ANNALES
POLONICI MATHEMATICI
88.2 (2006)

On absolutely monotone set-valued functions

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Abstract. We define absolutely monotone multifunctions and prove their analyticity on an interval $[0,b)$.

1. Let $f:[a,b)\to\mathbb{R}$. The pth order difference $\Delta_s^p f(t)$ of f is defined inductively as follows:

$$\Delta_s^0 f(t)=f(t),\qquad \Delta_s^{p+1} f(t)=\Delta_s^p f(t+s)-\Delta_s^p f(t)$$

for every nonnegative integer p, $t\in[a,b)$, $s>0$ such that $t+(p+1)s<b$.

We say that the function f is *absolutely monotone* in the interval $[a,b)$ if $\Delta_s^p f(t)\ge 0$ for $a\le t\le t+ps<b$, $p=0,1,\ldots.$ The following Bernstein theorem is well known (see e.g. [3, Theorem 2.3.2]):

THEOREM. *Every absolutely monotone function $f:[0,b)\to\mathbb{R}$ is analytic*:

$$f(t)=\sum_{n=0}^{\infty}a_n t^n$$

in $[0,b)$ with $a_n\ge 0, n=0,1,\ldots.$

2. In this paper we prove an analogue of S. Bernstein's theorem for absolutely monotone set-valued functions. Let Y be a real normed space and let $\mathrm{cc}(Y)$ denote the family of all nonempty compact convex subsets of Y. A set $C\in\mathrm{cc}(Y)$ is the *Hukuhara difference* of $A\in\mathrm{cc}(Y)$ and $B\in\mathrm{cc}(Y)$ if

$$A=B+C=\{b+c: b\in B,\ c\in C\}$$

(see [2]). If the difference $C=A-B$ exists, then it is unique. This is a consequence of the following:

LEMMA 1 (cf. [5]). *Let A,B and C be subsets of a real topological vector space such that*

$$A+B\subset C+B.$$

If C is convex closed and B is nonempty bounded, then $A\subset C$.

2000 *Mathematics Subject Classification*: 26A48, 26A51, 26E25.
Key words and phrases: absolutely monotone set-valued functions.

Now, let $-\infty < a < b \le \infty$ and let $H : [a,b) \to \mathrm{cc}(Y)$. We define the pth differences $\Delta_s^p H(t)$ by the recurrence

$$\Delta_s^0 H(t) = H(t), \qquad \Delta_s^{p+1} H(t) = \Delta_s^p H(t+s) - \Delta_s^p H(t)$$

for every nonnegative integer p, $t \in [a,b)$, $s > 0$ such that $t + (p+1)s < b$.

A set-valued function is said to be *absolutely monotone* if all differences $\Delta_s^p H(t)$ exist and each contains zero.

EXAMPLE. Let $A \in \mathrm{cc}(Y)$ be such that $0 \in A$. Suppose that $h : [a,b) \to [0,\infty)$. Then $H(t) = h(t)A$ is an absolutely monotone set-valued function if and only if h is an absolutely monotone real function.

We can observe the following:

REMARK. Let b and α be positive numbers, $H : [0,b) \to \mathrm{cc}(Y)$ and $G(t) = H(\alpha t)$ on $[0, b/\alpha)$. Then G is absolutely monotone if and only if H is absolutely monotone.

Let $G : [0,1] \to \mathrm{cc}(Y)$ be a given multifunction. The polynomial

$$B_n(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} G\left(\frac{i}{n}\right)$$

is called the nth *Bernstein polynomial* of G.

THEOREM 1. *If $G : [0,1] \to \mathrm{cc}(Y)$ is continuous* (*with respect to the Hausdorff metric d in* $\mathrm{cc}(Y)$), *then*

$$d(B_n(t), G(t)) \le \frac{3}{2}\omega\left(\frac{1}{\sqrt{n}}\right),$$

where

$$\omega(\delta) = \sup\{d(G(t''), G(t')) : |t'' - t'| < \delta\}.$$

The proof of this theorem runs similarly to the proof of Bernstein's approximation theorem (cf. [3]).

LEMMA 2. *Let $G : [0,1] \to \mathrm{cc}(Y)$ be a multifunction. Then*

$$B_n(t) = \sum_{i=0}^{n} \binom{n}{i} t^i \Delta_{1/n}^i G(0) \tag{1}$$

for positive integers n and $t \in [0,1]$.

Proof. Let "$\sim$" denote the Rådström equivalence relation between pairs of members of $\mathrm{cc}(Y)$ defined by the formula

$$(A,B) \sim (C,D) \Leftrightarrow A + D = B + C.$$

For any pair (A,B), $[A,B]$ denotes its equivalence class. All equivalence classes form a linear space $\widetilde{Y}$ with addition defined by the rule

$$[A,B] + [C,D] = [A+C, B+D]$$

and scalar multiplication

$$\lambda[A,B]=\begin{cases}[\lambda A,\lambda B] & \text{for } \lambda\ge 0,\\ \lambda[A,B]=[-\lambda B,-\lambda A] & \text{for } \lambda<0\end{cases}$$

(cf. [5]).

