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Bases in spaces of analytic germs

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Dedicated to Professor J. Siciak on the occasion of his 80th birthday

Abstract. We prove precise decomposition results and logarithmically convex estimates in certain weighted spaces of holomorphic germs near $\mathbb{R}$. These imply that the spaces have a basis and are tamely isomorphic to the dual of a power series space of finite type which can be calculated in many situations. Our results apply to the Gelfand–Shilov spaces S^1_α and S^α_1 for $\alpha > 0$ and to the spaces of Fourier hyperfunctions and of modified Fourier hyperfunctions.

1. Introduction. The structure theory of Fréchet spaces and especially the theory of power series spaces has proved to have many applications to linear problems in analysis such as existence of continuous linear right inverses or solvability of vector-valued or parameter dependent equations. The relevant modern tools to treat this type of problems are splitting theory for power series and homological techniques like the Proj or Ext functors. To apply these tools we often need to know that the spaces under consideration are isomorphic to power series spaces or at least share some of the properties of (DN) or (Ω) type which are typical for power series spaces.

For spaces of holomorphic functions defined on a fixed domain these properties have been intensively studied in the literature (see e.g. [14–16, 18, 23] and the references cited there), while for germs of holomorphic functions much less is known: the space of holomorphic germs near a compact set $K \subset \mathbb{C}^d$ is well studied, and we have shown in [10] that the Hermite functions are a basis in the space $P_*(\mathbb{R})$ of test functions for the Fourier hyperfunctions defining an isomorphism of $P_*(\mathbb{R})$ to $\Lambda_0(n^{1/2})'_b$, i.e. to the dual of a certain power series space of finite type. This result has recently

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been extended to expansions with respect to the eigenfunctions of certain elliptic differential operators (see [3] and also [1]). The method of proof used in [10] was limited to spaces invariant under Fourier transformation. So it cannot be applied to the Gelfand–Shilov spaces of holomorphic functions S^1_α for $\alpha > 0$ (see [2] and notice that $P_*(\mathbb{R}) = S^1_1$ in the notation of Gelfand and Shilov) and it also did not work for the test function space for the modified Fourier hyperfunctions (see [20]).

In the present paper we study this question for (DFS)-spaces $\mathcal{H}_v(\mathbb{R})$ of germs of holomorphic functions defined on strips near $\mathbb{R}$ as follows:

$$\mathcal{H}_v(\mathbb{R}) := \liminf_{n\to\infty} H_{1/n,1/n}(V_{1/n})$$

with

$$H_{1/n,1/n}(V_{1/n}) := \Big\{ f \in \mathcal{H}(V_{1/n}) \ \Big|\ \|f\|_n := \sup_{z\in V_{1/n}} |f(z)|e^{v(z)/n} < \infty \Big\}$$

for $V_{1/n} := \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < 1/n\}$ where v is a weight function satisfying some mild natural conditions (see 2.1).

We are working in the tame category since the splitting theory for power series spaces of finite type needs this restricted class of continuous linear mappings (see [19]). The basic tool of our considerations is the tame variant, developed in [9], of the Mityagin–Henkin result on existence of bases in power series spaces of finite type (see [22]). This means that we have to prove that the "norms" in the spaces in question (and in their duals) satisfy certain submultiplicative estimates. The latter means that we have to solve a decomposition problem with bounds for holomorphic functions near the real line. This is achieved in Section 2 (see Theorem 2.2) using suitable (holomorphic) cut-off functions (see Lemma 2.3) and the decomposition of holomorphic functions into summands defined on different strips including precise estimates for the summands (see Lemma 2.5). A useful logarithmically convex estimate is obtained in Section 3. We thus obtain the following main result in Section 4 (see Theorem 4.4)

Theorem. *For any weight function v the space $\mathcal{H}_v(\mathbb{R})$ is tamely isomorphic to some $\Lambda_0(\alpha)'_b$, i.e. to the dual of a power series space of finite type.*

Notice that our method of proof only gives the existence of a Schauder basis but not a concrete basis as in [1, 3, 10]. However, the coefficient space $\Lambda_0(\alpha)'_b$ (i.e. the sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$) may be calculated using estimates for the diametral dimension of $\mathcal{H}_v(\mathbb{R})$ given in [12]. This implies in particular that the Gelfand–Shilov spaces S^1_α are tamely isomorphic to $\Lambda_0(n^{1/(\alpha+1)})'_b$. More examples are provided in Section 5.

The method may be transferred to spaces defined on conic neighborhoods of $\mathbb{R}$, showing that the space of modified Fourier hyperfunctions is tamely isomorphic to $\Lambda_0(n/\ln(n))$. In particular, it has a basis. Moreover, the spaces of Fourier hyperfunctions and of modified Fourier hyperfunctions are not isomorphic.

2. Decomposition of holomorphic functions. Roughly speaking, proving a linear topological invariant of (Ω)-type (or the dual formulation of invariants of (DN) type) for a locally convex space E just means proving a decomposition in E with a certain control of the seminorms of the summands. In this section we will prove a rather general corresponding decomposition result for holomorphic functions defined on strips

$$V_t := \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < t\}$$

near $\mathbb{R}$ by weight functions in the following sense:

Definition 2.1. A continuous function $v : \mathbb{C} \to [0,\infty[$ is called a *weight function* if $v(x+iy) := v(|x|)$ on $\mathbb{C}$ where $v : [0,\infty[\ \to [0,\infty[$ is strictly increasing and satisfies

$$\ln(1+|x|) = o(v(x)) \tag{2.1}$$

and there are $\Gamma > 1$ and $C > 0$ such that

$$v(x+1) \le \Gamma v(x) + C \quad \text{ if } x \ge 0. \tag{2.2}$$

In the rest of the present paper v will always denote a weight function. We will also assume without loss of generality that $v(0) = 0$, i.e. that v is bijective on $[0,\infty[$.

In this section we consider the weighted Banach spaces of holomorphic functions given by

$$H_\tau(V_t) := \Big\{ f \in \mathcal{H}(V_t) \ \Big|\ \|f\|_{\tau,t} := \sup_{z \in V_t} |f(z)| e^{\tau v(z)} < \infty \Big\}$$

for $t > 0$ and $\tau \in \mathbb{R}$. The following decomposition theorem is the main result of this section:

Theorem 2.2. *There are $\widetilde{t}, K_1 > 0$ such that for any $\tau_0 < \tau < \tau_2$ there are $C_0 = C_0(\mathrm{sign}(\tau_0)) > 0$ and $K_0 = K_0(\mathrm{sign}(\tau)) > 0$ such that for any $0 < 2t_0 < t < t_2 < \widetilde{t}$ with*

$$t_0 \le \min\left[K_1, K_2 \sqrt{\frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau_0}}\right]$$

there is $C_1 \ge 1$ such that for any $r \ge 0$ and any $f \in H_\tau(V_t)$ with $\|f\|_{\tau,t} \le 1$ the following holds: there are $f_2 \in H(V_{t_2})$ and $f_0 \in H(V_{t_0})$ such that $f = f_0 + f_2$ on V_{t_0} and

$$\|f_0\|_{K_0\tau_0,t_0} \le C_1 e^{-Gr} \quad \textit{and} \quad \|f_2\|_{\tau_2,t_2} \le e^r \tag{2.3}$$

where

$$G := K_1 \min\left[1, \frac{t-t_0}{2\widetilde{t}}, \frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau_0}\right].$$

The proof of Theorem 2.2 will be obtained in several steps starting with the construction of appropriate holomorphic cut-off functions as follows: for $r > 0$ let

$$H_r(z) := \frac{1}{D_r}\int_{\gamma_z} \cosh(\xi)e^{-r\cosh(\xi)}\,d\xi, \qquad z \in V_1,$$

where $D_r := \int_{-\infty}^{\infty} \cosh(x)e^{-r\cosh(x)}\,dx$ and γ_z is a path in V_1 from $-\infty$ to z. Set

$$E_{r,A}(z) := H_r(A+z)H_r(A-z) \tag{2.4}$$

for $A > 0$.

