ANNALES
POLONICI MATHEMATICI
106 (2012)

Tangential Markov inequalities on semialgebraic curves and some semialgebraic surfaces

by Agnieszka Kowalska (Kraków)

Abstract. We give another proof of the fact that any semialgebraic curve admits a tangential Markov inequality. We establish this inequality on semialgebraic surfaces with finitely many singular points.

1. Introduction. The classical Markov inequality, which estimates the derivatives of polynomials on the line segment, has been generalized in many ways. The theory of the multivariate Markov inequality was developed in the seventies and eighties of the twentieth century. In particular, a Markov type inequality on convex compact subsets of $\mathbb{R}^N$ with a non-void interior and on uniformly polynomially cuspidal subsets of $\mathbb{R}^N$ was proved. For a detailed survey on this subject we refer the reader to [P]. Further important applications of Markov type inequalities to analysis were found. For semialgebraic sets we consider the following generalization of Markov's inequality.

A compact set $K \subset \mathbb{R}^N$ is said to admit a *tangential Markov inequality with exponent* l if there exists a positive constant M depending only on K such that for all polynomials p,

$$\|D_T p\|_K \leq M(\deg p)^l \|p\|_K,$$

where $D_T p$ denotes any (unit) tangential derivative of p along K, $\|p\|_K = \sup |p|(K)$ and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$.

The tangential Markov inequalities serve to characterize some subsets. According to [BLMT], a $\mathcal{C}^\infty$ submanifold K of $\mathbb{R}^N$ admits a tangential Markov inequality with exponent 1 if and only if K is algebraic.

Baran and Pleśniak characterized semialgebraic curves in $\mathbb{R}^N$ in terms of Bernstein and van der Corput–Schaake type inequalities (see [BP1]). Moreover in [BP2] they extended these results to the case of semialgebraic sets of higher dimensions in $\mathbb{R}^N$.

2010 *Mathematics Subject Classification*: Primary 41A17, 14P10; Secondary 41A25, 14P05.
Key words and phrases: tangential Markov inequality, semialgebraic sets.

DOI: 10.4064/ap106-0-17

In 2005 L. Gendre proved that every singular algebraic curve in $\mathbb{R}^N$ admits a local tangential Markov at each of its points. Moreover he showed that the Markov exponent at a point of a real algebraic curve A is less than or equal to twice the multiplicity of the smallest complex algebraic curve containing A.

Using the theorems proved by Baran and Pleśniak in [BP1] and [BP2], we show that semialgebraic curves and semialgebraic surfaces with finitely many singular points admit a tangential Markov inequality.

2. Preliminaries. Let K be a compact curve in $\mathbb{R}^N$ and let $I = [-1, 1]$. Following [BP1], K is said to admit an *analytic parametrization* if there exist $r \in \mathbb{N}$, $\gamma > 1$ and $\mathbb{R}$-analytic maps $\phi_j = (\phi_{j1}, \ldots, \phi_{jN}) : \gamma I \to K$, $j = 1, \ldots, r$, such that each $\phi_j|_I$ is a bijection onto $\phi_j(I)$ and

$$\bigcup_{j=1}^{r} \phi_j(I) = K.$$

We recall that a subset of $\mathbb{R}^N$ is *semialgebraic* if it is the union of finitely many subsets of the form

$$\{x \in \mathbb{R}^N : P(x) = 0 \wedge Q_1(x) > 0 \wedge \cdots \wedge Q_l(x) > 0\},$$

where $l \in \mathbb{N}$ and $P, Q_1, \ldots, Q_l \in \mathbb{R}[x_1, \ldots, x_N]$.

It is known that any semialgebraic curve in $\mathbb{R}^N$ admits an analytic parametrization. This is a consequence of the Puiseux theorem.