Consider the function $g:[0,1]\to\widetilde{Y}$ defined as follows:

$$g(t)=[G(t),\{0\}].$$

It can be proved by induction that

$$\Delta_s^p g(t)=[\Delta_s^p G(t),\{0\}] \tag{2}$$

and

$$\Delta_s^p g(t)=\sum_{i=0}^{p}\binom{p}{i}t^i(-1)^{p-i}g(t+is) \tag{3}$$

for nonnegative integers p, $t\in[0,1)$ and $s>0$ such that $t+ps<1$.

Let b_n be Bernstein's polynomials of g:

$$b_n(t)=\sum_{i=0}^{n}\binom{n}{i}t^i(1-t)^{n-i}g\left(\frac{i}{n}\right).$$

Then

$$b_n(t)=[B_n(t),\{0\}].$$

Using Newton's binomial formula, replacing j by $j-i$ in the second sum below, changing the order of summation, and then making use of the identity $\binom{n}{i}\binom{n-i}{j-i}=\binom{n}{j}\binom{j}{i}$ and equality (3) we obtain

$$\begin{aligned}b_n(t)&=\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}(-1)^j t^{i+j}g\left(\frac{i}{n}\right)\\&=\sum_{i=0}^{n}\sum_{j=i}^{n}\binom{n}{i}\binom{n-i}{j-i}(-1)^{j-i}t^j g\left(\frac{i}{n}\right)\\&=\sum_{j=0}^{n}\sum_{i=0}^{j}\binom{n}{j}\binom{j}{i}(-1)^{j-i}t^j g\left(\frac{i}{n}\right)=\sum_{j=0}^{n}\binom{n}{j}t^j\Delta_{1/n}^j g(0).\end{aligned}$$

According to (2) we have

$$\begin{aligned}\left[\sum_{i=0}^{n}\binom{n}{i}t^i\Delta_{1/n}^i G(0),\{0\}\right]&=\sum_{i=0}^{n}\binom{n}{i}t^i[\Delta_{1/n}^i G(0),\{0\}]\\&=\sum_{i=0}^{n}\binom{n}{i}t^i\Delta_{1/n}^i g(0)=b_n(t)=[B_n(t),\{0\}].\end{aligned}$$

Thus

$$\left(\sum_{i=0}^{n}\binom{n}{i}t^i\Delta^i_{1/n}G(0),\{0\}\right)\sim(B_n(t),\{0\})$$

and (1) holds.

LEMMA 3. *Let* $0<c\le b$ *and* $H:[0,b)\to \mathrm{cc}(Y)$. *If* $A_i,B_i\in\mathrm{cc}(Y)$, $i=0,1,\ldots,$ *are such that*

$$H(t)=\sum_{n=0}^{\infty}t^nA_n \quad \textit{for } t\in[0,b),$$

$$H(t)=\sum_{n=0}^{\infty}t^nB_n \quad \textit{for } t\in[0,c),$$

then $A_i=B_i$ *for* $i=0,1,\ldots.$

Proof. We see that $A_0=H(0)=B_0$. Suppose that

$$A_0=B_0,\ \ldots,\ A_k=B_k.$$

Then

$$A_{k+1}=\lim_{t\to 0+}\frac{H(t)-\sum_{i=0}^{k}t^iA_i}{t^{k+1}}=\lim_{t\to 0+}\frac{H(t)-\sum_{i=0}^{k}t^iB_i}{t^{k+1}}=B_{k+1}.$$

THEOREM 2. *A set-valued function* $H:[0,b)\to\mathrm{cc}(Y)$ *is absolutely monotone if and only if there exist sets* $A_i\in\mathrm{cc}(Y)$, $i=0,1,\ldots,$ *containing zero such that*

$$H(t)=\sum_{n=0}^{\infty}t^nA_n \quad \textit{for } t\in[0,b). \tag{4}$$

Proof. 1. Suppose that $H:[0,b)\to\mathrm{cc}(Y)$ is of the form (4) and that $0\in A_n\in\mathrm{cc}(Y)$. We see that

$$H(t+s)=\sum_{n=0}^{\infty}(t+s)^nA_n=H(t)+\sum_{n=0}^{\infty}((t+s)^n-t^n)A_n,$$

therefore

$$\Delta^1_sH(t)=\sum_{n=0}^{\infty}\Delta^1_st^nA_n.$$

By induction it may be shown that

$$\Delta^p_sH(t)=\sum_{n=0}^{\infty}\Delta^p_st^nA_n.$$

Thus all differences $\Delta^p_sH(t)$ exist. As they contain zero, H is an absolutely monotone multifunction.