Lemma 2.3. *H_r and $E_{r,A}$ are entire functions such that there are $B_j > 0$ and $C_1 > 0$ such that for any $t \in \,]0,1]$ and any $r, A > 0$,*

$$|E_{r,A}(z)| \le C_1 e^{B_1 r t^2} \qquad \text{if } z \in V_t, \tag{2.5}$$

$$|E_{r,A}(z)| \le C_1 e^{-\frac{r}{8}e^{|\mathrm{Re}(z)|-A}} \qquad \text{if } z \in V_1 \text{ and } |\mathrm{Re}(z)| \ge A + B_2, \tag{2.6}$$

$$|1 - E_{r,A}(z)| \le C_1 e^{-\frac{r}{8}e^{A-|\mathrm{Re}(z)|}} \qquad \text{if } z \in V_1 \text{ and } |\mathrm{Re}(z)| \le A - B_2. \tag{2.7}$$

Proof. (a) Since

$$\cosh(x+iy) = \cosh(x)\cos(y) + i\sinh(x)\sin(y) \qquad \text{for } x, y \in \mathbb{R} \tag{2.8}$$

and therefore

$$|\cosh(x+iy)|^2 = \cosh^2(x) - \sin^2(y) \qquad \text{for } x, y \in \mathbb{R}, \tag{2.9}$$

we have, for $t \in \,]0,1]$,

$$\begin{aligned} \exp(1 + e^{|x|}/2) \ge e^{\cosh(x)} &\ge |e^{\cosh(z)}| \\ &\ge e^{\cosh(x)\cos(t)} \ge \exp(e^{|x|}/4) \qquad \text{if } z = x+iy \in V_t. \end{aligned} \tag{2.10}$$

The integral defining H_r is thus convergent on V_1, and D_r is finite; H_r is well defined (by Cauchy's integral theorem) and holomorphic on V_1. Furthermore, H_r can be extended to an entire function since it is the primitive of $(1/D_r)\cosh(\xi)e^{-r\cosh(\xi)}$ on V_1 vanishing at $-\infty$.

(b) By (2.9) and (2.10) we get, for $z = x+iy \in V_t$ and $t \in \,]0,1]$,

$$\begin{aligned} |H_r(z)| &= \frac{1}{D_r}\Big|\int_{-\infty}^{x} \cosh(\xi+iy)e^{-r\cosh(\xi+iy)}\,d\xi\Big| \\ &\le \frac{1}{D_r}\int_{-\infty}^{x} \cosh(\xi)e^{-r\cosh(\xi)\cos(t)}\,d\xi. \end{aligned} \tag{2.11}$$

Notice that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \cosh(\xi) e^{-r\cosh(\xi)\cos(t)}\,d\xi \\
&\qquad \le 2\int_0^1 \cosh(\xi) e^{-r\cosh(\xi)}\,d\xi \sup_{\xi\in[0,1]} e^{r(1-\cos(t))\cosh(\xi)} \\
&\qquad\quad + 4\int_1^{\infty} \sinh(\xi) e^{-r\cosh(\xi)\cos(t)}\,d\xi \\
&\qquad \le 4\Big(\int_0^1 \cosh(\xi) e^{-r\cosh(\xi)}\,d\xi\, e^{r(1-\cos(t))\cosh(1)} + \frac{1}{r\cos(t)} e^{-r\cosh(1)\cos(t)}\Big) \\
&\qquad \le \frac{4}{\cos(1)} e^{r(1-\cos(t))\cosh(1)} D_r
\end{aligned}$$

since

$$\begin{aligned}
D_r &\ge \int_0^1 \cosh(\xi) e^{-r\cosh(\xi)}\,d\xi + 2\int_1^{\infty} \sinh(\xi) e^{-r\cosh(\xi)}\,d\xi \\
&= \int_0^1 \cosh(\xi) e^{-r\cosh(\xi)}\,d\xi + \frac{2}{r} e^{-r\cosh(1)}.
\end{aligned}$$

This shows that

$$|H_r(z)| \le C_1 e^{B_1 r t^2} \qquad \text{if } z \in V_t$$

and therefore $E_{r,A}$ satisfies (2.5).

(c) Let $z = x + iy \in V_1$ and $x \le -1$. Since cosh is even, we get as above

$$\begin{aligned}
|H_r(z)| &= \frac{1}{D_r}\Big| \int_{-\infty}^{x} \cosh(-\xi - iy) e^{-r\cosh(-\xi-iy)}\,d\xi \Big| \\
&= \frac{1}{D_r}\Big| \int_{|x|}^{\infty} \cosh(\xi - iy) e^{-r\cosh(\xi-iy)}\,d\xi \Big| \\
&\le \frac{1}{D_r} \int_{|x|}^{\infty} \cosh(\xi) e^{-r\cosh(\xi)\cos(1)}\,d\xi \le r e^{r} \int_{|x|}^{\infty} \sinh(\xi) e^{-r\cosh(\xi)/2}\,d\xi \\
&\le 2e^{r - r\cosh(|x|)/2} \le 2e^{r - r\exp(|x|)/4},
\end{aligned}$$

because

$$D_r \ge 2\int_0^{\infty} \sinh(x) e^{-r\cosh(x)}\,dx = 2e^{-r}/r.$$

Using also (2.5) (for H_r instead of $E_{r,A}$) this implies (2.6) for suitable B_2.

(d) Since cosh is even, we get by Cauchy's integral theorem

$$\begin{aligned}
1 - H_r(z) \\
&= \frac{1}{D_r}\Big(\int_{-\infty}^{\infty} \cosh(\xi+iy)e^{-r\cosh(\xi+iy)}\,d\xi - \int_{-\infty}^{x} \cosh(\xi+iy)e^{-r\cosh(\xi+iy)}\,d\xi\Big) \\
&= \frac{1}{D_r}\int_{x}^{\infty} \cosh(-\xi-iy)e^{-r\cosh(-\xi-iy)}\,d\xi \\
&= \frac{1}{D_r}\int_{-\infty}^{-x} \cosh(\xi-iy)e^{-r\cosh(\xi-iy)}\,d\xi = H_r(-x-iy) = H_r(-z)
\end{aligned}$$

and hence

$$\begin{aligned}
1 - E_{r,A}(z) &= (1 - H_r(z+A))H_r(A-z) + (1 - H_r(A-z)) \\
&= H_r(-A-z)H_r(A-z) + H_r(z-A)
\end{aligned}$$

satisfies (2.7) by (2.5) and the estimates given in (b) and (c) (applied to $H_r(-A-z)$ and $H_r(z-A)$). ∎

The bounds in the space $\mathcal{H}_v(\mathbb{R}) := \operatorname{lim\,ind}_{n\to\infty} H_{1/n,1/n}(V_{1/n})$ of germs of holomorphic functions are given by the functions $\exp(v(z)/n)$, $n \in \mathbb{N}$. We will now show that by (2.2) we can use the bounds $|\exp(w(z)/n)| = \exp(\operatorname{Re}(w(z))/n)$ instead with a holomorphic function w leading to a tame change of the seminorms.