For a compact curve K in $\mathbb{R}^N$ with an analytic parametrization $\{\phi_j\}$ (with parameters r and γ) Baran and Pleśniak gave conditions equivalent to K being semialgebraic. In Section 3, for a semialgebraic arc with parametrization Φ, we use the following two:

$$\text{(2.1)} \quad \exists M_1, M_2 > 0\ \forall P \in \mathbb{C}[x_1, \ldots, x_N] \quad |(P \circ \Phi)(\xi)| \le M_2 \|P\|_K \quad \text{if } \operatorname{dist}(\xi, I) \le \frac{M_1}{\deg P},$$

and

$$\text{(2.2)} \quad \exists M_3 > 0\ \forall P \in \mathbb{C}[x_1, \ldots, x_N] \quad |(P \circ \Phi)'(t)| \le M_3 \deg P \cdot \|P\|_K, \quad t \in I.$$

Let now $\mathbb{B}^m(R) := \{x \in \mathbb{R}^m : \|x\| \le R\}$, $\mathbb{B}^m := \mathbb{B}^m(1)$, $\mathbb{S}^{m-1}(R) := \partial\mathbb{B}^m(R)$ and $\mathbb{S}^{m-1} := \mathbb{S}^{m-1}(1)$.

DEFINITION 2.1 ([BP2, Definition 4.2]). Let K be a compact subset of $\mathbb{R}^n$. Then K is said to have an *analytic parametrization of dimension* m, $1 \le m \le n$, if there exist $\rho > 1$, $r \in \mathbb{N}$ and $\mathbb{R}$-analytic maps $\phi_j = (\phi_{j1}, \ldots, \phi_{jn}) : \mathbb{B}^m(\rho) \to K$, $j = 1, \ldots, r$, such that for each $j = 1, \ldots, r$ we have rank $\phi_j = m$ and

$$K = \bigcup_{j=1}^{r} \phi_j(\mathbb{B}^m).$$

Let $\mathbb{M}$ be an m-dimensional real-analytic manifold of $\mathbb{R}^n$. By the Hironaka Rectilinearization Theorem one can prove that every compact semialgebraic subset of $\mathbb{M}$ of pure dimension m admits an analytic parametrization in the sense of the above definition. Moreover in Definition 2.1, instead of considering an analytic parametrization defined in a neighbourhood of the unit ball $\mathbb{B}^m$, we may work with an analytic parametrization defined in an open neighbourhood of the cube $\mathbb{I}^m$ (see [BP2]).

3. Tangential Markov inequality on curves. In this section we prove a tangential Markov inequality on semialgebraic arcs. First we give a technical lemma.

LEMMA 3.1. *Let K be a semialgebraic arc which has an analytic parametrization*

$$\Phi(t) = (\varphi_1(t), \dots, \varphi_N(t))$$

in a neighbourhood of $I = [-1, 1]$ such that in a neighbourhood of 0 (*which is the only singular point*) $\varphi_i(t) = \alpha_{i0} + \sum_{n=k}^{\infty} \alpha_{in} t^n$ *and $\alpha_{1k} = 1$. Then there exists a positive constant C such that for every polynomial $P \in \mathbb{C}[x_1, \dots, x_N]$ with* $\deg P \le n$ *and for all $t \in I$,*

$$\left| \frac{1}{t^{k-1}} (P \circ \Phi)'(t) \right| \le C n^k \|P\|_K.$$

Proof. The proof is divided into two steps.

1. If $t \in I$ and $|t| > M_1/4n$, then from (2.2),

$$\left| \frac{1}{t^{k-1}} (P \circ \Phi)'(t) \right| \le \left(\frac{4n}{M_1} \right)^{k-1} |(P \circ \Phi)'(t)| \le \left(\frac{4}{M_1} \right)^{k-1} M_3 n^k \ \|P\|_K.$$

2. If $t \in I$ and $|t| \le M_1/4n$, then

$$\left| \frac{1}{t^{k-1}} (P \circ \Phi)'(t) \right| = \left| \frac{1}{2\pi i} \int_{|\xi - t| = r} \frac{1}{\xi^{k-1}} \frac{(P \circ \Phi)'(\xi)}{\xi - t} \, d\xi \right|$$

$$= \frac{1}{2\pi} \left| \int_{|\xi - t| = r} \frac{1}{\xi^{k-1}} \frac{1}{\xi - t} \frac{1}{2\pi i} \int_{|\eta - \xi| = \rho} \frac{(P \circ \Phi)(\eta)}{(\eta - \xi)^2} \, d\eta \, d\xi \right|.$$