2. Now, suppose that $H : [0,b) \to \mathrm{cc}(Y)$ is an absolutely monotone multifunction. The differences $\Delta_s^1 H(t)$ and $\Delta_s^2 H(t)$ exist and contain zero, therefore

$$H(t) \subset H(t) + \Delta_s^1 H(t) = H(t+s)$$

and

$$\begin{aligned} 2H(t+s) \subset 2H(t+s) + \Delta_s^2 H(t) &= H(t+s) + H(t) + \Delta_s^1 H(t) + \Delta_s^2 H(t) \\ &= H(t+s) + H(t) + \Delta_s^1 H(t+s) = H(t) + H(t+2s). \end{aligned}$$

Thus H is increasing and midconcave in $[0,b)$.

Fix a number $c \in (0,b)$. The function H, being midconcave and bounded on $[0,c]$, is continuous, according to Theorem 4.4 in [4]. Define $G(t) = H(ct)$ for $t \in [0,1]$. Then G is continuous and by Theorem 1 it is the uniform limit of the sequence of its Bernstein polynomials $B_n(t)$. By Lemma 2 we have

$$B_n(t) = \sum_{i=0}^{n} \binom{n}{i} t^i \Delta_{1/n}^i G(0) = \sum_{i=0}^{n} t^i A_i^n,$$

where $A_i^n = \binom{n}{i} \Delta_{1/n}^i G(0)$. We note that $0 \in A_0^n$ and

$$A_0^n \subset B_n(1) = G(1). \tag{5}$$

Since $G(1)$ is compact, the family of all closed subsets of $G(1)$ is compact (see [1, p. 41]). By (5) there exists a strictly increasing sequence (n_k^0) and $A_0(c) \in \mathrm{cc}(Y)$ such that

$$A_0^{n_k^0} \to A_0(c).$$

Similarly, since

$$A_1^{n_k^0} \subset B_{n_k^0}(1) = G(1),$$

there exists a strictly increasing subsequence (n_k^1) of (n_k^0) and $A_1 \in \mathrm{cc}(Y)$ such that

$$A_0^{n_k^1} \to A_1(c)$$

and so on. Applying the diagonalization procedure to the sequences $(A_0^{n_k^0})$, $(A_0^{n_k^1}), \ldots$ we obtain a strictly increasing sequence (n_k) such that

$$A_0^{n_k} \to A_0(c),\ A_1^{n_k} \to A_1(c),\ \ldots$$

Fix $t \in [0,1)$, $\varepsilon > 0$ and define

$$S_n(t) = \sum_{i=0}^{n} t^i A_i(c) \quad \text{for } n = 0, 1, \ldots.$$

Choose a positive integer k so large that $2\|G(1)\| t^k (1-t)^{-1} < \varepsilon/3$, where $\|G(1)\| = \sup\{\|y\| : y \in G(1)\}$, and then choose L large enough to get $d(B_{n_l}(t), G(t)) < \varepsilon/3$ and $\sum_{j=0}^{k-1} d(A_j^{n_l}, A_j) < \varepsilon/3$ for $l \ge L$. Then

$$d(S_{n_l}(t), G(t)) \le d(S_{n_l}(t), B_{n_l}(t)) + d(B_{n_l}(t), G(t)) \\ \le 2\varepsilon/3 + \sum_{i=k+1}^{n_l} t^i d(A_i^{n_l}, A_i) \le (2/3)\varepsilon + 2\|G(1)\| \frac{t^{k+1}}{1-t} < \varepsilon.$$

Thus

$$\lim_{l\to\infty} S_{n_l}(t) = G(t) \tag{6}$$

and according to Theorem II-2 in [1],

$$G(t) = \overline{\bigcup_{l=1}^{\infty} S_{n_l}(t)}.$$

Using the monotonicity of the sequence $(S_n(t))$ we get

$$S_l(t) \subset S_{n_l}(t) \subset G(t) \quad \text{for } l = 0, 1, \ldots.$$

Therefore the sequence $d(G(t), S_l(t))$ is decreasing. By (6),

$$\lim_{l\to\infty} d(G(t), S_l(t)) = \lim_{l\to\infty} d(G(t), S_{n_l}(t)) = 0.$$

Consequently,

$$G(t) = \lim_{n\to\infty} S_n(t) = \sum_{i=0}^{\infty} t^i A_i \quad \text{for } t \in [0, 1).$$

The definition of G leads to

$$H(t) = G(t/c) = \sum_{i=0}^{\infty} t^i c^{-i} A_i \quad \text{for } t \in [0, c).$$

Now (4) follows from Lemma 3.

References

[1] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math. 580, Springer, Berlin, 1977.

[2] M. Hukuhara, *Intégration des applications mesurables dont la valeur est un compact convexe*, Funcial. Ekvac. 10 (1967), 205–223;

[3] S. Łojasiewicz, *An Introduction to the Theory of Real Functions*, Wiley, 1988.

[4] K. Nikodem, *K-convex and K-concave set-valued functions*, Zeszyty Nauk. Politech. Łódz. Mat. 559, Rozprawy Naukowe 114, 1989.

[5] H. Rådström, *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc. 3 (1952), 165–169.

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Received 18.7.2005
and in final form 2.11.2005 (1596)