E \times F)^-_{D\times G} = E^-_D \times F^-_G.$$

Proof. It is enough to prove (6), since the proof of the other equality is similar. For (6), we only have to show

$$E^*_D \times F^*_G \subset (E \times F)^*_{D\times G}$$

as the reverse inclusion is trivial. To this end, we claim that

$$E^*_D \times F \subset (E \times F)^*_{D\times G}. \tag{7}$$

Indeed, let $(z_0, w_0) \in E^*_D \times F$ and $\varphi(z,w) \in \mathcal{PSH}(D\times G)$ and $\varphi|_{E\times F} = -\infty$. Then $\varphi(z, w_0) \in \mathcal{PSH}(D)$ and $\varphi|_E = -\infty$. Hence, $\varphi(z_0, w_0) = -\infty$ and $(z_0, w_0) \in (E \times F)^*_{D\times G}$. Thus (7) is proved. Replace F in (7) by F^*_G to obtain

$$E^*_D \times F^*_G \subset (E \times F^*_G)^*_{D\times G}. \tag{8}$$

On the other hand, interchanging the roles of E and F in (7) gives

$$E \times F^*_G \subset (E \times F)^*_{D\times G}. \tag{9}$$

Combining (8) and (9) we get (6).

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