By choosing $r = \rho = M_1/2n$ we have $\operatorname{dist}(\eta, I) \le M_1/n$. From (2.1) we conclude that

$$\left| \frac{1}{t^{k-1}} (P \circ \Phi)'(t) \right| \le \frac{1}{2\pi} \frac{2n}{M_1} M_2 \|P\|_K \int_{|\xi - t| = M_1/2n} \frac{1}{|\xi|^{k-1}} \frac{1}{|\xi - t|} \, d\xi.$$

Since $|t| \le M_1/4n$, we see that $|\xi| \ge M_1/4n$. Hence

$$\left|\frac{1}{t^{k-1}}(P\circ\Phi)'(t)\right| \le \frac{M_2}{2}\left(\frac{4}{M_1}\right)^k n^k\|P\|_K.$$

Taking $C = \max\{(4/M_1)^{k-1}M_3, (M_2/2)(4/M_1)^k\}$ we obtain our claim. ■

The main result of this section is the following

THEOREM 3.2. *Let K be a semialgebraic arc which has an analytic parametrization*

$$\Phi(t) = (\varphi_1(t), \ldots, \varphi_N(t))$$

in a neighbourhood of $I = [-1,1]$ such that in a neighbourhood of 0 (*which is the only singular point*) $\varphi_i(t) = \alpha_{i0} + \sum_{n=k}^{\infty} \alpha_{in}t^n$ *and* $\alpha_{1k} = 1$. *Then there exists a positive constant M such that for every polynomial $P \in \mathbb{C}[x_1, \ldots, x_N]$ with* $\deg P \le n$,

$$\|D_{\mathcal{T}}P\|_K \le Mn^k\|P\|_K,$$

where $\|P\|_K = \sup_{t\in I}|P(\Phi(t))|$ *and* $D_{\mathcal{T}}P(\Phi(t))$ *denotes the derivative of P in the direction of a unit vector of the tangent cone to K at $\Phi(t)$.*

Proof. We first prove that there exist positive constants M_1, M_2 such that for each $t \in I$,

$$M_1 \le \frac{|t|^{k-1}}{\|(\varphi_1'(t), \ldots, \varphi_N'(t))\|} \le M_2.$$

By assumption

$$\varphi_i'(t) = \sum_{n=k}^{\infty} \alpha_{in} n t^{n-1}.$$

Hence

$$\lim_{t\to 0}\frac{|t|^{k-1}}{\sqrt{\sum_{i=1}^{N}|\varphi_i'(t)|^2}} = \lim_{t\to 0}\frac{|t|^{k-1}}{|t|^{k-1}\sqrt{\sum_{i=1}^{N}\left|\sum_{n=k}^{\infty}\alpha_{in}nt^{n-k}\right|^2}} = \frac{1}{|k|\sqrt{1+\sum_{i=2}^{N}|\alpha_{ik}|^2}}.$$

Define

$$\Phi_1(t) := (\varphi_1(t^{1/k}), \ldots, \varphi_N(t^{1/k})), \quad t \in [0,1].$$

Then $\Phi_1(t)$ is a C^1 parametrization of the curve $K|_{[0,1]}$ and

$$\Phi_1'(t) := \left(\frac{1}{k}t^{1/k-1}\varphi_1'(t^{1/k}), \ldots, \frac{1}{k}t^{1/k-1}\varphi_N'(t^{1/k})\right), \quad t \in [0,1].$$

Moreover,

$$\lim_{t\to 0}\frac{1}{k}t^{1/k-1}\varphi_1'(t^{1/k}) = 1 \quad \text{and} \quad \lim_{t\to 0}\frac{1}{k}t^{1/k-1}\varphi_i'(t^{1/k}) = \alpha_{ik} \quad \text{for } i = 2, \ldots, N.$$