Lemma 2.4. *There are $0 < \widetilde{t} \le 1$ and $B_j \ge 1$ and a holomorphic function w on $V_{\widetilde{t}}$ such that*

$$v(z) \le \operatorname{Re}(w(z)) \le B_3 v(z) + B_4 \quad \text{if } z \in V_{\widetilde{t}}. \tag{2.12}$$

Proof. Considering $\widetilde{v} := v + A$ instead of v for large A we can assume that

$$v(x+1) \le \Gamma v(x) \quad \text{if } x \ge 0 \tag{2.13}$$

by (2.2) since $\Gamma > 1$. This implies that

$$v(x+y) \le \Gamma v(x)\Gamma^{y} \quad \text{if } x \ge 0 \text{ and } y \ge 0. \tag{2.14}$$

(a) Set $C_1 := 2\ln(\Gamma)$ and let

$$w(x+iy) := \int_{-\infty}^{\infty} v(t)/\cosh(C_1(x+iy-t))\,dt.$$

By (2.9) we have

$$\begin{aligned}
|\cosh(C_1(x-t+iy))|^2 &= \cosh^2(C_1(x-t)) - \sin^2(C_1 y) \\
&\ge \cosh^2(C_1(x-t)) - (C_1 y)^2 \ge \frac{1}{2}\cosh^2(C_1(x-t)) \\
&\ge \frac{1}{8}e^{2C_1|x-t|} \quad \text{if } |y| \le \widetilde{t} := 1/(2C_1).
\end{aligned} \tag{2.15}$$

The inequalities (2.15) and (2.14) imply that

$$|w(x+iy)| \le 4\int_{-\infty}^{\infty} v(|x|+|t-x|)e^{-C_1|t-x|}\,dt$$
$$\le 4\Gamma v(x)\int_{-\infty}^{\infty} e^{-\ln(\Gamma)|x-t|}\,dt \le 8\frac{\Gamma}{\ln(\Gamma)}v(x) \quad \text{if } x+iy \in V_{\widetilde{t}}$$

by the definition of C_1. Thus w is defined and holomorphic on $V_{\widetilde{t}}$ and satisfies the right inequality of (2.12).

(b) On the other hand we have, by (2.8) and (2.15),

$$\operatorname{Re}(w(x+iy)) = \int_{-\infty}^{\infty} v(t)\cosh(C_1(x-t))\cos(C_1y)/|\cosh(C_1(x-t+iy))|^2\,dt$$
$$\ge \int_{x}^{x+1/C_1} v(t)/\cosh(C_1(x-t))\,dt \ge v(|x|)/(2\cosh(1)\ln(\Gamma))$$

if $x \ge 0$ and $|y| \le \widetilde{t}$. For $x \le 0$ we argue with $\int_{x-1}^{x} v(t)\,dt$ instead and get the same estimate. We obtain (2.12) by multiplying w with $2\cosh(1)\ln(\Gamma)$. ∎

The following elementary but useful result on decomposition of holomorphic functions on strips is proved via Hörmander's solution of the weighted $\overline{\partial}$-problem. It is therefore convenient to switch to L_2-norms instead of sup-norms.

LEMMA 2.5. *Let $0 < t_0 < t_1 < t_2 < \infty$. Then for any $0 < \theta < (t_1 - t_0)/t_2$ there is $C_1 \ge 1$ such that for any plurisubharmonic function ψ on V_{t_2}, any $f \in \mathcal{H}(V_{t_1})$ satisfying*

$$\int_{V_{t_1}} |f(z)|^2 e^{-2\psi(z)}\,dz \le 1,$$

and any $r \ge 0$, there are $f_0 \in \mathcal{H}(V_{t_0})$ and $f_2 \in \mathcal{H}(V_{t_2})$ such that $f = f_0 + f_2$ on V_{t_0} and

$$\Big(\int_{V_{t_2}} |f_2(z)|^2 e^{-2\psi(z)}(1+|z|^2)^{-2}\,dz\Big)^{1/2} \le e^r,$$
$$\Big(\int_{V_{t_0}} |f_0(z)|^2 e^{-2\psi(z)}(1+|z|^2)^{-2}\,dz\Big)^{1/2} \le C_1 e^{-r\theta}.$$

Proof. (a) Since $0 < \theta < (t_1 - t_0)/t_2$ we may find $\tau_1 \in (t_0, t_1)$ such that $\theta < (\tau_1 - t_0)/t_2$. Choose $\varphi \in C_0^\infty((-t_1, t_1))$ such that $\varphi(y) = 1$ if $|y| \le \tau_1$ and extend φ to $\mathbb{C}$ by $\varphi(x+iy) := \varphi(y)$. Set

$$\psi_r(z) := r(|\operatorname{Im}(z)| - \tau_1)/t_2.$$

Clearly, ψ_r is plurisubharmonic on $\mathbb{C}$ and we have, by the choice of τ_1,

$$\psi_r(z) \le r(t_0 - \tau_1)/t_2 \le -r\theta \quad \text{on } V_{t_0}, \tag{2.16}$$

$$\psi_r(z) \le r \quad \text{on } V_{t_2} \quad \text{and} \quad \psi_r(z) \ge 0 \quad \text{if } z \notin V_{\tau_1}. \tag{2.17}$$

By [4, Theorem 4.4.2] there is a solution $g \in L^2_{\mathrm{loc}}(V_{t_2})$ of $\overline{\partial}(g) = \overline{\partial}(f\varphi)$ such that, by (2.17) and the assumption,

$$\begin{aligned} \int_{V_{t_2}} &|g(z)|^2 e^{-2\psi_r(z)-2\psi(z)}(1+|z|^2)^{-2}\,dz \\ &\le \int_{V_{t_2}} |\overline{\partial}(\varphi f)|^2 e^{-2\psi_r(z)-2\psi(z)}\,dz \le C_1 \int_{V_{t_1}\setminus V_{\tau_1}} |f(z)|^2 e^{-2\psi(z)}\,dz \le C_1. \end{aligned} \tag{2.18}$$

(b) Set $f_2 := \varphi f - g$ and $f_0 := g$. Then $f_2 \in \mathcal{H}(V_{t_2})$ and $f_0 \in \mathcal{H}(V_{\tau_1}) \subset \mathcal{H}(V_{t_0})$ and $f = f_1 + f_2$ on V_{t_0} since $\varphi = 1$ on $V_{\tau_1} \supset V_{t_0}$.

The claim for $f_0 = g$ holds since, by (2.16) and (2.18),

$$\begin{aligned} \Big(\int_{V_{t_0}} &|g(z)|^2 e^{-2\psi(z)}(1+|z|^2)^{-2}\,dz\Big)^{1/2} \\ &\le \Big(\int_{V_{t_0}} |g(z)|^2 e^{-2\psi_r(z)-2\psi(z)}(1+|z|^2)^{-2}\,dz\Big)^{1/2} e^{-r\theta} \le C_1 e^{-r\theta} \quad \text{for } r \ge 0. \end{aligned}$$

Similarly we get, by (2.17), (2.18) and the assumption on f,

$$\begin{aligned} \Big(\int_{V_{t_2}} &|f_2(z)|^2 e^{-2\psi(z)}(1+|z|^2)^{-2}\,dz\Big)^{1/2} \\ &\le \Big(\int_{V_{t_1}} |(f\varphi)(z)|^2 e^{-2\psi(z)}\,dz\Big)^{1/2} \\ &\quad + \Big(\int_{V_{t_2}} |g(z)|^2 e^{-2\psi_r(z)-2\psi(z)}(1+|z|^2)^{-2}\,dz\Big)^{1/2} e^r \\ &\le C_2 + C_1 e^r \le (C_1 + C_2)e^r \quad \text{for } r \ge 0. \end{aligned}$$

The lemma is proved. ■

Corollary 2.6. *There are $0 < \widetilde{t}$, $0 < C_{0,+} < 1$ and $1 < C_{0,-}$ such that for any $\tau \in \mathbb{R}$, any $0 < t_0 < t < t_2 < \widetilde{t}$ and any $0 < \theta < (t - t_0)/\widetilde{t}$ there is $C_1 \ge 1$ such that for any $r \ge 0$ and any $f \in H_\tau(V_t)$ the following holds for $C_0 := C_{0,\mathrm{sign}(\tau)}$: If $\|f\|_{\tau,t} \le 1$ then there are $f_0 \in H(V_{t_0})$ and $f_2 \in H(V_{t_2})$ such that $f = f_0 + f_2$ on V_{t_0} and*

$$\|f_0\|_{C_0\tau,t_0} \le C_1 e^{-r\theta} \quad \textit{and} \quad \|f_2\|_{C_0\tau,t_2} \le e^r.$$