It follows that the vector $r = (1, \alpha_{2k}, \ldots, \alpha_{Nk})$ is an element of the tangent cone to K at $(\varphi_1(0), \ldots, \varphi_N(0))$. We claim that for each $t \in I$,

$$D_{\mathcal{T}} P(\varphi_1(t), \ldots, \varphi_N(t)) = \frac{(P(\varphi_1(t), \ldots, \varphi_N(t)))'}{\|(\varphi_1'(t), \ldots, \varphi_N'(t))\|}. \tag{3.1}$$

It is sufficient to show (3.1) for homogeneous polynomials. Consider $P(x) = \prod_{j=1}^{N} x_j^{\beta_j}$, where $\beta_j \in \mathbb{N}$ for $j = 1, \ldots, N$. Then for $t \neq 0$ we have

$$\begin{aligned} D_{\mathcal{T}} P(\varphi_1(t), \ldots, \varphi_N(t)) &= \lim_{h \to 0} \frac{\prod_{j=1}^{N} \left(\varphi_j(t) + \frac{\varphi_j'(t)}{\sqrt{\sum_{i=1}^{N} |\varphi_i'(t)|^2}} h \right)^{\beta_j} - \prod_{j=1}^{N} (\varphi_j(t))^{\beta_j}}{h} \\ &= \frac{(P(\varphi_1(t), \ldots, \varphi_N(t)))'}{\sqrt{\sum_{i=1}^{N} |\varphi_i'(t)|^2}}. \end{aligned}$$

Moreover,

$$D_{\mathcal{T}} P(\varphi_1(0), \ldots, \varphi_N(0)) = \sum_{j=1}^{N} \beta_j \alpha_{j0}^{\beta_j - 1} \frac{\alpha_{jk}}{\sqrt{\sum_{m=1}^{N} |\alpha_{mk}|^2}} \prod_{i \neq j, i=1}^{N} \alpha_{i0}^{\beta_i}$$

and

$$\lim_{t \to 0} \frac{(P(\varphi_1(t), \ldots, \varphi_N(t)))'}{\sqrt{\sum_{m=1}^{N} |\varphi_m'(t)|^2}} = \frac{\sum_{i=1}^{N} k \beta_i (\alpha_{i0})^{\beta_i - 1} \alpha_{ik} \prod_{j \neq i,\, j=1}^{N} (\alpha_{j0})^{\beta_j}}{k \sqrt{\sum_{m=1}^{N} |\alpha_{mk}|^2}}.$$

From (3.1) and Lemma 3.1 we obtain

$$\begin{aligned} |D_{\mathcal{T}} P(\varphi_1(t), \ldots, \varphi_N(t))| &\leq M_2 \left| \frac{1}{t^{k-1}} (P(\varphi_1(t), \ldots, \varphi_N(t)))' \right| \\ &\leq M_2 C n^k \|P\|_K. \ \blacksquare \end{aligned}$$

Immediately from the above theorem and the structure of tangent cones for Cartesian products we have

Corollary 3.3. *Let $S = K_1 \times K_2$, where K_1 and K_2 are semialgebraic curves. Then there exists a positive constant M such that for each polynomial $P \in \mathbb{C}[x_1, \ldots, x_N]$ with* $\deg P \leq n$,

$$\|D_{\mathcal{T}} P\|_S \leq M n^k \|P\|_S.$$

Corollary 3.4. *Let $S = K \times S_1$, where K is a semialgebraic curve and S_1 is a C^1 non-singular semialgebraic surface. Then there exists a positive* *constant M such that for each polynomial $P \in \mathbb{C}[x_1, \ldots, x_N]$ with* $\deg P \le n$,

$$\|D_{\mathcal{T}} P\|_S \le M n^k \|P\|_S.$$

4. Tangential Markov inequality on surfaces. Another generalization of Theorem 3.2 is a tangential Markov inequality on semialgebraic surfaces with finitely many singular points. It is sufficient to prove this inequality for surfaces with one singular point. To simplify we describe it for a subset of $\mathbb{R}^3$.

THEOREM 4.1. *Let $\mathbb{V}$ be a C^1 semialgebraic surface with analytic parametrization*

$$\Phi(u) = (\varphi_1(u), \varphi_2(u), \varphi_3(u)), \qquad u \in \mathbb{B}^2(\rho),$$

such that $\operatorname{rank} \Phi = 2$ *on* $\mathbb{B}^2(\rho) \setminus \{0\}$ (0 *is the only singular point*). *Moreover, assume that $\Phi(0) = 0$ and there exists $\epsilon > 0$ such that $\Phi(u) \neq 0$ for $u \in \mathbb{B}^2(\epsilon) \setminus \{0\}$. Then there exist constants $D > 0$ and $k \in \mathbb{N}$ such that for each polynomial $P \in \mathbb{C}[x_1, x_2, x_3]$ with* $\deg P \le n$,

$$\|D_{\mathcal{T}} P\|_{\mathbb{V}} \le D n^k \|P\|_{\mathbb{V}}.$$

Proof. By assumptions

$$\varphi_i(u_1, u_2) = \sum_{j=1}^{\infty} \frac{1}{j!} \sum_{l=0}^{j} \alpha_{ijl} u_1^{j-l} {u_2}^l.$$

Let $v \in \mathbb{S}^1$ and $t \in I$. We have $\Phi(tv) = (\varphi_1(tv), \varphi_2(tv), \varphi_3(tv))$, where

$$\varphi_i(tv) = \sum_{j=1}^{\infty} P_{ij}(v) t^j \quad \text{with} \quad P_{ij}(v) = \frac{1}{j!} \sum_{l=0}^{j} \alpha_{ijl} {v_1}^{j-l} {v_2}^l.$$

By assumption, Φ is not equal to zero on $\mathbb{B}^2(\rho)$, so for each $v \in \mathbb{S}^1$ there exist $l, k(v) \in \mathbb{N}$ such that $P_{lk(v)}(v) \neq 0$ and $P_{ij} = 0$ for $i \in \{1,2,3\}$, $j \in \{1, \ldots, k(v) - 1\}$. Hence $\Phi(tv) = (\varphi_1(tv), \varphi_2(tv), \varphi_3(tv))$, where

$$\varphi_i(tv) = \sum_{j=k(v)}^{\infty} P_{ij}(v) t^j \qquad \text{for } i \in \{1,2,3\}.$$

We see at once that there exists a constant κ such that $k(v) \le \kappa$ for all $v \in \mathbb{S}^1$. Fix $v \in \mathbb{S}^1$. For $t \in [-\delta_3/4n, \delta_3/4n]$ we obtain

$$\begin{aligned} \left| \frac{(P(\Phi(tv))'}{t^{k(v)-1}} \right| &= \left| \frac{1}{2\pi i} \int_{|\xi - t| = r} \frac{1}{\xi^{k(v)-1}} \frac{(P(\Phi(\xi v)))'}{\xi - t} \, d\xi \right| \\ &= \frac{1}{2\pi} \left| \int_{|\xi - t| = r} \frac{1}{\xi^{k(v)-1}} \frac{1}{\xi - t} \frac{1}{2\pi i} \int_{|\eta - \xi| = \rho} \frac{(P \circ \Phi)(\eta v)}{(\eta - \xi)^2} \, d\eta \, d\xi \right|. \end{aligned}$$

If we take $r = \rho = \delta_3/2n$, then $\operatorname{dist}(\eta v, \mathbb{B}^2) \le \delta_3/n$. Hence (see [BP2, Theorem 4.5(iii)]) we get

$$\left|\frac{(P(\Phi(tv)))'}{t^{k(v)-1}}\right| \le \frac{1}{2\pi}\,\frac{2n}{\delta_3} C_3\|P\|_{\mathbb{V}} \int\limits_{|\xi-t|=\delta_3/2n} \frac{1}{|\xi|^{k(v)-1}}\,\frac{1}{|\xi-t|}\,d\xi.$$