Proof. (a) Let $\tau \geq 0$. By (2.1), $C_1 := \int_{V_t} e^{-\tau v(z)}\,dz < \infty$. By Lemma 2.4 we thus get

$$\int\limits_{V_t} |f(z)|^2 e^{\tau\,\mathrm{Re}(w(z))/B_3}\,dz \leq C_1 \|f\|_{\tau,t}^2.$$

Since $\psi(z) := -\tau\,\mathrm{Re}(w(z))/(2B_3)$ is plurisubharmonic we may apply Lemma 2.5 for $t_1 := t$. Using the mean value property of holomorphic functions with respect to discs, (2.2), (2.1) and Lemma 2.4 again, we may pass to the sup-norms $\|f_1\|_{C_0\tau,\widetilde{t}_2}$ for $\widetilde{t} > t_2 > \widetilde{t}_2$, and $\|f_2\|_{C_0\tau,\widetilde{t}_0}$ for $\widetilde{t}_0 < t_0 < t$. Here $C_0 := C_{0,+} := 1/(4\Gamma B_3) < 1$ for Γ from (2.2).

(b) For $\tau < 0$ we argue similarly, using first the left inequality of (2.12), then Lemma 2.5 and then the right inequality of (2.12) to switch to sup-norms again. Here $C_0 := C_{0,-} := 4\Gamma B_3 > 1$. ∎

Lemma 2.3 provides a decomposition of $f \in H_\tau(V_t)$ according to the weights $\{\tau v\}$, while Corollary 2.6 provides a decomposition according to the domains $\{V_t\}$. But a joint decomposition for both systems is needed to prove Theorem 2.2. A question of this kind appears in several analytical situations and we can solve it in the present case, i.e. we can now give

Proof of Theorem 2.2. (a) Let $f \in H_\tau(V_t)$ satisfy $\|f\|_{\tau,t} \leq 1$. Choose $F_0 \in H(V_{t_0})$ and $F_2 \in H(V_{t_2})$ for f by Corollary 2.6. We cut off the functions F_j using the functions $E_{r,A}$ from Lemma 2.3 for $A := A_r$ to be determined later: with $a := \ln(\Gamma) \geq 1$ for Γ from (2.2) let

$$\begin{aligned} f_0(z) &:= (1 - E_{r,A}(az))f(z) + F_0(z)E_{r,A}(az) \quad \text{if } z \in V_{t_0},\\ f_2(z) &:= F_2(z)E_{r,A}(az) \quad \text{if } z \in V_{t_2} \end{aligned}$$

where we assume that $\widetilde{t} \leq 1/a$ without loss of generality. Then we get

$$\begin{aligned} f_0(z) + f_2(z) &= (F_0(z) + F_2(z))E_{r,A}(az) + (1 - E_{r,A}(az))f(z)\\ &= f(z) \quad \text{if } z \in V_{t_0} \end{aligned}$$

since $F_0 + F_2 = f$ on V_{t_0} by Corollary 2.6.

(b) For $\widetilde{B}_2 \geq 1$ to be determined later, choose $r_0 > 0$ such that

$$A := av^{-1}(r/(\tau_2 - C_0\tau)) - \widetilde{B}_2 > 0 \quad \text{for } r \geq r_0 \tag{2.19}$$

for C_0 from Corollary 2.6 (notice that $\tau_2 - C_0\tau \geq \tau_2 - \tau > 0$ by Corollary 2.6). By Corollary 2.6 and (2.5) (for $t := t_2$) we then get for $r \geq r_0$, using also (2.19),

$$\begin{aligned} |f_2(z)|e^{\tau_2 v(x)} &\leq C_1|F_2(z)|e^{B_1 r + \tau_2 v(x)} \leq C_1 e^{(B_1+1)r + (\tau_2 - C_0\tau)v(x)}\\ &\leq C_1 e^{(B_1+2)r} \quad \text{if } z \in V_{t_2} \text{ and } a|x| \leq A + \widetilde{B}_2. \end{aligned} \tag{2.20}$$

Let $a|x| \geq A + \widetilde{B}_2$ and set $C := (A + \widetilde{B}_2)/a = v^{-1}(r/(\tau_2 - C_0\tau))$. Then

$\gamma := |x| - C \geq 0$ and we get, by (2.14) and (2.19),

$$(\tau_2 - C_0\tau)v(x) = (\tau_2 - C_0\tau)v(C+\gamma) \leq \Gamma(\tau_2 - C_0\tau)v(C)\Gamma^\gamma$$
$$= r\Gamma e^{a\gamma} = r\Gamma e^{a|x|-A-\widetilde{B}_2} \leq re^{a|x|-A}/8$$

if $\ln(8\Gamma) \leq \widetilde{B}_2$. If $\widetilde{B}_2 \geq B_2$, from (2.6) we thus get, for $z \in V_{t_2}$ and $a|x| \geq A + \widetilde{B}_2$, by, (2.6) and Corollary 2.6,

$$|f_2(z)|e^{\tau_2 v(x)} \leq C_2 e^{r - r\exp(a|x|-A)/8 + (\tau_2 - C_0\tau)v(x)} \leq C_2 e^r.$$

Summarizing we have shown that for $r \geq r_0$,

$$|f_2(z)|e^{\tau_2 v(x)} \leq C_3 e^{C_4 r} \quad \text{if } z \in V_{t_2}. \tag{2.21}$$

(c) To estimate f_0 we first notice that by (2.5) and Corollary 2.6 (and for θ defined there)

$$|F_0(z)E_{r,A}(az)|e^{C_0\tau_0 v(x)} \leq C_5 e^{a^2 B_1 r t_0^2 - \theta r} \leq C_5 e^{-\theta r/2} \quad \text{if } z \in V_{t_0} \text{ and } t_0 \leq T_0$$

where $T_0 := \min(t/2, 1/(4a^2 B_1 \widetilde{t}))$ (and also $\theta \geq (t-t_0)/(2\widetilde{t})$ without loss of generality) .

Since $\|f\|_{\tau,t} \leq 1$ by assumption, we have

$$|(1 - E_{r,A}(az))f(z)|e^{C_0\tau_0 v(x)} \leq e^{(C_0\tau_0 - \tau)v(x)}|1 - E_{r,A}(az)| \quad \text{if } z \in V_{t_0}.$$

For D to be determined later, we estimate the right hand side as follows, using (2.5) again:

$$|1 - E_{r,A}(az)|e^{(C_0\tau_0 - \tau)v(x)} \leq C_6 e^{a^2 B_1 t_0^2 r + (C_0\tau_0 - \tau)v(x)}$$
$$\leq C_6 e^{Dr + (C_0\tau_0 - \tau)v(x)} \leq C_6 e^{-Dr}$$

if $z \in V_{t_0}$, $t_0 \leq T_1 := \sqrt{D/(a^2 B_1)}$ and $|x| \geq v^{-1}\big(\frac{2rD}{\tau - C_0\tau_0}\big)$. Notice that again $\tau - C_0\tau_0 > 0$ by Corollary 2.6.