Since $|t| \le \delta_3/4n$ we have $|\xi| = |\xi - t + t| \ge |\xi - t| - |t| \ge \delta_3/2n - \delta_3/4n = \delta_3/4n$. Therefore

$$\left|\frac{(P(\Phi(tv)))'}{t^{k(v)-1}}\right| \le \left(\frac{4n}{\delta_3}\right)^{k(v)-1} \frac{2n}{\delta_3} C_3\|P\|_{\mathbb{V}} = \frac{C_3}{2}\left(\frac{4}{\delta_3}\right)^{k(v)} n^{k(v)}\|P\|_{\mathbb{V}}.$$

For $|t| > \delta_3/4n$ we get (see [BP2, Theorem 4.5(iv)]

$$\left|\frac{(P(\Phi(tv)))'}{t^{k(v)-1}}\right| \le \left(\frac{4n}{\delta_3}\right)^{k(v)-1} 2DC_4 n\|P\|_{\mathbb{V}} = \left(\frac{4}{\delta_3}\right)^{k(v)-1} 2DC_4 n^{k(v)}\|P\|_{\mathbb{V}},$$

where D is a constant depending only on $\mathbb{V}$.

Taking $C = \max\{2DC_4(4/\delta_3)^{\kappa-1}, (C_3/2)(4/\delta_3)^{\kappa}\}$ we obtain for $t \in I$ and $v \in \mathbb{S}^1$,

$$\left|\frac{(P(\Phi(tv)))'}{t^{k(v)-1}}\right| \le Cn^{\kappa}\|P\|_K.$$

Proceeding similarly to the proof of Theorem 3.2 we can show that $W_v = (P_{1k(v)}(v), P_{2k(v)}(v), P_{3k(v)}(v))$ for each $v \in \mathbb{S}^1$ is an element of the tangent cone to $\mathbb{V}$ at $\Phi(0)$. As before

$$|D_{\mathcal{T}}P(\varphi_1(tv), \varphi_2(tv), \varphi_3(tv))| = \left|\frac{(P(\varphi_1(tv), \varphi_2(tv), \varphi_3(tv)))'}{\|((\varphi_1(tv))', (\varphi_2(tv))', (\varphi_3(tv))')\|}\right|.$$

Finally, there exist constants D and k such that for each polynomial $P \in \mathbb{C}[x_1, x_2, x_3]$ we have

$$\|D_{\mathcal{T}}P\|_{\mathbb{V}} \le Dn^k\|P\|_{\mathbb{V}}. \ \blacksquare$$

References

[BP1] M. Baran and W. Pleśniak, *Bernstein and van der Corput–Schaake type inequalities on semialgebraic curves*, Studia Math. 125 (1997), 83–96.

[BP2] M. Baran and W. Pleśniak, *Characterization of compact subsets of algebraic varieties in terms of Bernstein type inequalities*, Studia Math. 141 (2000), 221–233.

[BLMT] L. P. Bos, N. Levenberg, P. Milman and B. A. Taylor, *Tangential Markov inequalities characterize algebraic submanifolds of* $\mathbb{R}^N$, Indiana Univ. Math. J. 44 (1995), 115–138.

[G] L. Gendre, *Inégalités de Markov tangentielles locales sur les courbes algébriques singulières de* $\mathbb{R}^n$, Ann. Polon. Math. 86 (2005), 59–77.

[PP] W. Pawłucki and W. Pleśniak, *Markov's inequality and* C^∞ *functions on sets with polynomial cups*, Math. Ann. 275 (1986), 467–480.

[P] W. Pleśniak, *Recent progress in multivariate Markov inequality*, in: Approximation Theory, Monogr. Textbooks Pure Appl. Math. 212, Dekker, New York, 1998, 449–464.

Agnieszka Kowalska
Institute of Mathematics
Pedagogical University
Podchorążych 2
30-084 Kraków, Poland
E-mail: kowalska@up.krakow.pl

Received 30.11.2011
and in final form 7.6.2012 (2624)