On the other hand, by (2.19), $|x| \leq v^{-1}\big(\frac{2rD}{\tau - C_0\tau_0}\big)$ implies $a|x| \leq A - \widetilde{B}_2$ if

$$v^{-1}\left(\frac{2rD}{\tau - C_0\tau_0}\right) \leq v^{-1}\left(\frac{r}{\tau_2 - C_0\tau}\right) - 2\widetilde{B}_2/a. \tag{2.22}$$

By (2.2) we may choose $\widetilde{\Gamma}$ and then D such that

$$v(y + 2\widetilde{B}_2/a) \leq \widetilde{\Gamma} v(y) \quad \text{for large } y \quad \text{and} \quad D \leq \frac{\tau - C_0\tau_0}{(\tau_2 - C_0\tau)2\widetilde{\Gamma}}.$$

By calculating the inverse functions we then find

$$v^{-1}(t/\widetilde{\Gamma}) \leq v^{-1}(t) - 2\widetilde{B}_2/a \quad \text{for large } t \tag{2.23}$$

and we get (2.22) by the choice of D and (2.23) as follows:

$$v^{-1}\left(\frac{2rD}{\tau - C_0\tau_0}\right) \le v^{-1}\left(\frac{r}{(\tau_2 - C_0\tau)\widetilde{\Gamma}}\right)$$
$$\le v^{-1}\left(\frac{r}{\tau_2 - C_0\tau}\right) - 2\widetilde{B}_2/a \quad \text{for large } r.$$

We thus may apply (2.7) for large r and for $|x| \le v^{-1}\big(\frac{2rD}{\tau - C_0\tau_0}\big)$ (since then $a|x| \le A - \widetilde{B}_2$ by the preceding reasoning) and get by the definition of A in (2.19), since $C_0\tau_0 - \tau < 0$,

$$|1 - E_{r,A}(az)|e^{(C_0\tau_0-\tau)v(x)} \le C_7 e^{-\frac{r}{8}\exp(A-a|x|)}$$
$$\le C_7 e^{-\frac{r}{8}\exp(av^{-1}(\frac{r}{\tau_2 - C_0\tau}) - \widetilde{B}_2 - av^{-1}(\frac{2rD}{\tau - C_0\tau_0}))} \le C_7 e^{-r}.$$

Here the last estimate holds if

$$\ln(8) + av^{-1}\left(\frac{2rD}{\tau - C_0\tau_0}\right) \le av^{-1}\left(\frac{r}{\tau_2 - C_0\tau}\right) - \widetilde{B}_2.$$

Calculating inverse functions again, the latter estimate holds if and only if

$$v(y + (\ln(8) + \widetilde{B}_2)/a) \le \widehat{\Gamma} v(y) \quad \text{and} \quad D \le \frac{\tau - C_0\tau_0}{(\tau_2 - C_0\tau)2\widehat{\Gamma}}.$$

Again, $\widehat{\Gamma}$ exists by (2.2). Summarizing we have the estimate

$$|f_0(z)|e^{C_0\tau v(x)} \le C_{11}e^{-Gr}, \tag{2.24}$$

where $G := \min(\theta/2, D, 1)$, for $z \in V_{t_0}$ and r sufficiently large. Theorem 2.2 is proved by rescaling r since (2.3) has to be proved only for large r. ■

3. Logarithmically convex estimates. Logarithmically convex estimates for the norms in $H_\tau(V_t)$ are obtained much easier than the decomposition results from the preceding section. We start with the space $A([-1,1])$ of analytic germs near $[-1,1]$: for $t > 0$ let W_t denote the ellipse with foci at ± 1 and half-axes $[0, \sqrt{1+t^2}]$ and $i[0,t]$, and let $\mathcal{H}^\infty(W_t)$ be the space of bounded holomorphic functions on W_t. The norm in $\mathcal{H}^\infty(W_t)$ is denoted by $\|\ \|_t$. Clearly, $A([-1,1]) := \operatorname{ind}_{t\downarrow 0} \mathcal{H}^\infty(W_t)$.

Moreover, it is well known that there is $A > 0$ such that, for any $0 < t_0 < t < t_2$ and any $f \in \mathcal{H}^\infty(W_{t_2})$,

$$\|f\|_{At} \le \|f\|_{t_0}^{1-\theta}\|f\|_{t_2}^{\theta} \quad \text{for } \theta \ge (t - t_0)/(t_2 - t_0) \tag{3.1}$$

(see e.g. [11, (3.1)] for a proof). This implies the following:

Proposition 3.1. *There are $\widetilde{t}, A > 0$ such that for any $0 < \tau_0 < \tau < \tau_2$ (respectively, for any $\tau_0 < \tau < \tau_2 < 0$) and any $0 < t_0 < t < t_2 < \widetilde{t}$ there is*

$C_1 \geq 1$ *such that for any* $f \in H_{\tau_2}(V_{t_2})$,

$$\|f\|_{A\tau,At} \leq C_1 \|f\|_{\tau_0,t_0}^{1-\theta} \|f\|_{\tau_2,t_2}^{\theta} \tag{3.2}$$

where $\theta \geq \max[(t-t_0)/(t_2-t_0), (\tau-\tau_0)/(\tau_2-\tau_0)]$.

Proof. Let $z = x+iy \in V_{At}$. By (3.1) we get

$$\begin{aligned}
|f(x+iy)|e^{\tau v(x)} &\leq \|\!|f(x+\cdot)|\!\|_{At} e^{\tau v(x)} \\
&\leq \|\!|f(x+\cdot)|\!\|_{t}^{1-\theta} e^{(1-\theta)\tau v(x)} \|\!|f(x+\cdot)|\!\|_{t_0}^{\theta} e^{\theta\tau v(x)} \\
&\leq \|\!|f(x+\cdot)|\!\|_{t}^{1-\theta} e^{(1-\theta)\tau_0 v(x)} \|\!|f(x+\cdot)|\!\|_{t_0}^{\theta} e^{\theta\tau_2 v(x)} \\
&\leq C_1 \|f\|_{G\tau_0,t_0}^{1-\theta} \|f\|_{G\tau_2,t_2}^{\theta}
\end{aligned}$$

where $G := F$ is chosen by (2.2) such that

$$v(x+2) \leq Fv(x) + C \quad \text{ if } \tau_0 > 0$$

(and $G := 1/F$ if $\tau_2 < 0$). The second to last estimate holds since $\theta \geq (\tau-\tau_0)/(\tau_2-\tau_0)$. ■

4. Bases in weighted spaces of holomorphic germs. The results of the preceding sections will now be used to show that certain spaces of weighted germs of holomorphic functions admit a basis and in fact are isomorphic to the dual of a power series space of finite type. More precisely, we are considering the following weighted spaces of holomorphic functions:

$$\mathcal{H}_v(\mathbb{R}) := \liminf_{n\to\infty} H_{1/n}(V_{1/n})$$

where

$$H_{1/n}(V_{1/n}) := \Big\{ f \in \mathcal{H}(V_{1/n}) \;\Big|\; \|f\|_n := \|f\|_{1/n,1/n} := \sup_{z \in V_{1/n}} |f(z)|e^{v(z)/n} < \infty \Big\}$$

as before. A typical example is the test function space $P_*(\mathbb{R})$ of Fourier hyperfunctions (here $v(x) = |x|$). More examples are provided in the next section.

REMARK 4.1. Let v and u be weight functions. Then $\mathcal{H}_v(\mathbb{R}) \subset \mathcal{H}_u(\mathbb{R})$ if and only if there is $C > 0$ such that $u(x) \leq Cv(x)$ for large x.

Proof. The sufficiency is obvious. If $\mathcal{H}_v(\mathbb{R}) \subset \mathcal{H}_u(\mathbb{R})$ then the inclusion is continuous by the closed graph theorem, and Grothendieck's factorization theorem [17, 24.33] implies that there is $k \in \mathbb{N}$ such that $H_\tau^v(V_\tau) \subset H_{1/k}^u(V_{1/k})$ with continuous inclusion (again by the closed graph theorem) for $\tau := \widetilde{t}$ from Lemma 2.4. Hence there is $C_1 > 0$ such that

$$\|f\|_{1/k,1/k}^u \leq C_1 \|f\|_{\tau,\tau}^v \quad \text{ if } f \in H_\tau^v(V_\tau).$$

This can be applied to $f(t) := 1/w(t)$ since $\|f\|_{\tau,\tau}^v \leq 1$ by Lemma 2.4. This shows that

$$u(x) \leq k(\mathrm{Re}(w(x)) + \ln(C_1)).$$

Applying Lemma 2.4 again we get

$$u(x) \leq 2kB_3 v(x) \qquad \text{for large } x$$

as desired. ■

Since we are aiming at power series spaces of finite type we will need to consider rather precise continuity estimates i.e. we will use graded spaces and tame linear mappings. For the convenience of the reader the basic related notions and tools are briefly recalled.

A Fréchet space E with a fixed increasing system $(|\ \ |_j)_{j\in\mathbb{N}}$ of seminorms defining the topology of E is called a *graded Fréchet space.* A linear mapping

$$T : (E, |\ \ |_j) \to (F, |\ \ |_j)$$

between two graded (F)-spaces $(E, |\ \ |_j)$ and $(F, |\ \ |_j)$ is called (linearly) *tame* if there is $A \in \mathbb{N}$ such that for any $j \in \mathbb{N}$ there is $C_1 > 0$ such that for any $f \in E$,

$$|T(f)|_j \leq C_1 |f|_{Aj}.$$

Finally, T is called a *tame isomorphism* if T is bijective and T and T^{-1} are tame.

The main tool used in this paper is the tame structure theory of power series spaces of finite type. Recall that power series spaces of finite type and their canonical gradings are defined as follows: Let $(a_k)_{k\in\mathbb{N}}$ be an increasing sequence of positive numbers. Then

$$\Lambda_0(a_k) := \Big\{ (c_k)_{k\in\mathbb{N}} \ \Big|\ \forall j \in \mathbb{N} : |(c_k)|_j := \sum_{k\in\mathbb{N}} |c_k| e^{-a_k/j} < \infty \Big\}.$$

The existence of a basis is provided by tame variants of the conditions $(\overline{\Omega})$ and $(\underline{\mathrm{DN}})$ of Vogt (see e.g. [17]) which were introduced in [9]: Let $(E, |\ \ |_j)$ be a graded Fréchet space and let U_n denote the unit ball with respect to $|\ \ |_n$. We say that E has *property* $(\overline{\Omega})_t$ if for any $k \in \mathbb{N}$ there is $B \in \mathbb{N}$ such that for any $n, j \in \mathbb{N}$ there is $C_1 > 0$ such that for any $r > 0$,

$$U_{Bn} \subset rU_j + C_1 r^{1-n} U_k. \tag{4.1}$$

Furthermore, E has property $(\underline{\mathrm{DN}})_t$ if there are $p, B \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ there are $m \in \mathbb{N}$ and $C_1 > 0$ such that

$$|f|_n \leq C_1 |f|_p^{1/(Bn)} |f|_m^{1-1/(Bn)}. \tag{4.2}$$

An easy calculation shows that power series spaces of finite type satisfy $(\overline{\Omega})_t$ and $(\underline{\mathrm{DN}})_t$ when endowed with their canonical grading from above. The following theorem states that the converse is also true, and it will be applied to $\mathcal{H}_v(\mathbb{R})'_b$ being the basic tool for our considerations:

THEOREM 4.2 ([9, Theorem 1.5]). *A nuclear graded Fréchet space E is tamely isomorphic to a power series space of finite type if E satisfies $(\overline{\Omega})_t$ and $(\underline{\mathrm{DN}})_t$.*

In this section we will prove $(\overline{\Omega})_t$ and $(\underline{\mathrm{DN}})_t$ in a dual formulation. Using [17, Lemma 29.13] the following is easily shown: A graded Fréchet space E satisfies $(\overline{\Omega})_t$ if and only if for any $k \in \mathbb{N}$ there is $B \in \mathbb{N}$ such that for any $n, j \in \mathbb{N}$ there is $C_1 > 0$ such that

$$| \ |_{Bn}^* \le C_1(| \ |_j^*)^{1-1/n}(| \ |_k^*)^{1/n} \tag{4.3}$$

where

$$|f|_k^* := \sup\{|\nu(f)| \mid \nu \in E, |\nu|_k \le 1\}, \qquad f \in E',$$

are the dual "seminorms" in E_b'.

Similarly (see e.g. [22, Lemma 2.4]), E has $(\underline{\mathrm{DN}})_t$ if and only if there are $p, B \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ there are $C_1 > 0$ and $m \in \mathbb{N}$ such that for any $r > 0$,

$$B_n \subset rB_p + C_1 r^{-1/(Bn)} B_m \tag{4.4}$$

where B_n are the unit balls with respect to $| \ |_n^*$.

Clearly, the "norms" $\| \ \|_n$ in $\mathcal{H}_v(\mathbb{R})$ could also be defined by taking L_2-norms instead of sup-norms leading to a tamely equivalent topology on $\mathcal{H}_v(\mathbb{R})$. This implies that $\mathcal{H}_v(\mathbb{R})$ is a (DFN)-space (compare e.g. [13, Section 2, Satz 2]). The Fréchet space $\mathcal{H}_v(\mathbb{R})_b'$ will always be considered with the canonical grading defined by

$$|\nu|_n := \sup\{|\nu(f)| \mid f \in H_{1/n,1/n}(V_{1/n}),\ \|f\|_n \le 1\} \quad \text{if } \nu \in \mathcal{H}_v(\mathbb{R})'.$$

Lemma 4.3. *Let v be a weight function. Then the space $\mathcal{H}_v(\mathbb{R})$ endowed with the canonical "norms" $\| \ \|_n$ is tamely isomorphic to $\mathcal{H}_v(\mathbb{R})$ endowed with the dual "norms" $| \ |_n^*$.*

Proof. Since v is a weight function, the dual "norms" $| \ |_n^*$ on $\mathcal{H}_v(\mathbb{R})$ satisfy

$$\begin{aligned}|g|_n^* &= \sup\{|\nu(g)| \mid \nu \in \mathcal{H}_v(\mathbb{R})',\ |\nu|_n \le 1\}\\ &= \sup\{|\nu(g)| \mid \nu \in \mathcal{H}_v(\mathbb{R})',\ \sup\{|\nu(f)| \mid \|f\|_n \le 1\} \le 1\}\\ &\le \|g\|_n \quad \text{if } g \in \mathcal{H}_v(\mathbb{R}).\end{aligned}$$

Moreover, for Γ from (2.2) we get

$$\begin{aligned}\|g\|_{\Gamma n} &= \sup\{|g(x+iy)|e^{v(x)/(\Gamma n)} \mid |y| < 1/(\Gamma n)\}\\ &\le \sup\Big\{\Big|\sum_{j=0}^k g^{(j)}(x)(iy)^j/j!\Big|e^{v(x)/(\Gamma n)} \ \Big| \ |y| < 1/(\Gamma n),\ k \in \mathbb{N}\Big\}\\ &= \sup\{|\nu_{k,x,y}(g)| \mid x \in \mathbb{R},\ |y| < 1/(\Gamma n),\ k \in \mathbb{N}\} \le C_1|g|_n^*\end{aligned}$$

where

$$\nu_{k,x,y}(g) := \sum_{j=0}^k g^{(j)}(x)((iy)^j/j!)e^{v(x)/(\Gamma n)}.$$

The last estimate follows since $\nu_{k,x,y} \in \mathcal{H}_v(\mathbb{R})'$ and since, for $n \geq 1$,

$$\begin{aligned}|\nu_{k,x,y}(g)| &\leq \sum_{j=0}^{\infty} |g^{(j)}(x)|((\Gamma n)^{-j}/j!)e^{v(x)/(\Gamma n)} \\ &\leq \sum_{j=0}^{\infty}(2/\Gamma)^j \sup\{|g^{(j)}(x)|((2n)^{-j}/j!)e^{v(x)/(\Gamma n)} \mid j \in \mathbb{N}\} \\ &\leq C_1\|g\|_n \sup\{e^{-v(x+z)/n+v(x)/(\Gamma n)} \mid x \in \mathbb{R},\ |z| \leq 1/n\} \\ &\leq C_1\|g\|_n \quad \text{if } |x| \geq x_0+1\end{aligned}$$

by Cauchy's estimate with radius $1/(2n)$ and (2.2) since we can assume that $\Gamma \geq 3$ in (2.2). ∎

We may thus use $\|f\|_n$ instead of the dual norms $|f|_n^*$ when proving $(\overline{\Omega})_t$ and $(\underline{\mathrm{DN}})_t$ for $\mathcal{H}_v(\mathbb{R})_b'$ via (4.3) and (4.4).

Theorem 4.4.

(a) *$\mathcal{H}_v(\mathbb{R})_b'$ is tamely isomorphic to a power series space $\Lambda_0(\alpha_n)$ of finite type.*
(b) *$\mathcal{H}_v(\mathbb{R})$ is tamely isomorphic to $\Lambda_0(\alpha_n)_b'$.*

Proof. (b) follows from (a) by duality. To prove (a) we have to show (4.3) and (4.4) for $\|f\|_n := \|f\|_{1/n,1/n}$ by Theorem 4.2 and the remarks above.

The estimate (4.3) now follows from Proposition 3.1 upon choosing $\tau = t = 1/(kn), \tau_0 = t_0 = 1/j$ if $j > kn$ and $\tau_2 = t_2 = 1/k$. We thus get

$$\theta = (1/(kn) - 1/j)/(1/k - 1/j) \leq 1/n$$

(for $j \leq kn$, (4.3) is trivially satisfied).

The proof of (4.4) follows from Theorem 2.2 by choosing $\tau = t = 1/n$, $\tau_0 = t_0 = 1/(K_0 m)$ for $m > 2n(C_0+1)/K_0$ and $\tau_2 = t_2 = 1/p$ for $2/p < \widetilde{t}$ (notice that K_0 is at most 1). ∎

Since we know by Theorem 4.4 that $\mathcal{H}_v(\mathbb{R})$ is tamely isomorphic to some $\Lambda_0(\alpha_n)_b'$ we can use the diametral dimension (see [5, p. 209]) to determine the sequence $(\alpha_n)_n$. For this we need to find suitable subspaces or quotients of $\mathcal{H}_v(\mathbb{R})$ (and represent $\mathcal{H}_v(\mathbb{R})$ as a subspace or quotient) of spaces for which the diametral dimension can be calculated. This is done in [12]. In fact, we also need the generalized diametral dimension introduced in [8] which is based on a linear topological invariant of $(\underline{\mathrm{DN}})$ type. Here the decomposition in Theorem 2.2 is used again. In this way the sequence (α_n) in Theorem 4.4 is calculated in [12, Theorem 4.6], giving the following result:

Theorem 4.5. *$\mathcal{H}_v(\mathbb{R})$ is tamely isomorphic to $\Lambda_0(n/g(n))_b'$ where g is the inverse function of $f(t) := tv(t)$.*

5. Examples. Since the assumptions needed in this paper are hardly restrictive, many examples are available, and we will mention some of them in this section. We start with an easy observation:

LEMMA 5.1. *A positive function $v \in C^1([0,\infty[)$ satisfies (2.2) if there is $C > 0$ such that*

$$v'(x) \le Cv(x) \quad \textit{for large } x. \tag{5.1}$$

If $v(x) = e^{w(\ln(x))}$ with $w \in C^1([0,\infty[)$ then (5.1) is equivalent to

$$w'(x) \le Ce^x \quad \textit{for large } x. \tag{5.2}$$

Proof. This is evident since

$$\ln(v(x+1)) - \ln(v(x)) = \int_x^{x+1} (\ln(v(t)))'\,dt = \int_x^{x+1} \frac{v'(t)}{v(t)}\,dt \le C \qquad \text{for large } x$$

and since $v'(x) = v(x)w'(\ln(x))/x$ if $v(x) = e^{w(\ln(x))}$. ∎

EXAMPLE 5.2. Each of the following is a weight function:

(i) $v(x) := v_{\alpha,\beta}(x) := (\ln(x))^\alpha(\ln(\ln(x)))^\beta$ for $x \ge x_0$ where $\alpha > 1$ and $\beta \in \mathbb{R}$ or $\alpha = 1$ and $\beta > 0$.
(ii) $v(x) := e^{v_{\alpha,\beta}(x)}$ for $x \ge x_0$ where $\alpha > 0$ and $\beta \in \mathbb{R}$.
(iii) $v(x) := v_{\alpha,\beta}(e^x) = x^\alpha(\ln(x))^\beta$ for $x \ge x_0$ where $\alpha > 0$ and $\beta \in \mathbb{R}$.
(iv) $v(x) := e^{ax^\alpha(\ln(x))^\beta}$ where $a > 0$ and $1 > \alpha > 0$ and $\beta \in \mathbb{R}$ or $\alpha = 1$ and $\beta \le 0$.

Proof. (2.1) is obviously satisfied. (2.2) directly follows from Remark 5.1. ∎

The functions $v(x) := e^{a|x|}$, $a > 0$, are the maximal weight functions satisfying (2.2) by (2.14).

Of course, products of the weight functions from Example 5.2 are also weight functions.

Two sequences (α_n) and (β_n) are said to be *equivalent* if there is $C > 1$ such that

$$\alpha_n/C \le \beta_n \le C\alpha_n \quad \text{for large } n.$$

Notice that $\Lambda_0(\alpha_n) = \Lambda_0(\beta_n)$ if (α_n) is equivalent to (β_n). Hence we only need to calculate the sequence $(n/g(n))$ from Theorem 4.5 up to equivalence.

We recall the results from [12, Example 5.3] for the examples from 5.2:

EXAMPLE 5.3. $(n/g(n))$ is equivalent to:

(i) $(v(n))$ if $v(x) := v_{\alpha,\beta}(x) := (\ln(x))^\alpha(\ln(\ln(x)))^\beta$ for $x \ge x_0$ where $\alpha > 1$ and $\beta \in \mathbb{R}$ or $\alpha = 1$ and $\beta > 0$.
(ii) $(v(n))$ if $v(x) := e^{v_{\alpha,\beta}(x)}$ for $x \ge x_0$ where $1/2 > \alpha > 0$ and $\beta \in \mathbb{R}$ or $\alpha = 1/2$ and $\beta \le 0$.
(iii) $(ne^{-(\ln(n))^{1/\alpha}})$ if $v(x) := e^{(\ln(x))^\alpha}$ where $\alpha \ge 2$.

(iv) $n^{\alpha/(\alpha+1)}$ if $v(x) := x^\alpha$ where $\alpha > 0$.
(v) $n(\ln(n))^{-1/\alpha}$ if $v(x) := e^{ax^\alpha}$ where $1 \geq \alpha > 0$ and $a > 0$.

Specifically the spaces S_α^1 of Gelfand–Shilov for $\alpha > 0$ satisfy the assumptions of this paper. Recall that the spaces S_α^β are defined as follows (for $\alpha, \beta > 0$, see [2, Chap. IV]):

$$S_\alpha^\beta := \{f \in C^\infty(\mathbb{R}) \mid \exists A, B > 0\ \forall k, j \in \mathbb{N}_0 : |x^k f^{(j)}(x)| \leq CA^k k^{k\alpha} B^j j^{j\beta}\}.$$

Example 5.4.

(a) Let S_α^1 be endowed with the grading defined by

$$\|f\|_n := \sup_{j,k\in\mathbb{N},\, x\in\mathbb{R}} |x^k f^{(j)}(x)|(kn)^{-k\alpha}(jn)^{-j}.$$

Then S_α^1 is tamely isomorphic to $\Lambda_0(n^{1/(\alpha+1)})_b'$ for $\alpha > 0$.

(b) Let S_1^β be endowed with the grading defined by

$$\|f\|_n := \sup_{j,k\in\mathbb{N},\, x\in\mathbb{R}} |x^k f^{(j)}(x)|(kn)^{-k}(jn)^{-j\beta}.$$

Then S_1^β is tamely isomorphic to $\Lambda_0(n^{1/(\beta+1)})_b'$ for $\beta > 0$.

Proof. (a) By [2, Chap. IV, Sect. 2], S_α^1 is tamely isomorphic to $\mathcal{H}_v(\mathbb{R})$ for the weight $v(x) := |x|^{1/\alpha}$ treated in Example 5.2(iii). The claim now follows by Theorem 4.5 and Example 5.2(iii).

(b) This follows from (a) since the Fourier transform is a tame isomorphism between S_1^β and S_β^1 by [2, Chap. IV, Sect. 6.2, formula (11)]. ∎

In particular, we have given a new proof for the result from [10] that the space $P_*(\mathbb{R})_b'$ of Fourier hyperfunctions on $\mathbb{R}$ is tamely isomorphic to $\Lambda_0(n^{1/2})$. Since $P_*(\mathbb{R}) = S_1^1$ this is the special case $\alpha = 1$ of Example 5.4(a) (see [6] for the respective definitions).

By [2, Chap. IV, Sect. 2.3]), S_1^β with the above grading can be tamely identified for $0 < \beta < 1$ with the following weighted space of entire functions:

$$\mathcal{H}_{1,\frac{1}{1-\beta}} := \Big\{f \in H(\mathbb{C}) \ \Big|\ \exists n \in \mathbb{N} : |f|_n := \sup_{z\in\mathbb{C}} |f(z)| e^{\frac{1}{n}|\mathrm{Re}(z)| - n^{\frac{\beta}{1-\beta}} |\mathrm{Im}(z)|^{\frac{1}{1-\beta}}} < \infty\Big\}.$$

Corollary 5.5. *When endowed with the above grading, $\mathcal{H}_{1,1/(1-\beta)}$ is tamely isomorphic to $\Lambda_0(n^{1/(\beta+1)})_b'$ for $1 > \beta > 0$.*

The following example shows that different spaces $\mathcal{H}_v(\mathbb{R})$ may be isomorphic.

Example 5.6. Let $v_a(x) := e^{a|x|^\beta}$ for fixed $0 < \beta \leq 1$. Then the spaces $H_{v_a}(\mathbb{R})$, $a > 0$, are a strictly decreasing scale of weighted spaces which are isomorphic for any $a > 0$.

Proof. By Remark 4.1 we have $H_{v_b}(\mathbb{R}) \subsetneq H_{v_a}(\mathbb{R})$ if $0 < a < b$. The spaces are isomorphic by Theorem 4.5 and Example 5.3(v). ∎

6. A modification. The space of modified Fourier hyperfunctions (see [7], [20]) does not fit in the setting used so far since the corresponding test functions are defined on conic neighborhoods of $\mathbb{R}$ defined by

$$W_{1/n} := \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < (1 + |\mathrm{Re}(z)|)/n\}.$$

However a slight modification of our arguments will also include this type of weighted holomorphic germs defined by weight conditions as before: let $\widetilde{\mathcal{H}}_v(\mathbb{R}) := \liminf_{n\to\infty} \mathcal{H}_{1/n}(W_{1/n})$, where

$$\mathcal{H}_{1/n}(W_{1/n}) := \Big\{ f \in \mathcal{H}(W_{1/n}) \Bigm| \|f\|_n := \sup_{z \in W_{1/n}} |f(z)| e^{v(|\mathrm{Re}(z)|)/n} < \infty \Big\}.$$

THEOREM 6.1. *Let $v : [0,\infty[\to [0,\infty[$ be continuous and strictly increasing and let $\ln(\ln(t)) = o(v(t))$. Also assume that v is stable, i.e. there is $C > 0$ such that*

$$v(2x) \le Cv(x) \quad \text{if } x \ge C.$$

(a) *$\widetilde{\mathcal{H}}_v(\mathbb{R})$ is tamely isomorphic to $\mathcal{H}_{v\circ\exp}(\mathbb{R})$.*
(b) *$\widetilde{\mathcal{H}}_v(\mathbb{R})'_b$ is tamely isomorphic to $\Lambda_0(n/\widetilde{g}(n))$ where $\widetilde{g}$ is the inverse function of $\widetilde{f}(t) := tv(e^t)$.*
(c) *$\widetilde{\mathcal{H}}_v(\mathbb{R})$ is tamely isomorphic to $\Lambda_0(n/\widetilde{g}(n))'_b$.*

Proof. We only need to show (a) since the remaining statements follow from Theorem 4.5, because $v \circ \exp$ is a weight function by the stability of v.

(a) follows from the fact that the mapping

$$T : \widetilde{\mathcal{H}}_v(\mathbb{R}) \to \mathcal{H}_{v\circ\exp}(\mathbb{R}), \qquad f \mapsto f \circ \sinh,$$

defines a tame isomorphism between $\widetilde{\mathcal{H}}_v(\mathbb{R})$ and $\mathcal{H}_{v\circ\exp}(\mathbb{R})$. Notice that

$$v(e^{|x|})/\Gamma \le v(e^{|x|}/2) \le v(|\mathrm{Re}(\sinh(x+iy))|) = v(|\sinh(x)\cos(y)|) \le v(e^{|x|})$$

for $|y| \le 1$ by the stability of v. ∎

From Example 5.2 we immediately get

EXAMPLE 6.2. The following functions v satisfy the assumptions of Theorem 6.1:

(i) $v(x) := (\ln(\ln(x)))^\alpha$ for $x \ge x_0$ where $\alpha > 1$.
(ii) $v(x) := e^{(\ln(\ln(x)))^\alpha}$ for $x \ge x_0$ where $\alpha > 0$.
(iii) $v(x) := (\ln(x))^\beta$ for $x \ge x_0$ where $\beta > 0$.
(iv) $v(x) := e^{a(\ln(x))^\beta}$ for $x \ge x_0$ where $1 \ge \beta > 0$ and $a > 0$.
(v) $v(x) := x^\beta$ for $\beta > 0$.

The corresponding functions $\widetilde{g}$ in Theorem 6.1 can be obtained from Example 5.3.

THEOREM 6.3. *The space of modified Fourier hyperfunctions on* $\mathbb{R}$ *is tamely isomorphic to* $\Lambda_0(n/\ln(n))$.

Proof. The space of test functions for the modified Fourier hyperfunctions on $\mathbb{R}$ is just $\widetilde{\mathcal{H}}_v(\mathbb{R})$ for $v(x) := |x|$ (see [7, 20] for the respective definitions). The conclusion thus follows from Theorem 6.1 and Example 5.3(iv). ∎

Since the space of Fourier hyperfunctions is isomorphic to $\Lambda_0(n^{1/2})$, the spaces of Fourier hyperfunctions and of modified Fourier hyperfunctions are not isomorphic.

The sequence $(n/\ln(n))_n$ is maximal for the sequences $(n/g(n))_n$ considered in Theorem 4.5(use the remark after Example 5.2). By [21, Corollary 4.3] this implies that $\Lambda_0(n/\ln(n))$ is isomorphic to a closed subspace of $\Lambda_0(n/g(n))$ for g as in Theorem 4.5 (notice that the stability of $\mathcal{H}_v(\mathbb{R})$ is proved in [12, Corollary 4.7]). Therefore, the modified Fourier hyperfunctions are contained as closed subspaces in all spaces $\mathcal{H}_v(\mathbb{R})'_b$ considered in this paper.

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